High resolution coding of stochastic processes

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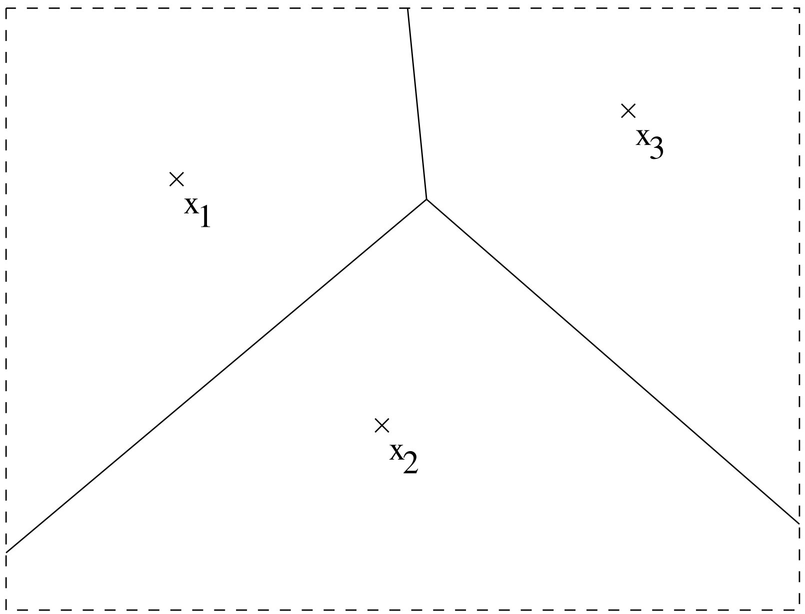
An example

Given: $\mathbb{R}^2$-valued r.v. X, $n \in \mathbb{N}$.

Problem: Find n points $x_1, \ldots, x_n$ in $\mathbb{R}^2$, such that

$$\mathbb{E}[\min_{i=1,\ldots,n} \|X - x_i\|^2]$$

is small.



E=$\mathbb{R}^2$

Codebuch { x_1, x_2, x_3 }

Voronoi Zellen

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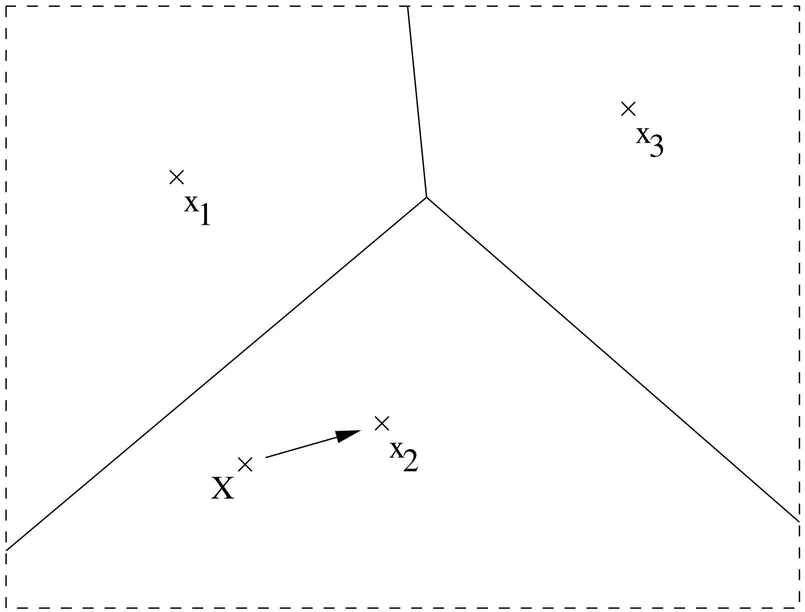

E=IR^2

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Quantization

Given: Random signal X (*original*) in some Banach space $(E, \|\cdot\|)$.

Aim: Minimize the *average distortion*

$$\mathbb{E}[\|X - \hat{X}\|^p]^{1/p}$$

among all discrete E-valued r.v. $\hat{X}$ (*reconstructions*) with

$$|\mathrm{range}(\hat{X})| \leq e^r.$$

Parameters: $r \geq 0$ - *rate*,

$p > 0$ - *moment*.

Quantization error: $D^{(q)}(r,p) = \inf \mathbb{E}[\|X - \hat{X}\|^p]^{1/p}$.

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$$\mathbb{H}(\hat{X}) = \sum_{\hat{x}} P(\hat{X} = \hat{x}) \log(1/P(\hat{X} = \hat{x})) \leq r.$$

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Relation: $D^{(e)}(r,p) \leq D^{(q)}(r,p)$

Applications

- Digitizing analog signals (Pulse-Code-Modulation)
- Variance reduction for Monte-Carlo schemes (Luschgy, Pagès, Printems)
- lower bounds for certain approximation quantities (D, Müller-Gronbach, Ritter)

Quantization in $E = \mathbb{R}^d$

- Research started in the 50th (motivated by Pulse-Code-Modulation)
- high resolution problem adressed by Zador '63, Bucklew and Wise '82, Graf and Luschgy '00 (LNM 1730)
- *high resolution formula* for absolutely continuous $\mu := \mathcal{L}(X)$ under a concentration assumption:

$$D^{(q)}(r,p) \sim \kappa(E,p) \left\| \frac{d\mu}{d\lambda^d} \right\|_{L^{d/(d+p)}(\mathbb{R}^d)}^{1/p} (e^r)^{-1/d}, \quad r \to \infty$$

- heuristics: good high resolution codebooks look locally like a good codebook for the uniform distribution with a particular point density depending on the density $\frac{d\mu}{d\lambda^d}$
- problem: optimal codebooks for the uniform distribution on $[0,1)^d$ hard to find
  – Euclidean space, $d = 2 \Rightarrow$ honeycomb structure

Gaussian originals

Notation:

- E *separable Banach space* (e.g. $E = \mathbb{C}[0,1]$)
- μ centered Gaussian measure on E
- X r.v. with $\mathcal{L}(X) = \mu$

Theorem: *(Fehringer '00, DFMS '03, D '03)*
Under a mild technical assumption on φ, one has for all $p > 0$:

$$\varphi^{-1}(r) \lesssim D^{(e)}(r,p) \le D^{(q)}(r,p) \lesssim 2\,\varphi^{-1}(r/2), \quad r \to \infty$$

where

$$\varphi(\varepsilon) := -\log \mu(B(0,\varepsilon)) = -\log \mathbb{P}(\|X\| \le \varepsilon) \qquad (\varepsilon > 0)$$

Small ball function
References: Li and Shao (2001): Gaussian processes: inequalities, small ball probabilities and applications.

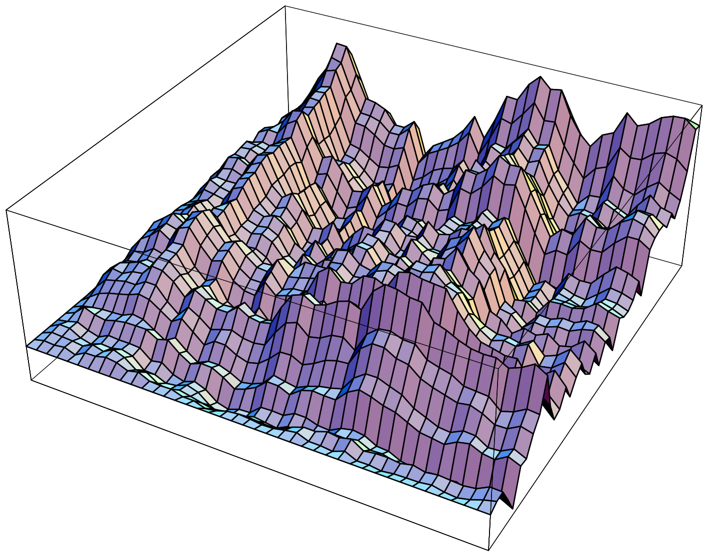
Conclusions

- $X = W$ *Wiener process* in $\mathbb{C}[0,1]$:

$$\frac{\pi}{\sqrt{8r}} \lesssim D^{(e)}(r,p) \leq D^{(q)}(r,p) \lesssim \frac{\pi}{\sqrt{r}}, \quad r \to \infty$$

- X 2-dimensional *Brownian sheet* in $\mathbb{C}([0,1]^2)$:

$$D^{(e)}(r,p) \approx D^{(q)}(r,p) \approx (\log r)^{3/2}\frac{1}{\sqrt{r}}, \quad r \to \infty$$



Random small deviations

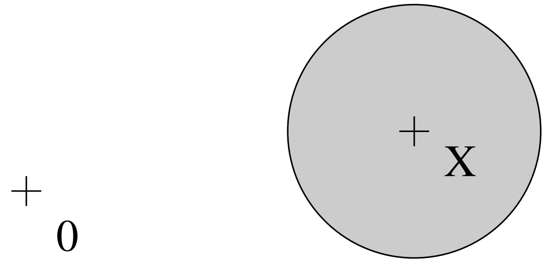
Quantization error of random codebooks:

$$D^{(R)}(r,p) = \mathbb{E}\Big[\min_{i=1,\dots,\lfloor e^r \rfloor} \|X - Y_i\|^p\Big]^{1/p},$$

where $X, Y_1, Y_2, \dots$ i.i.d. with $\mathcal{L}(X) = \mu$.

Random small ball function:

$$\ell_\varepsilon(X(\omega)) = -\log \mu(B(X(\omega), \varepsilon)) \qquad (\varepsilon > 0).$$



Reference: D, Lifshits '05

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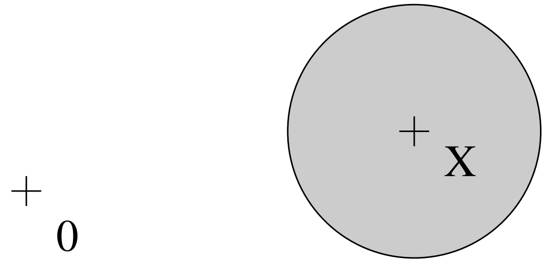

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Asymptotic properties

Problem: Connection of $\ell_\varepsilon(X)$ and $\varphi(\varepsilon) = \ell_\varepsilon(0)$.

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$$\varphi(\varepsilon) \leq \ell_\varepsilon(X) \lesssim 2\varphi(\varepsilon/2), \qquad \varepsilon \downarrow 0, \text{a.s.}$$

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Theorem:

$$\lim_{\varepsilon\downarrow 0} \frac{\ell_\varepsilon(X)}{\varphi_R(\varepsilon)} = 1, \qquad \text{a.s.}$$

for

- $\varphi_R(\varepsilon) = m_\varepsilon =$ median of $\ell_\varepsilon(X)$ or
- $\varphi_R(\varepsilon) = \mathbb{E}[\ell_\varepsilon(X)]$

(Asymptotic equipartition property)

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Theorem:

$$\varphi(\varepsilon/\sqrt{2}) \lesssim \ell_\varepsilon(X) \lesssim 2\varphi(\varepsilon/2), \qquad \varepsilon \downarrow 0, \text{a.s.}$$

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Proof of the SLT - 1

1st step:

- $V_\varepsilon := B(0,\varepsilon) + 3\sqrt{\varphi(\varepsilon)}\,\mathcal{K}$ (*enlarged ball (Talagrand)*)
- $\mu(V_\varepsilon^c) \le \exp\{-\varphi(\varepsilon)\}$ (V_ε is a *typical set*)
- $x \in V_\varepsilon \quad \Rightarrow \quad \log \mu(B(x,2\varepsilon)) \ge -5.5\,\varphi(\varepsilon)$

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2nd step:

- $\Psi(x) := \log \mu(B(x,2\varepsilon))$, for some fixed small $\varepsilon > 0$
- $h \in H_\mu,\ \ x, x+h \in V_\varepsilon \quad \Rightarrow \quad |\Psi(x+h) - \Psi(x)| \le 8\sqrt{\varphi(\varepsilon)}\,|h|_\mu$ (H_μ*-Lipschitz continuous*)

Proof of the SLT - 2

3rd step:

- to show: $|\ell_{2\varepsilon}(X) - m_{2\varepsilon}| = o(\varphi(\varepsilon))$ a.s.
- $\mathbb{P}(|\ell_{2\varepsilon}(X) - m_{2\varepsilon}| \geq 8\sqrt{\varphi(\varepsilon)}\, r) \leq \mu(V_\varepsilon^c) + \exp(-(r - r_\varepsilon)^2/2)$ $(r > r_\varepsilon)$, where $\lim_{\varepsilon \downarrow 0} r_\varepsilon = 0$ (*concentration principle (Ledoux)*)
- Application of the Borel-Cantelli Lemma for some $r(\varepsilon) = o(\sqrt{\varphi(\varepsilon)})$

Connection to quantization

Theorem: For all $q > 0$ one has

$$D^{(R)}(r,q) \sim \varphi_R^{-1}(r), \qquad r \to \infty.$$

- equivalence of all moments
- $\min_{i=1,\ldots,\lfloor e^r \rfloor} \|X - Y_i\|$ is concentrated around $\varphi_R^{-1}(r)$

Open problem: Is the equivalence of moments property valid for the quantization error for Gaussian underlyings?

Proof of the connection

Abridge $n = \lfloor e^r \rfloor$ and assume that $p = 1$. Note that for $x \in E$

$$\mathbb{E}[\min_{i=1,\ldots,n} \|x - Y_i\|] = \int_0^\infty \mathbb{P}(\min_{i=1,\ldots,n} \|x - Y_i\| \geq t)\, dt$$

$$= \int_0^\infty \underbrace{(1 - \mathbb{P}(\|x - Y_1\| < t)^n}_{\approx \exp(-n\,\mathbb{P}(\|x - Y_1\| < t))} \, dt$$

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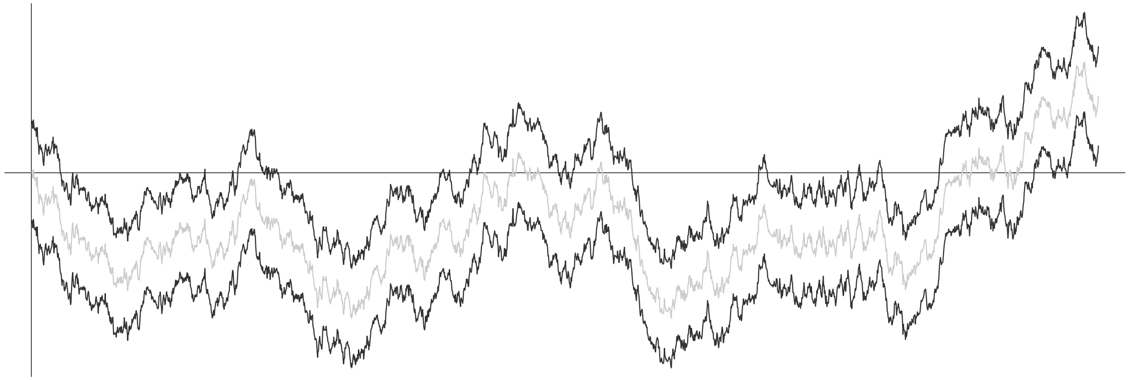
Therefore,

$$\mathbb{E}[\min_{i=1,\ldots,n} \|x - Y_i\|] \approx \ell_\cdot^{-1}(x)(r).$$

RSBP's (Wiener process)



For $E = \mathbb{C}[0,1]$ or $E = L^p[0,1]$ it is true that

$$\ell_\varepsilon(W) \sim \kappa_p \frac{1}{\varepsilon^2}, \quad \varepsilon \downarrow 0, \text{ a.s.},$$

for an appropriate constant $\kappa_p > 0$.

- κ_∞ - constant in $[2\lambda_1, 8\lambda_1]$
- λ_1 - *principal eigenvalue of the Dirichletproblem* on the unit disc of $\mathbb{R}^d$ $(d = 1 \Rightarrow \lambda_1 = \pi^2/8)$
- $\kappa_2 = \frac{9}{32}$

Fractional Brownian motion

- X fractional Brownian motion with Hurst index $H \in (0,1)$
- $E = \mathbb{C}[0,1]$ or $E = L^p[0,1] \;\Rightarrow\; \exists \kappa > 0$ s.th. for all $q > 0$

$$D^{(e)}(r,q) \sim D^{(q)}(r,q) \sim \kappa \frac{1}{r^H}$$

(D, Scheutzow '05)

- estimates for κ if X is Wiener process
  - $E = \mathbb{C}[0,1] \;\Rightarrow\; \kappa \in [\frac{\pi}{\sqrt{8}}, \pi]$
  - $E = L^2[0,1] \;\Rightarrow\; \kappa = \frac{\sqrt{2}}{\pi}$
- κ known for all H if $E = L^2[0,1]$.

Coding diffusions

Notation:

- drift coefficient: $b : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$
- diffusion coefficient: $\sigma : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ (*scalar*)
- $\tilde{W} = (\tilde{W}_t)_{t\geq 0}$ d-dim. Wiener process
- X solution of the SDE

$$X_t = \int_0^t \underbrace{\sigma(X_t,t)}_{=:\sigma_t} \, d\tilde{W}_t + \int_0^t \underbrace{b(X_t,t)}_{=:b_t} \, dt, \qquad t \geq 0.$$

Assumption: $\exists \beta \in (0,1]$, $L < \infty$ s.th. $\forall x, x' \in \mathbb{R}^d \, \forall t, t' \in [0,1]$:

$$|b(x,t)| \leq L(|x|+1)$$

$$|\sigma(x,t) - \sigma(x',t')| \leq L[|x-x'|^\beta + |x-x'| + |t-t'|^\beta].$$

Main result - $E = L^p[0,1]$

- $E = L^p[0,1]$, $p \geq 1$

Theorem: $\exists \kappa_p \in \mathbb{R}_+$ with

$$D^{(q)}(r,p|W) \sim \kappa_p \frac{1}{\sqrt{r}}.$$

and one has

$$D^{(q)}(r,p|X) \lesssim \kappa_p \, \mathbb{E}\left[\|\sigma_\cdot\|^p_{L^{2p/(p+2)}[0,1]}\right]^{1/p} \frac{1}{\sqrt{r}},$$

where

$$(\sigma_t)_{t\geq 0} := (\sigma(X_t,t))_{t\geq 0}.$$

- similar results valid for the pathwise interpolation problem (Hofmann, Müller-Gronbach, Ritter '01, Müller-Gronbach '03)

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Coding scheme

Representation for X (Doob-Meyer decomposition, time change):

$$X_t = A_t + W_{\tau(t)},$$

where

- $A_t = \int_0^t b(X_s, s)\, ds$ - process with bounded variation
- (W_t) - d-dimensional Wiener process
- $\tau(t) = \int_0^t \sigma^2(X_s, s)\, ds$ - time change.

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Coding scheme:

- approximate (A_t) by $(\hat{A}_t)$ and $(\tau(t))$ by $(\hat{\tau}(t))$ (*negligible complexity*)
- approximate $(W_t)_{t\in[0,\hat{\tau}(1)]}$ by $(\hat{W}_t)_{t\in[0,\hat{\tau}(1)]}$ (*dominates complexity*)
- Reconstruction: $\hat{X}_t = \hat{A}_t + \hat{W}_{\hat{\tau}(t)}$.

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Need:

- estimates for the complexity of regular processes
- regularity statement for $(\tau(t))$ (Hölder continuity of X)
- cope with dependencies:
  rate allocation $\leftrightarrow$ Wiener process.

Entropy numbers

- $(F, \|\cdot\|_F)$ Banach space compactly embedded into $(G, \|\cdot\|_G)$
- $e_n(F,G) = \inf\{\varepsilon > 0 : \exists x_1, \dots, x_{2^n} \in F$ s.th. $\bigcup_{i=1}^{2^n} B_G(x_i, \varepsilon) \supset B_F(0,1)\}$

  (n*-th entropy number*)
- Ass.: $e_n(F,G) \precsim n^{-\alpha}, \qquad n \to \infty$

Lemma: $\forall \tilde{p} > p > 0 \; \exists$ constant $C = C(p, \tilde{p})$ s.th. for all F-valued processe Z:

$$D^{(q)}(r|Z, \|\cdot\|_G, p) \leq C \, \mathbb{E}[\|Z\|_F^{\tilde{p}}]^{1/\tilde{p}} \, \frac{1}{1+r^\alpha}$$

Hölder continuity of X

- $M_t = \int_0^t \sigma_s \, d\tilde{W}_s$
- $|f|_\alpha := \sup_{0 \le s < t \le 1} \frac{|f(t)-f(s)|}{|t-s|^\alpha}$

Lemma: $\forall \alpha \in (0, 1/2) \; \forall p > 2/(1-2\alpha) \; \exists C = C(p, \alpha)$ s.th.

$$\mathbb{E}[|M|_\alpha^p] \le C \int_0^1 \mathbb{E}[|\sigma_s|^p] \, ds.$$

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Application to diffusions:

$\Rightarrow$ $\mathbb{E}[|\sigma_\cdot^2|_\varepsilon^p] < \infty$ for some $\varepsilon > 0$ and every $p \in \mathbb{R}_+$

$\Rightarrow$ $D^{(q)}(r|\tau, \|\cdot\|_{[0,1]}, p) \le \frac{\text{const}}{r^{1+\varepsilon}+1}$ for all $r \ge 0$

Garsia, Rodemich, Rumsey (70/71)

Theorem: $\forall f \in \mathbb{C}[0,1]$:

$$|f(s) - f(t)| \leq 8 \int_0^{|s-t|} \Psi^{-1}\Big(\frac{4B(f)}{\xi^2}\Big)\, dp(\xi) \qquad (s,t \in [0,1])$$

where

$$B(f) := \int_0^1 \int_0^1 \Psi\Big(\frac{|f(s)-f(t)|}{p(|s-t|)}\Big)\, ds\, dt$$

and

- $\Psi : [0,\infty) \to [0,\infty)$ bijectiv, $\nearrow$
- $p : [0,\infty) \to [0,\infty) \nearrow$ with $p(0) = 0$
- <u>Choose:</u> $\Psi(x) = x^\delta$, $p(x) = x^\gamma$ for some $\alpha, \gamma > 0$

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Theorem: $\forall f \in \mathbb{C}[0,1]:$

$$|f(s) - f(t)| \leq \text{const}\; |s-t|^{-2/\delta+\gamma} B(f)^{1/\delta} \qquad (s,t \in [0,1])$$

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(*Sobolev embedding*)

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Application:

- Choose appropriate values $\delta, \gamma > 0$ such that $\alpha = -2/\delta + \gamma$
- Control the moments of the random variable $B(M)$

Rate allocation problem

Problem: A good reconstruction $(\hat{W}_t)$ for (W_t) depends on the time change $(\tau(t))$

Technique:

- break the path $(W_t)_{t\in[0,\tau(1)]}$ into pieces
- assign to each part an optimal rate for coding

Problem: Interdependence between

$$\text{rate allocation} \longleftrightarrow \text{Wiener process}$$

Enlargement of filtration

Idea: rate allocation is an initial enlargement of filtration:

- $(\mathcal{F}_t^W)$-filtration generated by the Wiener process W
- $\|f\|_{\mathcal{H}} := \|\frac{df}{dt}\|_{L^2[0,\infty)}$ (*Cameron-Martin norm*)
- G discrete r.v., $\mathcal{G}_t = \mathcal{F}_t^W \vee \sigma(G)$ initial enlargement of $\mathcal{F}_t^W$

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Theorem: $\exists \mathcal{G}_t$-Wiener process $(\bar{W}_t)$, $\mathcal{G}_t$-adapted process $(\bar{A}_t)$:

$$W_t = \bar{W}_t + \bar{A}_t,$$

and

$$\mathbb{E}[\|\bar{A}\|_{\mathcal{H}}^{2p}] \le C(p)\, \mathbb{H}^p(G),$$

where

$$\mathbb{H}^p(G) = \sum_x p_x^G \left(\log(1/p_x^G)\right)^p.$$

Ref.: Jacod, Yor ('85). Grossissements de filtrations: exemples et applications, (LNM 1118); Ankirchner, Imkeller, D ('05)

Representation for X

Theorem: $\forall p \geq 1 \; \exists$ processes $(\bar{X}_t^{(r)})$, $(\hat{\bar{X}}_t^{(r)})$, $(\bar{W}_t^{(r)})$ and $(\hat{\tau}_t^{(r)})$ s.th.

$$X = \bar{X}^{(r)} + \bar{W}^{(r)}_{\hat{\tau}^{(r)}(\cdot)}$$

and

- $\exists \delta > 0 : \; \mathbb{E}[\|\bar{X}^{(r)} - \hat{\bar{X}}^{(r)}\|^p_{[0,1]}]^{1/p} = \mathcal{O}(r^{-\frac{1}{2}-\delta})$
- $\exists \gamma \in (0,1) : \; \log|\text{range}(\hat{\bar{X}}^{(r)}, \hat{\tau}^{(r)})| = \mathcal{O}(r^\gamma)$,
- $(\bar{W}_t^{(r)})$ is a Wiener process (independent of $\hat{\tau}^{(r)}$)
- $\mathbb{E}[\|\tau - \hat{\tau}\|_{[0,1]}] \to 0$