Error Bounds for Weak Approximation of SDEs

Thomas Müller-Gronbach, Uni Magdeburg

Joint work with

Steffen Dereich, TU Berlin

Klaus Ritter, TU Darmstadt

Given: r.v. X with values in a separable Banach space $(\mathfrak{X}, \|\cdot\|)$ and $E(\|X\|) < \infty$.

Let $F = \{f : \mathfrak{X} \to \mathbb{R} \mid f$ is Lip(1) - continuous$\}$

Quadrature problem: Approximate

$$S(f) = E(f(X)), \quad f \in F,$$

based on finitely many functional evaluations.

Here, in particular, **weak approximation of sdes:**

$$(\mathfrak{X}, \|\cdot\|) = \big(C([0,1]), \|\cdot\|_\infty\big),$$
$$dX_t = a(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \quad t \in [0,1].$$

Outline

- Worst case error analysis for deterministic and randomized algorithms
- General relations to quantization
- Results for sde's

General relations to quantization

Deterministic algorithms

$\mathbb{S}_n^{\text{det}}$: All mappings $\widehat{S} : F \to \mathbb{R}$

based on n (adaptive) functional evaluations

Worst case error:

$$e(\widehat{S}) = \sup_{f \in F} |S(f) - \widehat{S}(f)|, \quad \widehat{S} \in \mathbb{S}_n^{\text{det}}$$

n-th minimal error:

$$e_n^{\text{det}} = \inf_{\widehat{S} \in \mathbb{S}_n^{\text{det}}} e(\widehat{S})$$

Quantization

Approximate X by $\varphi(X)$ with

$$\varphi \in \Phi_n = \{\varphi : \mathfrak{X} \to \mathfrak{X} \mid \varphi \text{ measurable, } |\varphi(\mathfrak{X})| \le n\}$$

Error:

$$q(\varphi) = E\,\|X - \varphi(X)\|$$

n-th quantization error:

$$q_n = \inf_{\varphi \in \Phi_n} q(\varphi)$$

Theorem

$$\forall n \in \mathbb{N}: \ e_n^{\text{det}} = q_n$$

(Smolyak 1965; Bakhvalov 1971;

Kantorovich, Rubinstein 1958; Gray et al. 1975)

Good quantizers $\varphi \in \Phi_n$ yield good algorithms $\widehat{S}^\varphi \in \mathbb{S}_n^{\text{det}}$

$$\text{by} \qquad \widehat{S}^\varphi(f) = S(f \circ \varphi), \quad f \in F,$$

$$e(\widehat{S}^\varphi) = q(\varphi).$$

Results on e_n^{det}, q_n

for **compact** or **finite-dimensional spaces** $\mathfrak{X}$:

See, e.g.

Novak 1988; Graf, Luschgy 2000;

Wasilkowski, Woźniakowski 2000, 2001

for **function spaces** $\mathfrak{X}$:

Wasilkowski, Woźniakowski 1996

Since 2000: Dereich, Fehringer, Luschgy, Matoussi, Pagés, Scheutzow, ...

Randomized algorithms

$\mathbb{S}_n^{\text{ran}}$: All randomized algorithms based on n adaptive functional evaluations

Worst case error:

$$e(\widehat{S}) = \sup_{f \in F} \left(E\,|S(f) - \widehat{S}(f)|^2\right)^{1/2}, \quad \widehat{S} \in \mathbb{S}_n^{\text{ran}}$$

n-th minimal error:

$$e_n^{\text{ran}} = \inf_{\widehat{S} \in \mathbb{S}_n^{\text{ran}}} e(\widehat{S})$$

Theorem Dereich, M-G, Ritter 2005

$$\forall\, n \in \mathbb{N}: \quad e_n^{\text{ran}} \geq 1/2 \cdot n^{1/2} \cdot \sup_{k \geq 2n} (q_{k-1} - q_k)$$

Corollary

If q_n is regular varying with negative index then

$$e_n^{\text{ran}} \succcurlyeq n^{-1/2} \cdot q_n$$

Remark

For dim$(\mathfrak{X}) < \infty$ this bound is sharp in many cases, e.g., for X uniformly distributed on $\mathfrak{X} = [0,1]^d$:

$$e_n^{\text{ran}} \asymp n^{-1/2-1/d}$$

Results for SDEs

Now: $$(\mathfrak{X}, \|\cdot\|) = \big(C([0,1]), \|\cdot\|_\infty\big),$$
$$dX_t = a(X_t)\,dt + \sigma(X_t)\,dW_t,$$
$$X_0 = x_0 \in \mathbb{R}.$$

Assumptions

(i) a is Lipschitz continuous

(ii) $\sigma \in C^2(\mathbb{R})$ with bounded derivatives

(iii) $\sigma(x_0) \neq 0$

Dereich 2003: $e_n^{\text{det}} = q_n \approx c(x_0, a, \sigma) \cdot (\ln n)^{-1/2}$

Thus: Deterministic algorithms are inappropriate for the class of all Lip(1)-functionals

Theorem Dereich, M-G, Ritter 2005

$$n^{-1/2} \cdot (\ln n)^{-3/2} \preccurlyeq e_n^{\text{ran}} \preccurlyeq n^{-1/2} \cdot (\ln n)^{-1/2}$$

Lower bound: Bakhvalovs trick

Upper bound: Classical Monte-Carlo with variance reduction by quantization

Classical Monte-Carlo:

$$\widehat{S}(f) = \frac{1}{n} \sum_{i=1}^{n} f(X_i), \qquad f \in F,$$

with $X_1, \ldots, X_n$ i.i.d. as X. **Unfeasible!**

Model of computation

Assumptions so far include:

- Perfect generator for random elements in $\mathfrak{X}$ for any Borel probability measure
- Oracle provides functional evaluations $f(x)$ for every $x \in \mathfrak{X}$ at the same finite cost

More realistic:

- Perfect generator for random numbers in $[0, 1]$
- Oracle provides functional evaluations $f(x)$ for every x in a fixed finite-dimensional subspace of $\mathfrak{X}$.
  Cost of one call depends on dimension of subspace
  Hereby: Improved lower bound

Algorithms with functional evaluation in a subspace

$\widetilde{\mathfrak{X}}$: finite-dimensional subspace of $\mathfrak{X}$
e.g., piecewise linear functions with
k equidistant nodes

$\mathbb{S}_n^{\text{ran}}(\widetilde{\mathfrak{X}})$: All randomized algorithms based on
n adaptive functional evaluations in $\widetilde{\mathfrak{X}}$

Error and cost of $\widehat{S} \in \mathbb{S}_n^{\text{ran}}(\widetilde{\mathfrak{X}})$:

$$e(\widehat{S}) = \sup_{f \in F} (E\,|S(f) - \widehat{S}(f)|^2)^{1/2}$$

$$\text{cost}(\widehat{S}) = n \cdot \text{dim}(\widetilde{\mathfrak{X}})$$

Key quantities

For $n, k \in \mathbb{N}$ let

$$e_{n,k}^{\text{ran}} = \inf\{e(\widehat{S}) \mid \widehat{S} \in \mathbb{S}_n^{\text{ran}}(\widetilde{\mathfrak{X}}),\ \text{dim}(\widetilde{\mathfrak{X}}) = k\}$$

Minimal error for given cost bound N:

$$\varepsilon_N^{\text{ran}} = \inf_{n \cdot k \le N} e_{n,k}^{\text{ran}}$$

Theorem Dereich, M-G, Ritter 2005

$$\varepsilon_N^{\text{ran}} \succcurlyeq N^{-1/4} \cdot (\ln N)^{-3/4}$$

Sketch of Proof: Recall

$$e_{n,k}^{\text{ran}} = \inf\{e(\widehat{S}) \mid \widehat{S} \in \mathbb{S}_n^{\text{ran}}(\widetilde{\mathfrak{X}}),\ \text{dim}(\widetilde{\mathfrak{X}}) = k\}$$

Since $\mathbb{S}_n^{\text{ran}}(\widetilde{\mathfrak{X}}) \subset \mathbb{S}_n^{\text{ran}}$

$$\inf_k e_{n,k}^{\text{ran}} \geq e_n^{\text{ran}} \succcurlyeq n^{-1/2} \cdot (\ln n)^{-3/2}$$

Relation to average Kolmogorov widths

$$\inf_n e_{n,k}^{\text{ran}} = \inf_{\text{dim}(\widetilde{\mathfrak{X}})=k} E\,\text{dist}(X, \widetilde{\mathfrak{X}}) \succcurlyeq k^{-1/2}$$

Hence

$$\begin{aligned} \varepsilon_N^{\text{ran}} &= \inf_{n \cdot k \leq N} e_{n,k}^{\text{ran}} \\ &\succcurlyeq \inf_{n \cdot k \leq N} \max(n^{-1/2} \cdot (\ln n)^{-3/2}, k^{-1/2}) \end{aligned}$$

Upper bound by Monte-Carlo with Euler

$\widehat{X}^{(k)}$: Piecewise linear interpolation of Euler scheme with stepsize $1/k$.

Let $\widehat{X}_1^{(k)}, \ldots, \widehat{X}_n^{(k)}$ i.i.d. as $\widehat{X}^{(k)}$, and define

$$\widehat{S}_N(f) = \frac{1}{n} \sum_{i=1}^{n} f(\widehat{X}_i^{(k)}), \qquad f \in F,$$

with $k \asymp N^{1/2} \cdot (\ln N)^{1/2}, \; n \asymp N^{1/2}/(\ln N)^{1/2}$.

Then

$$\text{cost}(\widehat{S}_N) \asymp N \quad \wedge \quad e(\widehat{S}_N) \preccurlyeq N^{-1/4} \cdot (\ln N)^{1/4}$$

Recall: $\varepsilon_N^{\text{ran}} \succcurlyeq N^{-1/4} \cdot (\ln N)^{-3/4}$

Corollary

Up to logarithms, Monte-Carlo with Euler is optimal for integration of Lip(1)-functionals.

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