On the increments of principal value

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$\{W(t);\, t \geq 0\}$: one-dimensional standard Brownian motion.

$\{L(t,x);\, t \geq 0,\, x \in R\}$: local time.

$$\int_0^t f(W(s))\ ds = \int_{-\infty}^{\infty} f(x) L(t,x)\ dx$$

$$Y(t) := \int_0^t \frac{ds}{W(s)} = \int_{-\infty}^{\infty} \frac{L(t,x)}{x}\ dx$$

considered in the sense of Cauchy's principal value, i.e.,

$$Y(t) := \lim_{\varepsilon \to 0+} \int_0^t \frac{ds}{W(s)} I\{|W(s)| \geq \varepsilon\} =$$

$$= \int_0^{\infty} \frac{L(t,x) - L(t,-x)}{x} dx.$$

Last integral is almost surely absolutely convergent for all $t > 0$.

$\{Y(t),\, t \geq 0\}$: principal value of Brownian local time.

In general:

$$\int_0^t \frac{ds}{(W(s))^\alpha} = \int_{-\infty}^{\infty} \frac{L(t,x)}{x^\alpha}\, dx.$$

We consider only $\alpha = 1$.

Scaling: $\frac{Y(at)}{\sqrt{a}}$ has the same law as $Y(t)$.

Question: what (asymptotic) properties of W are inherited by Y?

Y has almost surely continuous sample path.

Exact distribution: Biane and Yor (1987)

$$\frac{P(Y(1) \in dx)}{dx} =$$

$$= \sqrt{\frac{2}{\pi^3}} \sum_{k=0}^{\infty} (-1)^k \exp\left(-\frac{(2k+1)^2 x^2}{8}\right).$$

Gaussian tail:

$$P(Y(1) \geq z) \sim \exp\left(-\frac{z^2}{8}\right), \quad z \to \infty.$$

Further results:

Fitzsimmons–Getoor (1992)

Bertoin (1995)

Yamada (1996)

Boufoussi–Eddahbi–Kamont (1997)

Yor (ed.): Exponential functionals and principal values related to Brownian motion (1997)

Ait Ouahra–Eddahbi (2001)

LIL: Hu–Shi (1997)

$$\limsup_{T\to\infty} \frac{Y(T)}{\sqrt{T\log\log T}} = \sqrt{8}, \qquad \text{a.s.}$$

Functional LIL: Csáki–Földes–Shi (2003)

$$\left\{\frac{Y(xT)}{\sqrt{8T\log\log T}},\ 0 \le x \le 1\right\}_{T\ge 3}$$

is almost surely relatively compact in $C[0,1]$ with limit set equal to

$$\begin{aligned}\mathcal{S} := \quad & \{f \in C[0,1] :\ f(0) = 0, \\ & f \text{ is absolutely continuous and} \\ & \int_0^1 (f'(x))^2\,dx \le 1\}.\end{aligned}$$

Chung type small deviation: Hu (2000)

$$c_1 \exp\left(-\frac{c_2}{z^2}\right) \le$$

$$\le P(\sup_{0\le s\le 1} |Y(s)| < z) \le c_3 \exp\left(-\frac{c_4}{z^2}\right),$$

where the best constants are unknown.

One-sided sup: Hu (2000)

$$C_1 z^{1/2} \le P(\sup_{0\le s\le 1} Y(s) < z) \le C_2 z^{1/2}.$$

Consequence: Chung's LIL, Hirsch type result.

Question: functional LIL with rate:
For $f \in \mathcal{S}$, $\int_0^1 f'^2 < 1$ is it true that

$$c_1 \exp\left(-\frac{c_2}{z^2}\right) \le$$

$$\le P(\sup_{0\le s\le 1} |Y(s) - f(s)| < z) \le$$

$$\le c_3 \exp\left(-\frac{c_4}{z^2}\right)$$

with some constants c_i ?

Increments

Csáki–Csörgő–Földes–Shi (2000)

Under

$$\lim_{T\to\infty} \frac{\log(T/a_T)}{\log\log T} = \infty$$

we have almost surely

$$\lim_{T\to\infty} \frac{I(T, a_T)}{\sqrt{a_T \log(T/a_T)}} = 2,$$

where

$$I(T, a_T) = \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s) - Y(t)|.$$

Modulus of continuity:

$$\lim_{h\to 0} \frac{\sup_{0\le t\le 1}\sup_{0\le s\le h} |Y(t+s) - Y(t)|}{\sqrt{h\log(1/h)}} = 2,$$

almost surely.

Wen, Jiwei (2002):

$$\limsup_{T\to\infty} \frac{I_1(T)}{\sqrt{t(\log\frac{T}{t} + 2\log\log t)}} = 2,$$

almost surely, where

$$I_1(T) = \sup_{0\le t\le T}\sup_{t\le s\le T} |Y(s) - Y(s-t)|.$$

$$\limsup_{T\to\infty} \frac{I(T, a_T)}{\sqrt{a_T(\log\frac{t+a_T}{a_T} + 2\log\log a_T)}} \le 2,$$

almost surely. If a_T is onto, then $\limsup = 2$.

New results

Csáki–Hu (2005):

Theorem 1

$$\limsup_{T\to\infty} \frac{I(T, a_T)}{\sqrt{a_T\left(\log\sqrt{T/a_T} + \log\log T\right)}} = \sqrt{8},$$

almost surely.

If $a_T > T(\log T)^{-\alpha}$, $\alpha < 2$, then

$$\liminf_{T\to\infty} \sqrt{\frac{\log\log T}{a_T}} I(T, a_T) = K_1,$$

almost surely (K_1 is unknown).

If $a_T \le T(\log T)^{-\alpha}$, $\alpha > 2$, then

$$\liminf_{T\to\infty} \frac{I(T, a_T)}{\sqrt{a_T \log(T/a_T)}} = K_2,$$

almost surely (K_2 is partly known).

Theorem 2 Let

$$J(T, a_T) = \inf_{0 \le t \le T - a_T} \sup_{0 \le s \le a_T} |Y(t+s) - Y(t)|.$$

We have

$$\liminf_{T\to\infty} \frac{\sqrt{T \log\log T}}{a_T} J(T, a_T) = K_3,$$

almost surely. If $a_T/T \to 0$, $T \to \infty$, then $K_3 = 1/\sqrt{2}$.

If $0 < \lim_{T\to\infty}(a_T/T) = \rho \le 1$, then

$$\limsup_{T\to\infty} \frac{J(T, a_T)}{\sqrt{T \log\log T}} = \rho\sqrt{8},$$

almost surely.

If $\lim_{T\to\infty} a_T (\log\log T)^2/T = 0$, then

$$\limsup_{T\to\infty} \frac{\sqrt{T}}{a_T\sqrt{\log\log T}} J(T, a_T) = K_4,$$

almost surely (K_4 is unknown).

Small deviation estimates

$$P(I(T,1) < z) \leq \frac{5}{T^{\kappa/2}} +$$

$$+\exp(-cT^{(1-\kappa)/2}e^{-(1+\delta)z^2/8}),$$

κ, δ arbitrary, $c = c(\delta)$.

$$P(I(T,1) < z) \geq \frac{c_1}{\sqrt{T}} \exp\left(-\frac{c_2}{z^2}\right) \qquad (1)$$

$$P(J(T,1) < z) \leq C_1 \exp\left(-\frac{(1-\delta)^2}{2(1+\delta)^2 z^2 T}\right) +$$

$$+C_1 \exp\left(-\frac{C_2\delta}{4(1+\delta)^2 z^2}\right) +$$

$$+\exp\left(\frac{C_3}{z^2} - \frac{C_4 z^2}{T} e^{C_5/z^2}\right)$$

A short proof of (1). We use the following properties.

Last zero before 1:

$$g = \sup\{t : \, t \le 1, \, W(t) = 0\}.$$

$$B(s) = \frac{W(sg)}{\sqrt{g}}, \, 0 \le s \le 1$$

is a Brownian bridge,

$$m(s) = \frac{|W(g + s(1-g))|}{\sqrt{1-g}}, \, 0 \le s \le 1$$

is a Brownian meander, g, B, m are independent.

Biane-Yor (1987):

$$P\left(\int_0^1 \frac{dv}{m(v)} < z \,|\, m(1) = 0\right) =$$

$$= \sum_{k=-\infty}^{\infty} (1 - k^2 z^2) \exp\left(-\frac{k^2 z^2}{2}\right) =$$

$$= \frac{8\pi^2\sqrt{2\pi}}{z^3} \sum_{k=1}^{\infty} \exp\left(-\frac{2k^2\pi^2}{z^2}\right), \quad z > 0.$$

$$P(I(T,1) < z) \geq$$

$$\geq P\left(Y^* < \frac{z}{4}, W(1) \geq \frac{4}{z}, \inf_{1 \leq u \leq T}(W(u) \geq \frac{2}{z}\right),$$

where

$$Y^* = \sup_{0 \leq s \leq 1} |Y(s)|.$$

It follows that

$$P(I(T,1) < z) \geq \frac{c}{\sqrt{T}} P\left(Y^* \leq \frac{z}{4}, W(1) \geq \frac{4}{z}\right).$$

Moreover,

$$P\left(Y^* \leq \frac{z}{4}, W(1) \geq \frac{4}{z}\right) \geq$$

$$P\left(Y^*(g) \leq \frac{z}{8}, Y(1) - Y(g) \leq \frac{z}{8}, W(1) \geq \frac{4}{z}\right) \geq$$

$$\geq P(\cdots, g < z^2),$$

where

$$Y^*(g) = \sup_{0 \leq s \leq g} |Y(s)|.$$

We can proceed as follows:

$$P\left(Y^*(g) \leq \frac{z}{8}|g < z^2\right) = const.$$

$$P\left(Y(1) - Y(g) \leq \frac{z}{8}, W(1) \geq \frac{4}{z}|g < z^2\right) \geq$$

$$\geq P\left(\int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8}, m(1) \geq \frac{4}{z\sqrt{1-z^2}}\right).$$

Using the inequality

$$P\left(\int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8} \,|\, m(1) = x\right) \geq$$

$$\geq P\left(\int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8} \,|\, m(1) = 0\right),$$

we finally obtain (1).

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