RANDOM SUMS, MELLIN TRANSFORM, GAUSSIAN RANDOM FIELDS, SMALL DEVIATIONS and REVERSION

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Probability Theory and Related Fields (2004).

Available from (e.g.)

http://www.ams.jhu.edu/∼fill/

§1. QUESTION ABOUT RANDOM SUMS

$$a_1 \geq a_2 \geq \cdots > 0, \qquad \sum a_n < \infty$$

$$Y_1, Y_2, \cdots \geq 0 \qquad \text{i.i.d.}$$

$$h_1(x) := -\log \mathbf{E} e^{-xY} \qquad (x \geq 0)$$

$$S := \sum_n a_n Y_n \geq 0$$

$$h(x) := -\log \mathbf{E} e^{-xS} \qquad (x \geq 0)$$

In terms of h_1, what are asymptotic properties of h ?

§2. HARMONIC SUMS

$$h(x) = \sum_n h_1(a_n x)$$

Harmonic sum ! Use Mellin transforms !

FIRST USE IN THIS KIND OF PROBLEM.

§3. MELLIN TRANSFORMS (see F& S book)

DEFN. $f^*(s) = \int_0^\infty f(x) x^{s-1} dx$.

EXAMPLES 1. $f(x) = e^{-x} \;\rightarrow\; f^*(s) = \Gamma(s)$.

2. $F(x) = (1+x)^{-1} \;\rightarrow\; f^*(s) = \frac{\pi}{\sin(\pi s)}$.

[Exercise in Hankel contours.]

SEPARATION for Harmonic sums: by using $y = a_n x$,

$$
\begin{aligned}
h^*(s) &= \sum_n b_n \int_0^\infty h_1(a_n x) x^{s-1} dx \\
&= \sum_n b_n a_n^{-s} \int_0^\infty h_1(y) y^{s-1} dy \\
&= h_1^*(s) \sum_n b_n a_n^{-s}
\end{aligned}
$$

where the first factor is MT for base function and the second factor is GDS.

INVERSION.

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) x^{-s} ds.$$

MT of derivative.

$$(f')^*(s) = -(s-1) f^*(s-1).$$

MAPPING PROPERTY. Asymptotic properties of $f(x)$ as $x \to 0$ or $x \to \infty$ correspond to singularities of f^*. Behavior $x^\xi (\log x)^k$ ($k = 0, 1, \ldots$) corresponds to $-\frac{(-1)^k \cdot k!}{(s+\xi)^{k+1}}$. Power ξ corresponds to the pole at $-\xi$.

"REASON" (k=0):

$$\int_1^\infty x^\xi x^{s-1} dx = \frac{-1}{s+\xi} \ .$$

EXAMPLE.

$$f(x) = \log(1+x) = \log x + x^{-1} - \frac{1}{2}x^{-2} + \ldots$$

$$f^*(x) = \frac{\pi}{s\ \sin(\pi s)}$$

FUNDAMENTAL STRIP $= \langle -1, 0 \rangle$,

$\exists$ mero. extension of f^* to $s \in \mathbf{C}$.

Look for singularities in $\langle -1, \infty \rangle$ (only !).

Double pole at $s = 0$:

$$f^*(s) = \frac{1}{s^2} + \frac{O}{s} + \ldots \ \leftrightarrow \log x + O.$$

Simple poles at $s = +j$:

$$f^* = \frac{(-1)^j}{j(s-j)} + \ldots \ \leftrightarrow \ \frac{(-1)^j}{j} x^{-j} \ .$$

MELLIN ASYMPTOTIC SUMMATION

1. Study singularities of

$$h_1^*(s) \qquad \text{and} \qquad \sum_n b_n a_n^{-s}$$

separately.

2.Multiply singular expansions to get singular expansion of $h^*(s)$.

3. Use (reverse) mapping property to get asy. exp. for $h(x)$.

Reduction of "d-dimensional harmonic sums" to "1-dimensional" in a special case: (say $h_1 \equiv 1$)

Suppose

$$a_{\bar{n}} = a_{n_1}(1) \cdots a_{n_d}(d) \qquad n_1, \ldots n_d = 1, 2, \ldots$$

Then

$$\bar{A}(s) = A_1(s) \cdots A_d(s).$$

§4. GAUSSIAN RANDOM FIELDS; KARHUNEN - LOÈVE

$$\begin{aligned} \bar{t} &= (t_1, \ldots, t_d) \in [0,1]^d \\ X &= (X(\bar{t})) = \text{ centered GRF} \\ K(\bar{s}, \bar{t}) &:= Cov(X(\bar{s}), X(\bar{t})) \end{aligned}$$

KARHUNEN-LOÈVE spectral decomposition:

$$K(\bar{s}, \bar{t}) = \sum_n a_n \phi_n(\bar{s}) \phi_n(\bar{t})$$

where $\sum a_n < \infty$, a_n are positive eigenvalues and ϕ_n are O.N. eigenfunctions. Then

$$X \stackrel{\mathcal{L}}{=} \left(\sum_n \sqrt{a_n}\ \phi_n(\bar{t})\ Z_n \right)$$

where Z_n's are i.i.d. $\sim N(0,1)$ and

$$||X||_2^2 := \int_{[0,1]^d} X^2(\bar{t}) d\bar{t} = \sum_n a_n Z_n^2 = S$$

Here

$$h_1(x) = -\log \mathbf{E} e^{-xZ^2} = \frac{1}{2} \log(1 + 2x).$$

From now on: Look at $L(x) = 2h(x/2)$.

EXAMPLE 1.$X =$ B.M. $(d = 1)$.

$$K(s,t) = s \wedge t$$

Integral equation $Kf = af (f \not\equiv 0)$ converts to

$$af'' + f = 0, \qquad f(0) = f'(1) = 0.$$

Soln. $\exists$ iff $a = a_n := [(n - 1/2)\pi]^{-2}, n = 1, 2, \ldots$
Then F.S.$= \langle -1, -1/2 \rangle$ and

$$L^*(s) = \frac{\pi}{s \sin \pi s}\left[(\pi/2)^{2s} - \pi^{2s}\right]\zeta(-2s)$$

- simple pole at $s = -1/2$
- <u>simple</u> pole at $s = 0$
- <u>analytic</u> at $s = j$

<u>BETTER</u>: $L(x) = \log \prod_{n=1}^{\infty}\left(1 + \frac{x}{(n - \frac{1}{2})^2\pi^2}\right)$

$$\begin{aligned} &\overset{exactly!}{=} \quad \log \cosh(\sqrt{x}) \\ &= \quad x^{-1/2} - \log 2 + O\left(\exp\{-x^{1/2}\}\right). \square \end{aligned}$$

EXAMPLE 2. $X =$ B.S. $(d \geq 2)$

$$K_{BS}(\bar{s}, \bar{t}) = K_{BM}(s_1, t_1) \cdots K_{BM}(s_d, t_d)$$

So F.S.$=\langle -1, -\frac{1}{2}\rangle$, evals. are $a_{\bar{n}} = a_{n_1} \cdots a_{n_d}$ and

$$L^*_{BS}(s) = \frac{\pi}{s \sin \pi s} \left[A_{BM}(s)\right]^d$$

- pole of order d at $s = \frac{1}{2}$
- <u>analytic</u> o.w. in $\langle -1, 0\rangle$

So

$$\begin{aligned} L_{BS}(x) &= \left[(2\pi)^{d-1}(d-1)!\right]^{-1} x^{1/2}(\log x)^{d-1} \\ &+ \_\ x^{1/2}(\log x)^{d-2} + \ldots \\ &+ \_\ x^{1/2} \\ &+ O(x^{-R}) \qquad \text{for any } R \end{aligned}$$

In fact the last term is exponentially small in a power of x. $\square$

EXAMPLE 3. X = integrated B.M. ($d = 1$). Studied in Kh&Shi(1998),Ch&Li(2003).

$$X(t) := \int_0^t B(u)du = \int_0^t (t-u)dB(u)$$

$$K(s,t) = \int_0^{s\wedge t} (s-u)(t-u)du.$$

Eigenvalues are known only as reciprocal roots of

$$\cosh\left(2^{1/2}(-z)^{1/4}\right) + \cos\left(2^{1/2}(-z)^{1/4}\right) + 2 = 0.$$

with F.S.$=\langle -1, -\frac{1}{4}\rangle$. But MIRACULOUSLY (Hadamard's factorization thm.: GHT(2003), GHLT(2003)

$$\begin{aligned} L(x) \quad &\overset{exactly}{=} \quad \log\Big[\cosh\left(2^{1/2}x^{1/4}\right) \\ &\qquad + \cos\left(2^{1/2}x^{1/4}\right) + 2\Big] - 2\log 2 \\ &= \quad 2^{1/2}x^{1/4} - 3\log 2 \\ &\qquad + O\left(\exp\{-2^{-1/2}x^{1/4}\}\right). \end{aligned}$$

By (direct) mapping property L^* has simple poles at $s = -\frac{1}{4}$ and at $s = 0$; otherwise it is analytic in $\langle -1, \infty\rangle$.

So (or from GHLT study of eigenvalue asymptotics):

$A(s)$ has simple pole at $s = -\frac{1}{4}$ and simple ZERO at $s = 0$. $\square$

EXAMPLE 4. X= m-times integrated B.M. ($d = 1$).

Similarly, $L_m(x)$ has explicit expression involving the characteristic determinant of the boundary-value problem,

$$L_m(x) = c_m x^{1/(2m+2)} + \tilde{c}_m + R_m$$

where c_m admits a simple explicit expression, $\tilde{c}_m$ involves determinant of a Vandermonde matrix and R_m is exponentially small in $x^{1/(2m+2)}$.

So:

$A_m(s)$ has simple pole at $s = -\frac{1}{2m+2}$ and has simple <u>ZERO</u> at $s = 0$. $\square$

EXAMPLE 5. $X = \bar{m}$-integrated B.S. ($d \geq 2$). Notation:

$$\begin{aligned} 0 &\leq m_1 \leq m_2 \leq \dots m_d \\ g &:= \text{\# of groups of tied } m_j\text{'s} \quad (1 \leq g \leq d) \\ t_\nu &:= \text{size of } \nu^{\text{th}} \text{ group } (1 \leq \nu \leq g) \\ \bar{m}_\nu &:= \text{common value of } m_j \text{ in } \nu^{\text{th}} \text{ group} \\ \xi_\nu &:= \frac{1}{2\bar{m}_\nu + 2} \end{aligned}$$

$L^*_{\bar{m}}(s)$ has pole of order t_ν at $-\xi_\nu$ ($\forall \nu$) and is analytic elsewhere in $\langle -1, \infty \rangle$.

<u>THEOREM</u>. $L_{\bar{m}}(x)$ has an asymptotic expansion of the form

$$L_{\bar{m}}(x) = \sum_{\nu=1}^{g} \sum_{k=0}^{t_\nu - 1} c_{\nu,k}\ x^{\xi_\nu}\ (\log x)^k + R_{\bar{m}}$$

Here we can compute coefficients $c_{\nu,k}$. The remainder $R_{\bar{m}}$ is exponentially small.

This expression can be differentiated term by term, any number of times. $\square$

§5. SMALL DEVIATIONS VIA SYTAYA (1974)

$$S = ||X||_2^2 = \sum_n a_n Z_n^2,$$

$$h(x) := -\log \mathbf{E} e^{-xS} = \frac{1}{2} L(2x).$$

<u>THEOREM</u>. (Sytaya) As $\epsilon \to 0$,

$$\begin{aligned} \mathbf{P}(S \le \epsilon) &= (1 + o(1)) \left[-2\pi (x^*)^2 h''(x^*) \right]^{-1/2} \\ &\quad \times \exp\{ -[h(x^*) - \epsilon x^*]\} \end{aligned}$$

where $x^* = x^*(\epsilon)$ is defined by

$$h'(x^*) = \epsilon. \qquad \square$$

REMARKS. 1.) Many applications of small-ball estimates in various norms: connections with metric entropy, Hausdorff dimension, LILs, empirical processes,... Excellent surveys: W.Li & Q-M Shao (2001), Lifshits (1997).

2.) Sytaya: STRONG SDs.

3.) Lifshits (1997) extended to general sums as at start of talk.

4.) Fred & I know how to get <u>full</u> asymptotic expansion (Compare Bahadur & Ranga Rao (1960), Fill (1989).) $\square$

If we want EXPLICIT expansions in ϵ, we must "reverse" (invert)

$$h'(x^*) = \epsilon$$

to get an asymptotic expansion for x^* (and thus for $h(x^*), h''(x^*)$).

UNTIL FURTHER NOTICE, RESTRICT ATTENTION TO $\bar{m}$-INTEGRATED B.S.

§6. REVERSION

Outline:

1.) Exponentially small remainders can be ignored.

2.) lead-order asymptotics for x^* in terms of solution x_0 to

$$\epsilon = \sum_{k=0}^{t_1-1} c_{1,k}(1)\ x^{\xi_1 - 1}\ (\log x)^k$$

Here the sum contains the terms involving highest power x^{ξ_1-1} in asymptotic expansion of $h'(x)$. Denote $-\eta_1 := \xi_1 - 1$.

3.) <u>complete</u> asymptotic expansion for x^* in terms of elementary functions & x_0.

4.) exact computation of x_0.

REVERSION

1.) Exponentially small remainders in expansions for h can be ignored:

LEMMA. Let

$$\widehat{h}(x) := \sum_{\nu=1}^{g} \sum_{k=0}^{t_\nu - 1} c_{\nu,k}\; x^{\xi_\nu}\; (\log x)^k,$$

and let $\widehat{x}$ satisfy

$$\widehat{h}'(\widehat{x}) = \epsilon.$$

Then the following are each exponentially small in a power of $1/\epsilon$:

$$\widehat{x} - x^*, \qquad \widehat{h}^{(j)}(\widehat{x}) - h^{(j)}(x^*) \quad (\forall j).$$

Proof. EASY. $\square$

REVERSION

For lead-order LOGARITHMIC SDs, it's enough to get

2.) lead-order asymptotics for x^*:

$$
\begin{aligned}
x^* &\sim \widehat{x} && \text{(solution dropping exp. rem.)} \\
&\sim x_0 && \text{(keeping only terms with } x^{-\eta_1}\text{)} \\
&\sim \tilde{x}_0 && \text{(keeping only single largest term).}
\end{aligned}
$$

So: Solve

$$\epsilon = c\ x^{-\eta_1}\ (\log x)^{t_1 - 1}$$

for $\tilde{x}_0$.

<u>LEMMA</u>.

$$
\begin{aligned}
x^* &\sim x_0 \sim \tilde{x}_0 \\
&\sim \left[\frac{c}{\eta_1^{t_1-1}} \cdot \frac{1}{\epsilon} \cdot \left(\log \frac{1}{\epsilon} \right)^{t_1 - 1} \right]^{1/\eta_1}
\end{aligned}
$$

<u>Proof</u>. EASY. $\square$

REVERSION

3.) complete asymptotic expansion for x^* (or $\hat{x}$) in terms of elementary functions & x_0:

See paper for general case: we get a complete asymptotic expansion for $\hat{x}$ / x_0 in which remainder estimates drop off at least by powers of $\epsilon^{\eta_2/\eta_1 - 1}$. (This is good enough).

EXAMPLE. $d = 2, 0 = m_1 < m_2$. Then $\hat{x}$ satisfies

$$\epsilon = c_1 \hat{x}^{-1/2} + c_2 \hat{x}^{-\eta} \qquad (1)$$

where $1/2 < \eta = \eta_2 = \frac{2m_2+1}{2m_2+2} < 1$), and x_0 satisfies

$$\epsilon = c_1 x_0^{-1/2} \qquad \text{(i.e., } x_0 = \frac{c_1^2}{\epsilon^2}\text{)}$$

Technique: We know $\hat{x} = x_0 + (1 + o(1)) x_0 y_0$ for some $y_0 = o(1)$. Plug this into eq. (1) and solve for y_0:

REVERSION

$$\begin{aligned}\epsilon &= c_1 x_0^{-1/2}\,[1+(1+o(1))y_0]^{-1/2} \\ &\quad +(1+o(1))c_2 x_0^{-\eta} \\ &= \epsilon\ \left[1+(1+o(1))\frac{1}{2}y_0\right]+(1+o(1))\frac{c_2}{c_1^{2\eta}}\epsilon^{2\eta},\end{aligned}$$

So

$$y_0=(1+o(1))\ \frac{2c_2}{c_1^{2\eta}}\ \epsilon^{2\eta-1}.$$

(Recall here $\frac{\eta_2}{\eta_1}=\frac{\eta}{1/2}=2\eta$.)

Now we know

$$\widehat{x}=x_0\left[1+\frac{2c_2}{c_1^{2\eta}}\ \epsilon^{2\eta-1}+(1+o(1))y_1\right]$$

for some $y_1=o\left(\epsilon^{2\eta-1}\right)$.

Plug this into equation (1) and solve for y_1,

ETC. !

$\square$

4. exact computation of solution x_0 to

$$\begin{aligned} \epsilon &= \sum_{k=0}^{t_1-1} c_{1,k}(1)\ x^{-\eta_1}\ (\log x)^k \\ &:= x^{-\eta} \sum_{k=0}^{t_1-1} a_k\ (logx)^k \end{aligned}$$

We find a (conv.) series representation for x_0.

EXAMPLE. $t = 1$. Solution is trivial:

$$\epsilon = a_0 x^{-\eta} \rightarrow x_0 = (\frac{a_0}{\epsilon})^{\eta}. \quad \square$$

EXAMPLE. $t = 2$. Solution is instructive.

Change variables, from x to

$$w := -\eta \left(\log x + \frac{a_0}{a_1} \right)$$

We need to solve for w the equation

$$w\ e^w = -\frac{\eta}{a_1} \exp\left(-\eta \frac{a_0}{a_1} \right) \epsilon =: -z$$

Solve

$$w\,e^{w} = -z \qquad \left(= -\frac{\eta}{a_1}\ \exp\left(-\eta\,\frac{a_0}{a_1}\right)\epsilon\right) \quad \text{for } w.$$

No elementary solution, but absolutely and uniformly convergent series (also an asymptotic expansion):

$$\begin{aligned} w &= W_{-1}(-z) = \log z - \log\log\frac{1}{z} \\ &\quad + \sum_{r=0}^{\infty}\sum_{s=1}^{\infty} d_{rs}(\log\log\frac{1}{z})^{s}(\log z)^{-(r+s)} \end{aligned}$$

with d_{rs} given simply in terms of Stirling numbers [:]. Here W is a branch of Lambert W-function

(see CGHJK(1996))

REVERSION

$(t = 2)$

The solution can be rearranged into the form

$$(*) \quad \tilde{x}_0 \times \left[1 + \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \tilde{d}_{rs} (\log\log \frac{1}{\epsilon})^s (\log \frac{1}{\epsilon})^{-(r+s)} \right]$$

with $\tilde{d}_{\infty} = 0$. $\square$

GENERAL t RESULT:

Again a result of the form (*) holds. $\square$

e have completed the reversion.

§7. EXPLICIT FORMS OF ASYMPTOTIC EXPANSIONS FOR SDs ($\bar{m}$-integrated B.S., $d \geq 2$)

A.) <u>WEAK RESULTS</u> ($t := t_1, m := m_1$)

<u>THM</u>.1. (independently, and in somewhat different form, by Karol, Nazarov, & Nikitin (2003))

$$
\begin{aligned}
-\log \mathbf{P}(S \leq \epsilon) &= (1 + o(1)) C(d,t,m) \left(\frac{1}{\epsilon}\right)^{\frac{1}{2m+1}} \\
&\quad \times \left(\log \frac{1}{\epsilon}\right)^{(t-1)\frac{2m+2}{2m+1}} . \qquad \square
\end{aligned}
$$

Here $C(d,t,m)$ has a complicated but explicit expression !

<u>THM</u>.2. The $o(1)$ term in Thm.1 has a complete asymptotic expansion of the form

$$
\sum_{r=1}^{\infty} \sum_{s=0}^{r} D_{rs} \left(\log \log \frac{1}{\epsilon}\right)^{s} \left(\log \frac{1}{\epsilon}\right)^{-r}
$$

Here $D_{rs} = D_{rs}(d,t,m)$. In particular, $D_{11} = (t-1)^2 \ \frac{2m+2}{2m+1}$.

B.) STRONG RESULTS

For simplicity, assume distinct m_j's $(g = d)$. From Thm.1, recall (here $t = t_1 = 1$)

$$-\log \mathbf{P}(S \le \epsilon) = (1 + o(1))E(\epsilon)$$

with

$$E(\epsilon) = E_{d,m_1}(\epsilon) = C(d, 1, m_1)\left(\frac{1}{\epsilon}\right)^{\frac{1}{2m_1+1}}.$$

THEOREM 3. The small-ball prob. $\mathbf{P}(S \le \epsilon)$ satisfies

$$\begin{aligned}\mathbf{P}(S \le \epsilon) &= (1 + o(1))\left[\frac{\pi}{m_1+1}E(\epsilon)\right]^{-1/2} \\ &\quad \times \exp\left\{-E(\epsilon)\left[1 + \sum(\epsilon)\right]\right\}\end{aligned}$$

for some finite linear combination $\sum(\epsilon)$ of ϵ^{powers}, wherein each power is a nonzero nonnegative integer combination of the numbers

$$\frac{(2m_1+2)(2m_\nu+1)}{(2m_1+1)(2m_\nu+2)} - 1 > 0, \qquad \nu = 2, \ldots, d. \square$$