Stochastic Differential Equations

with

Additive Fractional Noise:

Approximation at a Single Point

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SDEs with Additive Fractional Noise

$a : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$	drift coefficient,
$\sigma : [0,1] \longrightarrow \mathbb{R}$	diffusion coefficient,
$x_0 \in \mathbb{R}$	initial value,
$B^H(t), t \in [0,1]$	fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$.

(SDE)
$$\begin{aligned} dX(t) &= a(t, X(t))dt + \sigma(t)dB^H(t) \\ X(0) &= x_0 \end{aligned}$$

Pathwise Riemann-Stieltjes integral equation.

Lyons (1994); Lin (1995); Klingenhöfer, Zähle (1999); Mikosch, Norvaiša (2000);
Ruzmaikina (2000); Nualart, Răşcanu (2002); Nourdin (2004); ...

Existence and uniqueness under standard assumptions.

Numerical Schemes

Problem. Approximation of $X(1)$ based on evaluations of B^H at n points,

$$(*) \qquad B^H(t_1), B^H(t_2), \ldots, B^H(t_n).$$

Error criterion. Mean square error:

$$e_2(\widehat{X}(1)) = (\mathrm{E}\,|X(1) - \widehat{X}(1)|^2)^{1/2}.$$

In this talk:

- Analysis of equidistant Wagner-Platen-type approximation scheme.
- Comparison with **minimal error** for approximation schemes based on $(*)$:

$$e_2(n) = \inf\{e_2(\,\mathrm{E}(\,X(1)\,|\,B^H(t_i\,), i = 1, \ldots, n\,)\,) : 0 \le t_1 \le \ldots \le t_n \le 1\}.$$

Assumptions.

(A1) $a \in C^{2,3}([0,1] \times \mathbb{R})$ with bounded derivatives,

(A2) $\sigma \in C^2([0,1])$.

An Integration Problem

Special case.

$$a = 0,\ \sigma = \mathrm{id}: \quad X(1) = x_0 + \int_0^1 t\, dB^H(t).$$

Equidistant Euler approximation:

$$e_2(\widehat{X}_n^E(1)) \approx \frac{1}{2} \cdot n^{-1}.$$

Equidistant trapezoidal approximation:

$$e_2(\widehat{X}_n^T(1)) \approx \sqrt{|\zeta(-2H)|} \cdot n^{-1/2-H}.$$

Integration problems for B^H with deterministic weight functions:

- Stein (1995), Benhenni (1998): stationary processes with local behaviour like B^H.
- Istas (1997, 2003): multi-fractional Brownian motion, non-Gaussian processes.
- Ritter (2000).

Wagner-Platen-type Scheme

Equidistant discretization:

$$\begin{aligned} \Delta &:= 1/n, \\ t_j &:= j \cdot \Delta, \\ \Delta_j B^H &:= B^H(t_{j+1}) - B^H(t_j), \\ \bar\sigma(t_j) &:= (\sigma(t_j) + \sigma(t_{j+1}))/2. \end{aligned}$$

$$\begin{aligned} \widehat{X}_n^{WPt}(0) &= x_0, \\ \widehat{X}_n^{WPt}(t_{j+1}) &= \widehat{X}_n^{WPt}(t_j) + a(t_j, \widehat{X}_n^{WPt}(t_j)) \cdot \Delta + \bar\sigma(t_j) \cdot \Delta_j B^H \\ &\quad + \frac{1}{2} \cdot (a_t + a a_x)(t_j, \widehat{X}_n^{WPt}(t_j)) \cdot \Delta^2 \\ &\quad + \frac{1}{2} \cdot \bar\sigma(t_j) a_x(t_j, \widehat{X}_n^{WPt}(t_j)) \cdot \Delta_j B^H \cdot \Delta, \qquad j = 0, \ldots, n-1. \end{aligned}$$

Error: $e_2(\widehat{X}_n^{WPt}(1)) = (\mathrm{E}\,|X(1) - \widehat{X}_n^{WPt}(1)|^2)^{1/2}$.

Exact rate of convergence? Asymptotic constant?

Theorem 1. (N. (2005))

$$e_2(\widehat{X}_n^{WPt}(1)) \approx \sqrt{|\zeta(-2H)|} \cdot \left(\int_0^1 \mathrm{E}\,|\mathcal{M}'(t)|^2\,dt\right)^{1/2} \cdot n^{-1/2-H},$$

where

ζ : Riemann Zeta function,

$$\mathcal{M}(t) = \sigma(t)\exp\left(\int_t^1 a_x(\tau, X(\tau))\,d\tau\right), \quad t \in [0,1],$$

Malliavin derivative of $X(1)$.

Remarks.

- Related to integration problem for B^H with random weight $\mathcal{M}'$.
- Non-equidistant discretizations:
  - (i) same exact rate of convergence,
  - (ii) asymptotic constant depending on the discretization.

Lower Bounds

Minimal error.

$$e_2(n) = \inf\{e_2(\,\mathrm{E}(\,X(1)\,|\,B^H(t_i\,), i=1,\ldots,n\,)\,) : 0 \le t_1 \le \ldots \le t_n \le 1\}$$

Theorem 2. (N. (2005))

$$e_2(n) \asymp e_2(\widehat{X}_n^{WPt}(1)) \asymp n^{-1/2-H},$$

if $\mathrm{E}\,\mathcal{M}'(t) \neq 0$ for all $t \in [0,1]$.

Remarks.

- Wagner-Platen-type scheme order optimal.
- Additional assumption of Theorem 2 satisfied, e.g., for autonomous equations with monotone drift.
- Known for $H = 1/2$ (Itô-SDE): $e_2(n) \approx \frac{1}{\sqrt{12}} \cdot \left(\int_0^1 (\mathrm{E}\,|\mathcal{M}'(t)|^2)^{1/3}\right)^{3/2} \cdot n^{-1}$.

  Cameron, Clark (1980); Cambanis, Hu (1996); Müller-Gronbach (2004).

So far: equations with additive noise.

- Minimal error for approximation of $X(1)$: $\qquad e_2(n) \asymp n^{-1/2-H}$.
- Wagner-Platen-type scheme order optimal.
- Malliavin derivative of $X(1)$: key quantity for upper and lower bounds.

Equations with non-additive noise.

$$dX(t) = a(X(t))dt + \sigma(X(t))dB^H(t), \qquad X(0) = x_0.$$

Proposition 1. (N. (2005))

$$e_2(n) \asymp n^{-1/2-H},$$

if $a, \sigma \in C_b^3(\mathbb{R})$, $\inf_{x \in \mathbb{R}} \sigma(x) > 0$ and Malliavin derivative of $X(1)$ ”regular”.

Open Problem.

- Upper bound in Proposition 1: conditional expectation.
- Implementable optimal approximation schemes?

Further Results

In this talk: approximation of X at $t = 1$.

Global approximation of X on $[0, 1]$.

Mean square L^2- error: $\mathrm{e}_2(\widehat{X}) = (\mathrm{E} \int_0^1 |X(t) - \widehat{X}(t)|^2 \, dt)^{1/2}$.

Here:

$$\gamma_H \cdot \|\sigma\|_{1/(H+1/2)} \cdot n^{-H} \preceq \mathrm{e}_2(n) \preceq (\sqrt{7\pi^2}/6) \cdot \gamma_H \cdot \|\sigma\|_{1/(H+1/2)} \cdot n^{-H},$$

if $\sigma > 0$.

- Upper bound: Euler scheme with discretization related to local smoothness of the solution and piecewise linear interpolation (Seleznjev (2000)).
- Key for lower bound: strong asymptotics of the eigenvalues of the Karhunen-Loève expansion of B^H (Bronski (2003); Nazarov, Nikitin (2003); Luschgy, Pagès (2004)).

References.

Alòs, E. and Nualart, D. (2003) Stochastic integration with respect to the fractional Brownian motion. *Stoch. Stoch. Rep.*, **75**, 129-152.

Benhenni, K. (1998) Approximating integrals of stochastic processes: Extensions. *J. Appl. Prob.*, **35**, 843-855.

Bronski, J. C. (2003) Small ball constants and tight eigenvalue asymptotics for fractional Brownian motions. *J. Theor. Probab.*, **16**(1), 87-100.

Clark, J.M.C. and Cameron, R.J. (1980) The maximum rate of convergence of discrete approximations for stochastic differential equations. In Flemming, W.H. and Gorostiza, L.D. (eds.), *Lecture Notes in Control and Information sciences*, **25**, Springer, Berlin.

Cambanis, S. and Hu, Y. (1996) Exact convergence rate of the Euler-Maruyama scheme, with application to sampling design. *Stoch. Stoch. Rep.*, **59**, 211-240.

Duncan, T.E., Hu, Y. and Pasik-Duncan, B. (2000) Stochastic calculus for fractional Brownian motion. I: Theory. *SIAM J. Control Optim.*, **38**(2), 582-612.

Istas, J. (1997) Estimation d'intégrales de processus multi-fractionnaires. *C. R. Acad. Sci., Paris, Sér. I*, **324**(5), 565-568.

Istas, J. (2003) L^p-loss and limit distribution for predicting integrals of some non-Gaussian second order processes. *Stat. Inference Stoch. Process.*, **6**(3), 237-246.

Klingenhöfer, F. and Zähle, M. (1999) Ordinary differential equations with fractal noise. *Proc. Amer. Math. Soc.*, **127**(4), 1021-1028.

Lin, S.J. (1995) Stochastic analysis of fractional Brownian motions. *Stoch. Stoch. Rep.*, **55**, 121-140.

Luschgy, H. and Pagés, G. (2004) Sharp asymptotics of the functional quantization problem for Gaussian processes. *Ann. Probab.*, **32**(2), 1574-1599.

Lyons, T. (1994) Differential equations driven by rough signals (I): An extension of an inequality of L.C. Young. *Math. Res. Lett.*, **1**, 451-464.

Mikosch, T. and Norvaiša, R. (2000) Stochastic integral equations without probability. *Bernoulli*, **6**(3), 401 - 434.

Müller-Gronbach, T. (2004) Optimal pointwise approximation of SDEs based on Brownian motion at discrete points. *Ann. Appl. Probab.*, **14**(4), 1605-1642.

Nazarov, A.I. and Nikitin, Y.Y. (2003) Logarithmic L_2-small ball asymptotics for some fractional Gaussian processes. Working paper.

Nourdin, I. (2004) *Calcul stochastique généralisé et applications au mouvement brownien fractionnaire; Estimation non-paramétrique de la volatilité et test d'adéquation*. Thése de doctorat, Nancy.

Nualart, D. and Răşcanu, R. (2002) Differential equations driven by fractional Brownian motion. *Collect. Math.*, **53**(1), 55-81.

Ritter, K. (2000) *Average-Case Analysis of Numerical Problems*. Berlin: Springer.

Ruzmaikina, A.A. (2000) Stieltjes integrals of Hölder continuous functions with applications to fractional Brownian motion. *J. Stat. Phys.*, **100**, 1049-1069.

Seleznjev, O. (2000) Spline approximation of random processes and design problems. *J. Statist. Plann. Inference*, **84**, 249-262.

Stein, M.L. (1995) Predicting integrals of stochastic processes. *Ann. Appl. Probab.*, **5**(1), 158-170.

Zähle, M. (1998) Integration with respect to fractal functions and stochastic calculus. I. *Probab. Theory Relat. Fields*, **111**, 333-374.