Representations of isotropic random fields with homogeneous increments, with applications to spacial fractional Brownian motion

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Isotropic random field with homogeneous increments. The results presented here are taken over from the current work by Dzhaparidze, van Zanten and Zareba [9] in which some of the results of the previous papers [5]–[8] are extended to isotropic Gaussian random fields with homogeneous increments. Our work, inspired in large extend by Malyarenko [13], does improve upon his general result à la our representation (8) and somewhat simplify applications to the spacial fractional Brownian motion.

The departure point in the aforementioned papers is the spectral representation of the covariance function of a mean zero random process $X_t, t \in \mathbb{R}^1$, with stationary increments:

$$EX_sX_t = \int_{\mathbb{R}^1} \big(e^{i\lambda t} - 1\big)\big(e^{-i\lambda s} - 1\big) d\varrho(\lambda) \qquad s, t \in \mathbb{R}^1 \tag{1}$$

where ϱ is a spectral function. In the special case of fBm(H) with

$$EX_sX_t = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big), \tag{2}$$

the spectral function is known to be given by

$$d\varrho(\lambda) = C_H^2 \frac{d\lambda}{|\lambda|^{1+2H}}$$

with a certain positive constant C_H^2. Malyarenko [13] treats at once a multi-dimensional case, a random field with homogeneous increments characterized by the spectral representation

$$EX_tX_s = \int_{\mathbb{R}^N} \left(e^{i(v,t)} - 1\right)\left(e^{-i(v,s)} - 1\right) d\varrho(v) \qquad s, t \in \mathbb{R}^N \tag{3}$$

under the additional isotropy requirement in the sense that $X_{\Omega t} \stackrel{d}{=} X_t$ for any orthogonal matrix Ω. The spectral function $\varrho(v)$ then depends only on the length $|v|$ of the vector $v \in \mathbb{R}^N$ (for simplicity we use the same symbol ϱ also for the resulting scalar function). The main object of study in [13] is a spacial version of fBm(H) that is characterized by the covariance of the same form (2) with $s, t \in \mathbb{R}^N$, whose spectral representation (3) holds with

$$d\varrho(v) = C_H^{N\,2} \frac{dv}{|v|^{N+2H}},$$

$C_H^{N\,2}$ a certain positive constant. The intention is to extend results of [5] to the multi-dimensional case. The first step is finding the multi-dimensional analogue to the following simple reformulation of (1): for $s, t \in \mathbb{R}^1$

$$EX_sX_t = \int_0^\infty \left(\frac{\cos\lambda s - 1}{\lambda}\frac{\cos\lambda t - 1}{\lambda} + \frac{\sin\lambda s}{\lambda}\frac{\sin\lambda t}{\lambda}\right)\mu(d\lambda) \tag{4}$$

with $\mu(d\lambda) = 2\lambda^2\varrho(d\lambda)$. However, this step is not elementary. It turns out (see [13] for details or the books [19] or [12] where the basic methodology can be found) that (3) can be rewritten in the following form: with $s, t \in \mathbb{R}^N$

$$EX_sX_t = \frac{\pi^N}{\Gamma^2(\frac{N}{2})} \sum_{\ell=0}^{\infty} \sum_{m=1}^{h(\ell,N)} \int_0^\infty \frac{g_\ell^m(s,\lambda) - g_\ell^m(0,\lambda)}{\lambda} \frac{g_\ell^m(t,\lambda) - g_\ell^m(0,\lambda)}{\lambda}\mu(d\lambda) \tag{5}$$

with

$$\mu(d\lambda) = \frac{2\pi^N|\lambda|^{N+1}\varrho(d\lambda)}{\Gamma^2(N/2)}.$$

It will be seen in the concluding section how the system of functions g_ℓ^m is defined and how the numbers

$$h(\ell, N) = \frac{(2\ell + N - 2)(\ell + N - 3)!}{(N-2)!\ell!} \tag{6}$$

occur (in case $N = 1$ only two terms will remain in the series (5) and it turns into (4), of course). Meanwhile in the next section we present an interesting consequence of the representation (5).

Series expansion. In this section we restrict our attention to the unit ball $|t| < 1$. As is demonstrated in [8], in the scalar case one can make use of Krein's spectral theory of vibrating strings (see [11] or [4]) that allows us to switch over to the discrete spectrum. The discrete counterpart of the spectral function μ in (4) is defined in terms of eigenvalues and eigenfunctions of the corresponding string equation, and we have the representation of the following form: for $s, t \in \mathbb{R}^1$

$$EX_sX_t = \sum_{n=1}^{\infty} \left(\frac{\cos \lambda_n s - 1}{\lambda_n} \frac{\cos \lambda_n t - 1}{\lambda_n} + \frac{\sin \lambda_n s}{\lambda_n} \frac{\sin \lambda_n t}{\lambda_n} \right) \sigma_n^2.$$

Krein has developed procedures for finding the numbers λ_n and σ_n^2 in many practically important cases (see the list of rules in [4], section 6.9), except in the fBm(H) case. The latter is discussed in [8] where one can see that λ_n's in that case are the positive zero's of the Bessel function J_{-H} and that

$$\sigma_n^2 = \frac{2C_H^2}{\lambda_n^{2H} J_{1-H}^2(\lambda_n)}.$$

In the paper [9] under preparation we show how the same method of vibrating strings does extend to the multi-dimensional case and does lead to the representation of the following form: for $s, t \in \mathbb{R}^N$

$$EX_sX_t = \frac{\pi^N}{\Gamma^2(\frac{N}{2})} \sum_{\ell=0}^{\infty} \sum_{m=1}^{h(\ell,N)} \sum_{n=1}^{\infty} \frac{g_\ell^m(s, \lambda_n) - g_\ell^m(0, \lambda_n)}{\lambda_n} \frac{g_\ell^m(t, \lambda_n) - g_\ell^m(0, \lambda_n)}{\lambda_n} \sigma_n^{N\,2} \tag{7}$$

with the same spectrum $\{\lambda_n, n = 1, 2, \ldots\}$ as before and the jumps $\{\sigma_n^{N\,2}, n = 1, 2, \ldots\}$ that are determined in the similar way as before. For the spacial fBm(H), for instance, λ_n's are again the positive zero's of the Bessel function J_{-H}, while

$$\sigma_n^{N\,2} = \frac{2C_H^{N\,2}}{\lambda_n^{2H} J_{1-H}^2(\lambda_n)}.$$

Precisely as in the scalar case, it follows from the representation (7) that the following series expansion holds a.s. and uniformly in $|t| \le 1$:

$$X_t = \frac{\pi^N}{\Gamma^2(\frac{N}{2})} \sum_{\ell=0}^{\infty} \sum_{m=1}^{h(\ell,N)} \sum_{n=1}^{\infty} \frac{g_\ell^m(t, \lambda_n) - g_\ell^m(0, \lambda_n)}{\lambda_n} \eta_{\ell\,n}^m \tag{8}$$

with independent Gaussian $N(0, \sigma_n^{N\,2})$ random variables $\eta_{\ell\,n}^m$. We have to skip the details on the necessary arguments and conclude this note by characterizing the system of functions g_ℓ^m.

Wave propagation in space These functions naturally occur in the spectral theory of random fields (as developed in [19], [12], etc.) Actually, this stems from the basic rôle they play in harmonic analysis in $L^2(\mathbb{R}^N)$ (see in particular [17], section 10.2.5; see also [3] or [10]). Another context to be mentioned briefly in the sequel, is from mathematical physics (see e.g. [1], [2], [15], [14], [18]).

The wave equation that describes the propagation of sound in a media with density $\rho(x), x \in \mathbb{R}^N$ takes the form of a hyperbolic equation for the induced pressure p:

$$\rho \frac{\partial^2 p}{\partial t^2} = \Delta p \qquad\qquad x \in \mathbb{R}^N$$

where $\Delta = \sum_{i=1}^{N} \partial^2/\partial x_j^2$ is the Laplace operator. To separate the time and space variables t and x, substitute $p(t,x) = \Theta(t)\,\Xi(x)$ and use the separation constant λ^2 to get two equations $\Theta'' = \lambda^2\Theta$ and $\Delta\Xi = \lambda^2\rho\Xi$. The general solution of the first equation in the time component t is expressed in terms of the linear combination of trigonometric functions $\sin\lambda t$ and $\cos\lambda t$. Therefore we focus our attention to the second equation in the space component $x \in \mathbb{R}^N$. It is handy to rewrite the Laplace operator in terms of the spherical coordinates $(r, \theta_1, \ldots, \theta_{N-2}, \phi)$ related to the vector $x = (x_1, \ldots, x_N)$ by

$$\begin{aligned}
x_1 &= r\cos\theta_1 \\
x_2 &= r\sin\theta_1\cos\theta_2 \\
\cdots & \quad \cdots \\
x_{N-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-2}\cos\phi \\
x_N &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-2}\sin\phi
\end{aligned}$$

where $r = |x|$. In these coordinates

$$\Delta = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\big(r^{N-1}\frac{\partial}{\partial r}\big) + \frac{1}{r^2}\Delta_0$$

where Δ_0 is the Laplace–Beltrami operator on the unit sphere S^{N-1}, given by

$$\begin{aligned}
\Delta_0 = {} & \frac{1}{\sin^2\theta_{N-2}}\frac{\partial}{\partial\theta_{N-2}}\big(\sin^{N-2}\theta_{n-2}\frac{\partial}{\partial\theta_{N-2}}\big) \\
& + \frac{1}{\sin^2\theta_{N-2}\sin^2\theta_{N-3}}\frac{\partial}{\partial\theta_{N-3}}\big(\sin^{N-3}\theta_{N-3}\frac{\partial}{\partial\theta_{N-3}}\big) \\
& + \cdots + \frac{1}{\sin^2\theta_{N-2}\cdots\sin^2\theta_1}\frac{\partial}{\partial\phi},
\end{aligned}$$

that is a symmetric operator possessing the complete orthonormal set of squire integrable eigenfunctions on the unit sphere, the so-called spherical harmonics, corresponding to the eigenvalues $-\ell(\ell+N-2)$, $\ell = 0, 1, \ldots$, with multiplicities (6). Denoting these eigenfunctions by $\{Y_\ell^m, m = 1, \ldots, h(\ell, N)\}$, we thus have

$$\Delta_0 Y_\ell^m + \ell(\ell+N-2)Y_\ell^m = 0.$$

Let us turn back to our problem of solving the equation

$$\Delta\Xi = \lambda^2\rho\Xi. \tag{9}$$

By treating only the case of a radial density $\rho(r)$, we can separate the radial and angular coordinates r and (θ, ϕ) by substituting $\Xi(x) = u(r)v(\theta,\phi)$ and using a separation constant k^2. We get then two equations: $\Delta_0 v + k^2 v = 0$ for the angular coordinates and $r^{3-N}\big(r^{N-1}u'\big)' + (\lambda^2 r^2\rho - k^2)u = 0$ for the radius. As was already said, the former equation is integrated for $k^2 = \ell(\ell+N-2)$ in terms of the spherical harmonics. Prescribe therefore the same value to k^2 also in the radial equation. We get

$$r^{3-N}\big(r^{N-1}u'\big)' + [\lambda^2 r^2\rho - \ell(\ell+N-2)]u = 0. \tag{10}$$

Equation (10) is the subject of hard study in the literature, but the explicit solutions are known only in several particular cases of the density $\rho(r)$ (see, for instance, the books on mathematical physics and quantum mechanics, cited above). The simplest case is $\rho \equiv 1$, of course. The solution in this case is well-known, since (10) is reducible to the Bessel equation. Subject to the initial conditions $u_\ell(0) = \delta_{\ell\,0}$, we have

$$u_\ell(r) = 2^{\frac{N-2}{2}} \Gamma\left(\tfrac{N}{2}\right) \frac{J_{\ell+\frac{N-2}{2}}(\lambda r)}{(\lambda r)^{\frac{N-2}{2}}}$$

(see e.g. [18], p. 351, or [15], p. 231). Thus the product

$$g_\ell^m(x,\lambda) = 2^{\frac{N-2}{2}} \Gamma\left(\tfrac{N}{2}\right) \frac{J_{\ell+\frac{N-2}{2}}(\lambda |x|)}{(\lambda |x|)^{\frac{N-2}{2}}}\, Y_\ell^m\left(\tfrac{x}{|x|}\right)$$

satisfy equation (9) for $\rho \equiv 1$, that is in fact the characteristic equation for the Laplace operator (the so-called Helmholtz equation; see e.g. [18], section 31, or [16], section 2.6).

Thus the explicit expression is given for functions g_ℓ^m in the representations (5), (7) and (8).

As was already mentioned, in the scalar case $N = 1$ the representation (5) turns into (4), because the unit sphere in this degenerate case is interpreted as the two point set $\{-1, 1\}$ and we have only two spherical harmonics $Y_0^1(x) = 1$ and $Y_1^1(x) = x$. Besides, in this case $u_0(r) = \cos \lambda r$ and $u_1(r) = \sin \lambda r$.

In the planar case $N = 2$ we have for each ℓ only $h(\ell, 2) = 2$ spherical harmonics $Y_\ell^1(\phi) = \cos \ell\phi/\sqrt{2\pi}$ and $Y_\ell^2(\phi) = \sin \ell\phi/\sqrt{2\pi}$, while $u_\ell(r) = J_\ell(\lambda r)$.

Finally, in the case $N = 3$, most important for physical applications, for each ℓ we have

$$u_\ell(r) = \sqrt{\frac{\pi}{2\lambda r}}\, J_{\ell+\frac{1}{2}}(\lambda r)$$

and $h(\ell, 3) = 2\ell + 1$ spherical harmonics

$$Y_\ell^m(\theta,\phi) = (-1)^m \sqrt{\frac{(1+2\ell)(\ell-m)!}{4\pi\,(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi} \qquad |m| \le \ell$$

with the associated Legendre functions $P_\ell^m(\cos\theta)$ (see e.g. [14], Appendix B, or [16], section 2.4).

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