On probabilities of small deviations for compound sum processes

Andrei N. Frolov and Alexander I. Martikainen

Department of Mathematics and Mechanics,
St. Petersburg State University

This talk is based mainly on the results obtained in our joint work with Prof. J. Steinebach (Köln, Germany).

Abstract. The logarithmic asymptotics of small deviation probabilities for Poisson and compound Poisson processes are well known because they have independent increments. We study such asymptotics for compound sum processes of much more general structure. Our results are also new for renewal processes. In our theorems, the renewal process and summands, generating the compound sum process, may be dependent.

1. Introduction

Let $(X, Y), (X_1, Y_1), (X_2, Y_2), \ldots$ be a sequence of i.i.d. random vectors with

$$\mathbb{E}X = \mu \in (0, \infty),$$
$$\mathbb{P}(Y > 0) = 1 \text{ and } \mathbb{E}Y = \lambda \in (0, \infty).$$

Note that X and Y may be dependent.

Put $S_0 = R_0 = 0$,

$$S_n = X_1 + \cdots + X_n,\ R_n = Y_1 + \cdots + Y_n,\ n \geqslant 1.$$

Define the compound sum process as

$$S(t) = S_{N(t)}, \ \text{ where } \ N(t) = \max\{n : R_n \leqslant t\}, \ \ t \geq 0.$$

$N(t)$ is the renewal process generated by Y's.

If $\mathbb{P}(Y_1 > x) = e^{-x/\lambda}$, $x > 0$, then $N(t)$ is a Poisson process. If $N(t)$ and $\{X_i\}$ are independent, then $S(t)$ is called a compound Poisson process.

We will describe the logarithmic asymptotics of

$$P_T = \mathbb{P}\Big(\sup_{0 \leqslant t \leqslant T} \Big| S(t) - \frac{\mu}{\lambda}\, t \Big| \leqslant x_T \Big),$$

where $\{x_T\}$ is a positive real function such that $x_T \to \infty$ and $x_T^2 = o(T)$, as $T \to \infty$.

Note that if X and Y are independent, $\mathbb{E}S(t) \sim t\mu/\lambda$ as $t \to \infty$ and $\mathbb{E}S(t) = t\mu/\lambda$ for compound Poisson process.

P_T is the probability that a trajectory of the compound sum process, centered at $t\mu/\lambda$, lies within a symmetric strip (around zero) of width $2x_T$. The same results may be stated for a more general class of sets instead of strips.

If $\mathbb{P}(Y_1 = 1) = 1$, then $S(t) = S_n$, for $t \in [n, n+1)$, the problem has been investigated by Mogul'skii (1974).

Theorem A (Mogul'skii). *Let $\{\xi_i\}$ be a sequence of i.i.d. zero mean random variables from the domain of attraction of a strictly stable law F_α with index $\alpha \in (1,2]$, i.e. $(\xi_1 + \cdots + \xi_n)/B_n$ converges weakly to F_α, where $\{B_n\}$ is a sequence of positive constants. Then, for every sequence $\{x_n\}$ of positive numbers such that $x_n \to \infty$ and $x_n = o(B_n)$,*

$$\log \mathbb{P}\Big(\max_{1 \leqslant k \leqslant n} |\xi_1 + \cdots + \xi_k| \leqslant x_n \Big) \sim -C \, \frac{n}{x_n^\alpha} \, L(x_n),$$

where the function $L(x) = x^{\alpha-2}\, \mathbb{E}\xi_1^2 I\{|\xi_1| < x\}$ is slowly varying (at infinity), and C is an absolute positive constant, depending only on the distribution F_α. If $\alpha = 2$, then $C = \pi^2/8$.

Here and in the sequel,
$I\{B\}$ denotes the indicator function of an event B,
O, o, $\sim$ $\to$ etc. are used for $n \to \infty$ or $T \to \infty$.

Such results for the Poisson process and compound Poisson processes follow from results for processes with independent increments in Mogul'skii (1974). (See, also Borovkov and Mogul'ski (1989, 1991).)

For compound sum processes our results are new, even in the case $\mathbb{P}(X_1 = 1) = 1$, that is for renewal processes.

2. Results

Theorem 1. *Assume that either* $\mathbb{E}e^{hY} < \infty$, *for some* $h > 0$, *or that* X *is positive and* $\mathbb{E}e^{hX} < \infty$, *for some* $h > 0$. *Moreover, let the random variables* $\xi_i = X_i/\mu - Y_i/\lambda$, $i \geqslant 1$, *satisfy the conditions of Theorem A.*
Then, for every positive function $\{x_T\}$ *such that* $x_T \to \infty$ *and* $x_T = o(B_{[T]})$,

$$\log \mathbb{P}\Big(\sup_{0\leqslant t\leqslant T} \Big| S(t) - \frac{\mu}{\lambda}\, t \Big| \leqslant x_T \Big) \leqslant -C \frac{\mu^\alpha}{\lambda} \frac{T}{x_T^\alpha} L(x_T)\big(1+o(1)\big),$$

where C *is the constant from Theorem A.*

We cannot omit the one-sided Cramér condition in Theorem 1. Nevertheless one can obtain a weaker upper bound.

Remark 1. *Assume that the random variables* $\xi_i = X_i/\mu - Y_i/\lambda$ *satisfy the conditions of Theorem A.*
Then, for every positive function $\{x_T\}$ *such that* $x_T \to \infty$, $x_T = o\big(B_{[T]}\big)$ *and* $x_T = o\big(T^{1/(\alpha+1)}\big)$,

$$\log \mathbb{P}\Big(\sup_{0\leqslant t\leqslant T} \Big| S(t) - \frac{\mu}{\lambda}\, t \Big| \leqslant x_T \Big) \leqslant -C \frac{\mu^\alpha}{2\lambda} \frac{T}{x_T^{\alpha+1}} L(x_T)\big(1+o(1)\big),$$

where C *is the constant from Theorem A.*

Let us now turn to lower bounds for P_T. We first consider the case of finite variances.

Theorem 2. *Assume that one of the following two conditions holds:*
(i) *at least one of the random variables X and Y is bounded, and another one has a finite variance;*
(ii) $\mathbb{E}X^3 < \infty$, $\mathbb{E}Y^3 < \infty$ *and*

$$\mathbb{E}X I\{Y > u\} \leqslant \frac{\mu}{\lambda}\Big(1 - \frac{A}{u}\Big)\,\mathbb{E}Y I\{Y > u\}, \quad (1)$$

$$\mathbb{E}Y I\{X > w\} \leqslant \frac{\lambda}{\mu}\Big(1 - \frac{A}{w}\Big)\,\mathbb{E}X I\{X > w\}, \quad (2)$$

for some $A > 0$, all large u and all w close to $\operatorname{ess\,sup} X$.

Then, for every positive function $\{x_T\}$ such that $x_T \to \infty$ and $x_T = o\big(\sqrt{T}\big)$,

$$\log \mathbb{P}\Big(\sup_{0 \leqslant t \leqslant T} \Big|S(t) - \frac{\mu}{\lambda}\,t\Big| \leqslant x_T\Big) \geqslant -\,\frac{\pi^2}{8}\,\frac{\mu^2 v^2}{\lambda}\,\frac{T}{x_T^2}\big(1 + o(1)\big),$$

if $v^2 = \mathbf{Var}(X/\mu - Y/\lambda) > 0$.

Conditions (1) and (2) are satisfied, if either
(i') X and Y are independent and X is nondegenerate;
(ii') Y is bounded.

Theorems 1 and 2 yield the following result.

Corollary 1. *Assume that at least one of the following three conditions holds:*

(i") X *is bounded and* $\mathbb{E}e^{hY} < \infty$, *for some* $h > 0$,

(ii") Y *is bounded and* $\mathbb{E}X^2 < \infty$,

(iii") $\mathbb{E}e^{hY} < \infty$, *for some* $h > 0$, *and* (ii) *is satisfied.*

Then, for every positive function $\{x_T\}$ *such that* $x_T \to \infty$ *and* $x_T = o\big(\sqrt{T}\big)$,

$$\log \mathbb{P}\Big(\sup_{0 \leqslant t \leqslant T} \Big| S(t) - \frac{\mu}{\lambda}\, t \Big| \leqslant x_T \Big) \;\sim\; -\frac{\pi^2}{8}\,\frac{\mu^2 v^2}{\lambda}\,\frac{T}{x_T^2},$$

if $v^2 = \mathbf{Var}(X/\mu - Y/\lambda) > 0$.

The next result follows immediately from Corollary 1 and duality principle in Theorem 4.

Corollary 2. *Assume that* $\mathbb{E}X^2 < \infty$, $\mathbb{E}Y^2 < \infty$ *and* $v^2 > 0$. *Moreover, let* X *and* Y *be positive and either* X *or* Y *be bounded.*

Then, for every positive function x_T *such that* $x_T \to \infty$ *and* $x_T = o\big(\sqrt{T}\big)$,

$$\log \mathbb{P}\Big(\sup_{0 \leqslant t \leqslant T} \Big| S(t) - \frac{\mu}{\lambda}\, t \Big| \leqslant x_T \Big) \;\sim\; -\frac{\pi^2}{8}\,\frac{\mu^2 v^2}{\lambda}\,\frac{T}{x_T^2}.$$

For the special case of $\mathbb{P}(X = 1) = 1$, Corollary 2 implies the following result for renewal counting processes.

Corollary 3. *Assume that* $0 < \mathbf{Var}(Y) = \sigma_Y^2 < \infty$. *Then, for every positive function* $\{x_T\}$ *such that* $x_T \to \infty$ *and* $x_T = o\big(\sqrt{T}\big)$,

$$\log \mathbb{P}\Big(\sup_{0 \leqslant t \leqslant T} \Big| N(t) - \frac{t}{\lambda} \Big| \leqslant x_T \Big) \;\sim\; -\frac{\pi^2}{8} \frac{\sigma_Y^2}{\lambda^3} \frac{T}{x_T^2}.$$

We now turn to the case, when $\mathbb{E}X^2 = \infty$ or $\mathbb{E}Y^2 = \infty$.

Theorem 3. *Assume that X and Y are independent. If the random variables X and Y belong to the domains of attraction of stable laws, i.e.*

$$\mathbb{P}\left(S_n - n\mu < xB_n\right) \Rightarrow F_\alpha(x), \quad \alpha \in (1,2] \text{ and } B_n > 0,$$

and

$$\mathbb{P}\left(R_n - n\lambda < xB'_n\right) \Rightarrow F_\beta(x), \quad \beta \in (1,2] \text{ and } B'_n > 0,$$

then, for every positive function $\{x_T\}$ such that $x_T \to \infty$ and $x_T = o\big(\max\{B_{[T]}, B'_{[T]}\}\big)$,

$$\log \mathbb{P}\Big(\sup_{0 \leqslant t \leqslant T} \Big|S(t) - \frac{\mu}{\lambda}\, t\Big| \leqslant x_T\Big)$$
$$\geqslant -C\,\frac{\mu^2}{\lambda}\,\frac{T}{x_T^2} \int\limits_{-x_T/\mu}^{x_T/\mu} u^2\, d\big(F_{X/\mu}(u) + F_{Y/\lambda}(u)\big)\big(1 + o(1)\big),$$

where C is an absolute positive constant, depending only on the distributions F_α and F_β.

Moreover, if $\alpha \neq \beta$, then the latter bound holds with the same absolute constant C as in Theorem A, depending on $F_{\min\{\alpha,\beta\}}$.

If X is positive, then the proof of Theorem 1 is based on the duality between the compound sum processes $S(t)$ and

$$R(t) = R_{M(t)+1}, \quad \text{where} \quad M(t) = \max\{n : S_n \leqslant t\},\ t \geq 0.$$

This duality is of independent interest.

Theorem 4. *Assume that* $\mathbb{P}(X > 0) = 1$*. Then,*

$$\mathbb{P}\Big(\sup_{0\leqslant t\leqslant T}\Big|S(t) - \frac{\mu}{\lambda}t\Big| < x_T\Big) \leqslant \mathbb{P}\Big(\sup_{0\leqslant t\leqslant \frac{\mu}{\lambda}T - x_T}\Big|R(t) - \frac{\lambda}{\mu}t\Big| \leqslant \frac{\lambda}{\mu}x_T\Big),$$

$$\mathbb{P}\Big(\sup_{0\leqslant t\leqslant T}\Big|S(t) - \frac{\mu}{\lambda}t\Big| \leqslant x_T\Big) \geqslant \mathbb{P}\Big(\sup_{0\leqslant t\leqslant \frac{\mu}{\lambda}T + x_T}\Big|R(t) - \frac{\lambda}{\mu}t\Big| < \frac{\lambda}{\mu}x_T\Big).$$