Approximation vs. small deviation of SαS Lévy process

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Approximation Quantities

The Gaussian Case

Example: Fractional Brownian Motion

Linear vs. Kolmogorov Widths

Links to Quantization and Small Deviation

Symmetric stable r.v.

General Results

Example: Symmetric Stable Lévy motion

Outline

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Classical Widths

Let

- E be a Banach space,
- $B \subseteq E$ precompact,
- $Q_N : E \to E/N$ quotient mapping ($N \subseteq E$ closed linear subspace)

define *linear widths*

$$a_n(B,E) := \inf\left\{\sup_{x\in B} \|x - u(x)\| \;:\; u : E \to E \text{ linear, } \mathrm{rk}(u) \le n\right\}$$

and *Kolmogorov widths*

$$d_n(B,E) := \inf\left\{\sup_{x\in B} \|Q_N(x)\|_{E/N} \;:\; N \subseteq E, \dim N \le n\right\}.$$

Average Widths

Let now

- E be a separable Banach space,
- X r.v. on E with $\mathbb{E}\|X\|^q < \infty$ ($q \in (0, \infty)$),
- $Q_N : E \to E/N$ quotient mapping

define *linear widths*

$$a_n(X, E, q) := \inf\left\{ \left(\mathbb{E}\|X - u(X)\|^q\right)^{1/q} \ : \ \mathrm{rk}(u) \leq n \right\}$$

and *Kolmogorov widths*

$$d_n(X, E, q) := \inf\left\{ \left(\|Q_N(X)\|^q_{E/N}\right)^{1/q} \ : \ N \subseteq E, \dim N \leq n \right\} .$$

Quantization numbers

For $C \subseteq E$, $x \in E$ set

$$d(x, C) := \inf_{c \in C} \|x - c\| \ .$$

The *n–th quantization number* is

$$e_n(X, E, q) := \inf\left\{ \mathbb{E}\Big(d(X, E)^q\Big)^{1/q} \ : \ C \subseteq E, \#C \leq 2^n \right\} .$$

Also denote

$$\varphi(X, E, \varepsilon) := -\log \mathbb{P}(\|X\| < \varepsilon) \ .$$

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Let B^γ denote the fractional Brownian Motion on $[0,1]$.

Theorem (Maiorov, Wasilkowski)

If $p < \infty$, then

$$a_n(B^\gamma, L_p, q) \asymp d_n(B^\gamma, L_p, q) \asymp n^{-\frac{\gamma}{2}} .$$

Furthermore,

$$d_n(B^\gamma, C, q) \asymp n^{-\frac{\gamma}{2}} , \qquad a_n(B^\gamma, C, q) \asymp n^{-\frac{\gamma}{2}} (\log n)^{\frac{1}{2}} .$$

- Polynomial decay, p, q irrelevant for rate.
- Kolmogorov approximation does not help much.

Denote now B_d^γ the d–dimensional fractional Brownian sheet defined on $[0,1]^d$.

Theorem (Kühn, Linde)

For $p < \infty$,

$$a_n(B_d^\gamma, L_p, q) \asymp d_n(B_d^\gamma, L_p, q) \asymp n^{-\frac{\gamma}{2}} (\log n)^{\frac{\gamma}{2}(d-1)}$$

while

$$a_n(B_d^\gamma, C, q) \asymp n^{-\frac{\gamma}{2}} (\log n)^{\frac{\gamma}{2}(d-1)+\frac{1}{2}} \; .$$

Theorem (Talagrand)

$$d_n(B_2^\gamma, C, q) \asymp a_n(B_2^\gamma, C, q) \; .$$

Many further results by Wasilkowski, Ritter, Buslaev/Seleznjev, Maiorov, Gensun, . . .

Linear vs. Kolmogorov

Theorem (Pisier)

For $p \in (1, \infty)$ there is C_p such that for Gaussian X,

$$a_n(X, L_p, q) \le C_p \cdot d_n(X, L_p, q) \ .$$

Theorem

There is $K > 0$ such that for any E, Gaussian X

$$a_n(X, E, q) \le K \cdot d_n(X, E, q) \cdot (1 + \log n) \ .$$

Conclusion: Using Kolmogorov-type approximations never improves the rate for Gaussian processes.

Question: Is $(1 + \log n)$ improvable to $\sqrt{1 + \log n}$?

Quantization and Small Deviation

Theorem (Dereich et al)

For any centered Gaussian X and $\varrho > 0$, we have

$$e_n(X, E, q) \asymp n^{-\varrho} \qquad \textit{iff} \qquad \varphi(\varepsilon, X, E) \asymp \varepsilon^{-\frac{1}{\varrho}} .$$

Key Lemma for lower bounds: Define the pseudo–inverse

$$b_n(X, E) = \varphi^{-}(X, E)(n^{-1}) .$$

Lemma (Dereich et al)

If X is a r.v. with the Anderson property

$$\mathbb{P}(\|X - y\| < \varepsilon) \leq \mathbb{P}(\|X\| < \varepsilon) , \qquad y \in E, \varepsilon > 0$$

then

$$b_{n+2}(X, E) \leq 2^{\frac{1}{q}} \cdot e_n(X, E, q) .$$

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Relations to Widths

Theorem (Kuelbs, Li, Linde, C)

For X Gaussian,

$$d_n(X,E,q) \asymp n^{-\varrho} \qquad \textit{iff} \qquad \varphi(\varepsilon,X,E) \asymp \varepsilon^{-1/\varrho} \ .$$

Corollary

For X Gaussian,

$$d_n(X,E,q) \asymp n^{-\varrho} \qquad \textit{iff} \qquad e_n(X,E,q) \asymp n^{-\varrho} \ .$$

- a_n, d_n, e_n, b_n all 'walk hand in hand'.

SαS processes

Definition

A r.v. ξ is SαS iff

$$\hat{\xi}(\lambda) := \mathbb{E}e^{\mathrm{i}\lambda\xi} = \exp\{|\lambda|^\alpha \sigma^\alpha\}$$

for some $\sigma \geq 0$.
An E–valued r.v. X is SαS iff for any $a \in E'$, $a(X)$ is SαS.

Note: S2S $=$ centered Gaussian.

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Features

- Stability: Sum of two independent SαS processes is SαS again.
- Different smoothness properties than Gaussian counterparts.
- Different integrability: $\mathbb{E}\|X\|_F^q < \infty$ only iff $q < \alpha$ for $\alpha < 2 \quad \Rightarrow \quad$ No second–order theory.
- Conditional laws more delicate to handle.
- Representation as mixture of Gaussian processes.
- Connection with type/cotype theory of Banach spaces.

Link to Quantization

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Theorem

For X $S\alpha S$ in E,

$$d_n(X,E,q) \preceq n^{-\varrho} \quad \Rightarrow \quad e_n(X,E,q) \preceq n^{-\varrho} .$$

- Valid in slightly weaker form for arbitrary process X.
- Semi–constructive: Proof shows how to construct 'good' quantizers using only information about 'good subspaces' for d_n.
- Optimal, as will be shown below.

Ordering of the quantities

X SαS variable mixture of centered Gaussian
$\Rightarrow$ has Anderson property

$$\mathbb{P}(\|X-y\|<\varepsilon) \le \mathbb{P}(\|X\|<\varepsilon) \qquad \varepsilon>0, y \in E \ ,$$

Hence, the Lemma of Dereich et al is applicable.

Corollary

For X SαS and $q<\alpha$ we have

$$b_{n+2}(X,F) \preceq e_n(X,F,q) \text{``}\preceq\text{''} d_n(X,F,q) \le a_n(X,F,q) \ .$$

Consider a spectral measure ν_X of X, i.e., a Borel measure on F satisfying

$$\mathbb{E}e^{\mathrm{i}\langle\lambda,X\rangle} = \exp\left\{-\frac{1}{2}\int_F |\langle\lambda,x\rangle|^\alpha \,\mathrm{d}\nu_X(x)\right\}, \qquad \lambda \in F^* .$$

Let S_X be a r.v. distributed after $\nu_X/\nu_X(F)$. Then $\|S_X\|_F$ is α–integrable, and considerations of type/cotype lead to

Theorem

If $p > \alpha$, then

$$d_n(X, L_p, q) \asymp n^{-\varrho} \qquad \textit{iff} \qquad d_n(S_X, L_p, \alpha) \asymp n^{-\varrho} .$$

Same result for a_n as well.

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SαS Lévy motion

The SαS Lèvy motion is a SαS process $X = (X_t)_{t\in[0,1]}$ with

- independent increments, $X_0 = 0$,
- $(X_{ct}) \stackrel{d}{=} c^{1/\alpha}(X_t)$ for $c > 0$,
- X has cádlàg trajectories.

Theorem (Folklore)

$$b_n(X, L_p) \asymp n^{-1/\alpha} \ .$$

Theorem (Dereich)

$$e_n(X, L_p, q) \asymp n^{-1/\alpha} \ .$$

SαS Lévy motion

Theorem

$$a_n(X, L_p, q) \asymp d_n(X, L_p, q) \asymp \begin{cases} n^{-1/\alpha}, & p < \alpha, \\ n^{-1/p}, & \alpha < p \leq 2, \\ n^{-1/2}, & \alpha > 2\,. \end{cases}$$

- Same asymptotics for b_n, e_n and for a_n, d_n.
- *Different* rates for approximation and quantization/small deviations when $p > \alpha$.
- Optimal rates for approximation in the case $p \leq 2$ achievable by naive sampling, for $p > \alpha$ needs results of Gluskin about random projections.

Summary

Questions

- Conditions when

$$a_n(X, E, q) \preceq d_n(X, E, q)$$

  does hold?
- Examples for $a_n \not\asymp d_n$?
- Is $b_n \asymp \varphi_n$ a general principle for Lévy processes?
- Generalizations/Applications for SDE after SαS motion?

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