LOGARITHMIC SMALL BALL ASYMPTOTICS IN L_2-NORM WITH RESPECT TO SELF-SIMILAR MEASURE FOR SOME GAUSSIAN PROCESSES

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$X(t)$, $0 \le t \le 1$, is zero mean Gaussian process with covariance function $G_X(t,s) = EX(t)X(s)$. μ is a measure on $[0,1]$, $\mu([0,1]) = 1$.

$$||X||_\mu = ||X||_{L_2(0,1;\mu)} = (\int_0^1 X^2(t)\ \mu(dt))^{\frac{1}{2}}$$

Small ball behavior problem: to define the asymptotics of $P\{||X||_\mu \le \varepsilon\}$ as $\varepsilon \to 0$.

Logarithmic s.b. asymptotics: the asymptotics of $\ln P\{||X||_\mu \le \varepsilon\}$ as $\varepsilon \to 0$.

By Karhunen-Loève expansion, we have in distribution

$$||X||_\mu^2 = \sum_{n=1}^{\infty} \lambda_n \xi_n^2.$$

ξ_n, $n \in \mathbb{N}$, are independent standard normal r.v.'s

while $\lambda_n > 0$, $n \in \mathbb{N}$, $\sum\limits_n \lambda_n < \infty$, are the eigenvalues of the integral equation on $[0, 1]$

$$\lambda y(t) = \int_0^1 G_X(s,t) y(s) \mu(ds). \quad (1)$$

If μ contains absolutely continuous component, Birman and Solomyak (1970) derived one-term asymptotics of λ_n for wide class of kernels G_X. Using this result Nazarov and Nikitin (2004) obtained the **explicit** logarithmic small ball asymptotics in L_2-norm for general class of processes with weights.

Birman and Solomyak also showed that **in this case the asymptotics of λ_n does not depend on singular component of μ.**

Let μ be **singular** with respect to Lebesgue measure. For G_X being the Green function of BVP for the operator $(-1)^{\ell}y^{(2\ell)}$ Borzov (1970) showed that $\lambda_n = o(n^{-2\ell})$ instead of usual asymptotics $\lambda_n \sim C \cdot n^{-2\ell}$ in the case of non-singular μ. Also he obtained better estimates for some special classes of μ.

The case of **self-similar** measure μ.

For the Green function of simplest operator $-y''$ Fujita (1985) derived the exact power order of eigenvalues decreasing. Further, Kigami and Lapidus (1993), Solomyak and Verbitsky (1995) showed that in the case of **non-arithmetic self-similarity** the eigenvalues have the asymptotics $\lambda_n \sim C \cdot n^{-p}$ while in the case of **arithmetic self-similarity** the asymptotics of λ_n is more complicated: besides power term it can contain a periodic function of $\ln(n)$.

Vladimirov and Sheipak generalized this result for μ being a self-similar distribution of more general class.

We generalize this result in another direction and establish the one-term spectral asymptotics for the Green function of self-adjoint ordinary differential operator with the main term $(-1)^\ell y^{(2\ell)}$, $\ell \in \mathbb{N}$. Then, on this basis, we obtain the logarithmic L_2-s.b. asymptotics with respect to self-similar measure for zero mean Gaussian process X under condition that G_X is such Green function.

Recall that this class of processes contains Brownian motion, Brownian bridge, Slepian process, Ornstein – Uhlenbeck process, centered and integrated counterparts of these processes. Unfortunately, our method cannot give explicit expression for the small ball constants.

Recently Lifshits, Linde and Shi derived **the order** of logarithmic s.b.a. in **arbitrary L_q-norm** for more wide class of Gaussian processes. Their approach is essentially more complicated and the results are much more general than our ones. However, the specific character of L_2 allows to obtain more detailed results.

We call the function f **asymptotically T-periodic** if there exists T-periodic function g such that $f(t) \sim g(t)$ as $t \to \infty$.

Recall the construction of self-similar probability measure on $[0,1]$. Consider $k \geqslant 2$ nonempty non-intersecting intervals in $]0,1[$:

$$I_j =]a_j, b_j[, \quad j = 1, \ldots, k;$$

$$a_1 \geqslant 0; \quad b_k \leqslant 1; \quad b_j \leqslant a_{j+1}.$$

Consider also a vector of positive numbers (ρ_j), $j = 1, \ldots, k$, such that $\sum_j \rho_j = 1$.

Define a family of affine functions (contractions) S_j moving $[0,1]$ onto I_j, $j = 1, \ldots, k$.

There exists the only probability measure μ s.t. for any Lebesgue-measurable set $E \subset [0,1]$

$$\mu(E) = \sum_j \rho_j \cdot \mu(S_j^{-1}(E)).$$

This measure is called **self-similar measure generated by the system** (S_j, ρ_j), $j = 1, \ldots, k$.

When $\sum_j |I_j| < 1$ the support of μ (minimal closed set $\mathcal{E} \subset [0,1]$ such that $\mu([0,1] \setminus \mathcal{E}) = 0$) is called **Cantor set generated by the system** (S_j). Its Hausdorff dimension $\alpha \in]0,1[$ is equal to the unique solution of the equation

$$\sum_j |I_j|^\alpha = 1.$$

In the case $\sum\limits_j |I_j| = 1$ the support of μ is $[0,1]$, and $\alpha = 1$. If, in addition, $\rho_j = |I_j|$, $j = 1, \ldots, k$, then μ is usual Lebesgue measure. However in all other cases μ is singular.

Recall that the Hausdorff dimension of the measure μ is the least Hausdorff dimension of a set $\mathcal{E} \subset [0,1]$ (not necessarily closed) such that $\mu([0,1] \setminus \mathcal{E}) = 0$. For our measure μ this dimension is equal to

$$\beta = \frac{\sum_j \rho_j \ln(\rho_j)}{\sum_j \rho_j \ln(|I_j|)}.$$

Clearly, $\beta \leqslant \alpha$, and $\beta = \alpha$ iff $\rho_j = |I_j|^\alpha$ for any $j = 1, \ldots, k$. In particular, it's the case if μ is simplest **Cantor measure** (in this case $\alpha = \beta = \ln(2)/\ln(3)$).

Let consider the self-adjoint, positive definite operator L generated by differential expression

$$\mathcal{L}y \equiv (-1)^\ell y^{(2\ell)} + \left(\mathcal{P}_{\ell-1} y^{(\ell-1)}\right)^{(\ell-1)} + \cdots + \mathcal{P}_0 y$$

with proper boundary conditions. Here $\mathcal{P}_i \in L_1(0,1)$, $i = 0, \ldots, \ell - 1$.

We are interested in the behavior of the eigenvalues of BVP

$$\lambda \mathcal{L} y = \mu y \ \text{(+ boundary conditions)}, \quad (2)$$

where μ is a self-similar probability measure.

If G_X is the Green function for operator $\mathcal{L}$ then (2) is equivalent to (1). Denote λ_n the eigenvalues of (2) enumerated in the decreasing order.

Theorem 1. *Given self-similar probability measure μ, define*

$$c_j = \rho_j \cdot |I_j|^{2\ell-1}, \quad j = 1, \dots, k,$$

and define $p \geqslant 2\ell$ as the unique solution of

$$\sum_j c_j^{1/p} = 1.$$

In the case of **"arithmetic"** *self-similarity, when all* $\ln(1/c_j)$ *are mutually commensurable, there exists a function $\varphi \in C(\mathbb{R})$, bounded and separated from 0 such that*

$$\lambda_n \sim \frac{\varphi(\ln(n))}{n^p}, \quad n \to \infty.$$

Moreover, φ is $\frac{T}{p}$-periodic, where T is the greatest common divisor of $\ln(1/c_j)$, $j = 1, \dots, k$.

In the case of **"non-arithmetic"** *self-similarity, when at least one ratio* $\ln(c_i)/\ln(c_j)$ *is irrational, there exists a constant $M > 0$ such that*

$$\lambda_n \sim \frac{M^p}{n^p}, \quad n \to \infty.$$

Remarks. 1. The exponent p satisfies

$$p = 1 + \frac{2\ell - 1}{\gamma}, \qquad \beta \leqslant \gamma \leqslant \alpha, \tag{3}$$

where β and α are Hausdorff dimensions of the measure μ and of its support, correspondingly. Moreover, both inequalities in (3) are strict if $\beta < \alpha$.

2. The statement of Theorem 1 in the "arithmetic" case does not exclude that function φ is a constant, i.e. generally speaking λ_n can have classical power asymptotics as in "non-arithmetic" case.

We conjecture that it's not the case, i.e. $\varphi \neq const$ for any non-Lebesgue arithmetically self-similar measure μ. This conjecture was proved recently by Vladimirov and Sheipak in particular case for the second order operator $\mathcal{L}$ and the simplest Cantor measure μ. In general case this question remains open.

The idea of the proof:

If $\mathcal{L}$ is the simplest operator $(-1)^\ell d^{2\ell}/dx^{2\ell}$ with Dirichlet boundary conditions then by variational principle and self-similarity we reduce the relation for λ_n to the renewal equation. Then the well-known asymptotics for the solution of this equation gives us the asymptotics for λ_n.

In general case we apply a new variant of Weyl theorem which shows that the low-order terms do not influence on the asymptotics of λ_n.

Now we connect given asymptotic behavior of λ_n with the logarithmic s.b.a. for corresponding process. The non-arithmetic case gives pure power asymptotics considered by Nazarov and Nikitin (2004).

Theorem 2. *Let the eigenvalues λ_n from (1) have the form*

$$\lambda_n = \frac{\varphi(\ln(n))}{n^p},$$

where $p > 1$, and the positive function φ is uniformly continuous on $\mathbb{R}$, bounded and separated from 0.

Then, as $\varepsilon \to 0$,

$$\ln P\{||X||_\mu \le \varepsilon\} \sim -\varepsilon^{-\frac{2}{p-1}} \cdot \zeta(\ln(1/\varepsilon)),$$

where the positive function ζ expressed explicitely in terms of φ and p is bounded and separated from 0. Moreover, if the function φ is asymptotically $\frac{T}{p}$-periodic then the function ζ is asymptotically $\frac{T(p-1)}{2p}$-periodic.

The proof is based on the result of Lifshits (1997).

Remark. The order of logarithmic s.b.a. equals

$$-\frac{2}{p-1} = -\frac{2\gamma}{2\ell-1}.$$

The exponent γ introduced in (3) is called the spectral dimension of order $2\ell-1$ of the self-similar measure μ. Recall that if $\alpha = \beta$ it coincides with α and β and therefore it does not depend on $2\ell-1$. Otherwise $\gamma(t)$ is strictly increasing function, with

$$\lim_{t\to+0} \gamma(t) = \beta,$$

$$\lim_{t\to+\infty} \gamma(t) = \alpha.$$

Lifshits, Linde and Shi showed that the spectral dimension plays the key role in logarithmic s.b.a. in L_q-norms for all q.