Small Deviations of Fractional Processes in L_q-Spaces with Respect to Fractal Measures

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Random vector in normed space: $X \in (E, \|\cdot\|)$

Small ball (small deviation) probabilities:

$$\mathbf{P}\left\{\|X\| \leq \varepsilon\right\}, \qquad \varepsilon \to 0.$$

Usually: X - sample path of a process, E - a functional space $C[0,1], L_q[0,1]$ *etc*.

Connections:

- entropy of compact operators;

- quantization of random vectors;

- approximation of random processes ...

Typically:

$$\mathbf{P}\left\{\|X\| \le \varepsilon\right\} \sim C\varepsilon^{\beta} \exp\left\{-K\varepsilon^{-\gamma}\right\}, \qquad \varepsilon \to 0,$$

or

$$\log \mathbf{P}\left\{\|X\| \le \varepsilon\right\} \sim -K\varepsilon^{-\gamma}, \qquad \varepsilon \to 0.$$

K - small ball constant, γ - small ball rate.

Examples for Brownian motion:

$$\mathbf{P}\left\{\sup_{t\in[0,1]} |W(t)| \le \varepsilon\right\} \sim \frac{4}{\pi} \exp\left\{-\frac{\pi^2}{8}\varepsilon^{-2}\right\}$$

$$\mathbf{P}\left\{\int_0^1 |W(t)|^2\,dt \le \varepsilon^2\right\} \sim \frac{4\varepsilon}{\sqrt{\pi}} \exp\left\{-\frac{1}{8}\varepsilon^{-2}\right\}$$

Problem:

What can we say about

$$\mathbf{P}\left\{\sup_{t\in T}|W(t)| \le \varepsilon\right\}$$

for arbitrary $T \subset [0,1]$ or about

$$\mathbf{P}\left\{\int_0^1 |W(t)|^q \, \mu(dt) \le \varepsilon^q\right\}$$

for arbitrary finite measure μ on $[0,1]$?

The answer depends on *entropy properties* of T and μ, respectively.

Extension: other processes.

Fractional processes.

Fractional Brownian Motion(FBM),0 $< H <$ 1.

$$W^H(t) = \int_{-\infty}^{t} \left[(t-u)^{H-1/2} - (-u)_{+}^{H-1/2} \right] dW(u)$$

Riemann-Liouville process (RL), 0 $< H$.

$$R^H(t) = \int_0^t (t-u)^{H-1/2} dW(u).$$

RL has no stationary increments but has three advantages: well defined for $H > 1$, closed w.r. to integration and has an extrapolation homogeneity.

Recently:

M.Lifshits and T.Simon found small ball rates for W^H and R^H with respect to fairly general *self-similar* norms. This includes the sup-norm and L^q-norm w.r.t. Lebesgue measure but does not cover the case of general sets and measures.

The sup-**norm ($q = \infty$).**

A result of W.Linde (2004). Let $T \subset [0, 1]$. Let $N_T(\varepsilon)$ be the metric entropy of T, that is the minimal number of intervals of length ε sufficient to cover T. Then

$$N_T(\varepsilon) \approx \varepsilon^{-\beta}$$

is necessary and sufficient for

$$-\log \mathbf{P}\left\{\sup_{t\in T} |R^H(t)| < \varepsilon\right\} \approx \varepsilon^{-\beta/H}.$$

Example: Let $T = [0, 1]$, $H = 1/2$, then $R^H = W$ and $\beta = 1$. We get the small ball rate 2, in accordance to the classical result.

For smaller sets, β is smaller, and the small ball rate is smaller as well.

Mixed entropy of a measure.

Let μ be a measure on $[0,1]$, let $H > 0$ and $q \in [1,\infty)$. Define a "magic number" $r > 0$ by

$$1/r := H + 1/q.$$

The normed mixed entropy numbers of μ are defined as follows:

- take an integer $m > 0$;

- cover the interval $[0,1]$ with any m closed intervals Δ_j, $1 \le j \le m$.

- minimize over coverings:

$$\sigma_\mu(m) := \inf_{(\Delta_j)} \left\{ \left(\sum_{j=1}^{m} |\Delta_j|^{H\,r} \mu(\Delta_j)^{r/q} \right)^{1/r} \right\}.$$

Example: μ - Lebesgue measure, $\sigma_\mu(m) \equiv 1$.

Main result.

Notation: for positive functions f, g notation $f \preceq g$ (resp. $f \succeq g$) means $\limsup f/g < \infty$, (resp. $\liminf f/g > 0$). Let

$$\|f\|_{q,\mu} = \left[\int |f(t)|\, \mu(dt)\right]^{1/q}.$$

Theorem. *Let μ be a finite continuous measure on $[0,1]$ and let R^H be the Riemann–Liouville process of index $H > 0$. Then*

(a). *If $\sigma_\mu(m) \succeq m^{-\nu} (\log m)^\beta$ for some $\nu \geq 0$ and $\beta \in \mathbf{R}$, then*

$$-\log \mathbf{P}\{\|R^H\|_{q,\mu} < \varepsilon\} \succeq \varepsilon^{-1/(H+\nu)} \cdot |\log \varepsilon|^{\beta/(H+\nu)}.$$

(b). *If $\sigma_\mu(m) \preceq m^{-\nu} (\log m)^\beta$, then*

$$-\log \mathbf{P}\{\|R^H\|_{q,\mu} < \varepsilon\} \preceq \varepsilon^{-1/(H+\nu)} \cdot |\log \varepsilon|^{\beta/(H+\nu)}.$$

For $0 < H < 1$, both assertions also hold for the FBM B^H instead of R^H.

Application 1: fractal measure

For $N \geq 2$, take some positive weights $\rho_1, \ldots, \rho_N$ such that

$$\sum_{k=1}^{N} \rho_k = 1,$$

and N intervals with disjoint interiors $[a_1, b_1], \ldots,$ $[a_N, b_N]$ in $[0, 1]$. Let $S_k : [0,1] \to [a_k, b_k]$ be affine isomorphisms. The self–similar measure μ is defined by the equation

$$\mu = \sum_{k=1}^{N} \rho_k \ \ [\mu \circ S_k^{-1}].$$

On every interval $[a_k, b_k]$ it behaves like on $[0, 1]$ up to the numeric factor ρ_k and, eventually, the space inversion.

Example: Cantor measure, $N = 2$.

Let $f \approx g$ mean that $f \succeq g$ and $f \preceq g$. We have

Theorem. *Let* $\lambda_k := b_k - a_k$ *and let* $\gamma > 0$ *be the unique solution of the equation*

$$\sum_{k=1}^{N} \lambda_k^{H\gamma} \rho_k^{\gamma/q} = 1.$$

Then

$$\sigma_\mu(m) \approx m^{-(1/\gamma - 1/r)} ,$$

hence

$$-\log \mathbf{P} \left\{ \|R^H\|_{q,\mu} < \varepsilon \right\} \approx \varepsilon^{-1/(1/\gamma - 1/q)} .$$

Remark 1. The small ball rate does not depend on the special choice of S_k.

Remark 2. For a fixed μ the small ball rate depends on q, unlike for Lebesgue measure. More precisely, there is no dependence on q iff $\lambda_k = \rho_k^s$, for some $s > 0$, $1 \le k \le N$.

Remark 3. Hilbert space case, $q = 2$ (Nazarov).

Application 2: random fractal measures

Subordinator $A = (A(t))_{t \geq 0}$: a non–decreasing process with homogeneous independent increments. Its Laplace exponent Φ_A is defined by

$$\mathbf{E} e^{-A(t) \cdot x} = e^{-t \cdot \Phi_A(x)}\,, \qquad t, x \geq 0\,.$$

Every subordinator A generates random measures by $\mu_\omega([0,s]) = Leb\,(\{t \in [0,1] : A(t,\omega) \leq s\})\,.$

Theorem. *For any $H > 0$ and any $q \in [1,\infty)$ there exist constants $c_1, c_2 > 0$ such that for any A and almost all measures μ_ω*

$$\left(\frac{m}{\Phi_A^{-1}(c_1 m)} \right)^H \preceq \sigma_{\mu_\omega}(m) \preceq \left(\frac{m}{\Phi_A^{-1}(c_2 m)} \right)^H \,.$$

Remark. Here the left and the right hand sides are not necessary equivalent.

Hence, for small ball probabilities, we have

Theorem. *Let A be a subordinator such that*

$$\Phi_A(x) \approx x^{\beta} (\log x)^{\kappa}, \qquad x \to \infty,$$

for certain $\beta \in (0,1]$ and $\kappa \in \mathbf{R}$. If R^H is an RL–process, $H > 0$, independent of A, then for almost all ω and each $q \in [1, \infty)$ we have

$$-\log \mathbf{P} \left\{ \|R^H(A(\cdot, \omega))\|_q < \varepsilon \right\} \approx \varepsilon^{-\beta/H} \, |\log \varepsilon|^{\kappa}.$$

Example: symmetric stable process with independent increments.

Work in progress: multi-parametric case

We consider now $W(t), t \in \mathbf{R}^d$, the fractional Brownian function with d parameters, defined by $W(0) = 0$ and

$$\mathbf{E}|W(t) - W(s)|^2 = \|t - s\|^{2H}.$$

This is a natural extension of FBM. No reasonable extension for RL process is known to us.

Now the measure μ is concentrated on a compact subset of $\mathbf{R}^d$. The entropy notions and their properties are considerably more involved in multi-dimensional setting.

Roughly speaking, we can give some entropy bounds for small deviation probabilities but the entropy notions used are, in general, different in the lower and in the upper bound.

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