Tensor products of operators with regular spectral asymptotics and small deviations of Gaussian fields.

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1. Introduction

Let $X(x)$, $x \in \Omega \subset \mathbb{R}^d$, be a Gaussian random field with zero mean and covariance function

$$G_X(x, y) = EX(x)X(y),\ x, y \in \Omega.$$

Let $\rho \geq 0$ be a summable weight on Ω. Put

$$||X||_\rho = (\int\limits_\Omega X^2(x)\rho(x)dx)^{1/2}.$$

Problem:

What is the behavior of $\mathbf{P}\{||X||_\rho \leq \varepsilon\}$ *as* $\varepsilon \to 0$*?*

Easier problem : *To describe as* $\varepsilon \to 0$

$$\ln \mathbf{P}\{||X||_\rho \leq \varepsilon\}$$

(logarithmic asymptotics)

According to Karhunen-Loève expansion, we have equality in distribution

$$||X||_\rho^2 = \int\limits_\Omega X^2(x)\rho(x)dx = \sum_{n=1}^{\infty} \lambda_n \xi_n^2,$$

where ξ_n, $n \in \mathbb{N}$, are independent standard normal r.v.'s, and $\lambda_n > 0$, $n \in \mathbb{N}$, $\sum\limits_n \lambda_n < \infty$ are the eigenvalues of the integral equation

$$\lambda f(x) = \int\limits_\Omega G_X(x,y)\sqrt{\rho(x)\rho(y)}f(y)dy, \quad x \in \Omega.$$

Thus we arrive to the problem of studying the asymptotic behavior as $\varepsilon \to 0$ of

$$\mathbf{P}\left\{\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \le \varepsilon^2\right\} \text{ or } \quad \ln \mathbf{P}\left\{\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \le \varepsilon^2\right\}.$$

We study small deviations of Gaussian fields X having the covariance of "tensor product" type.

Let $X(x)$, $Y(y)$ be zero mean Gaussian fields with covariances $G_X(x,u)$ and $G_Y(y,v)$. Consider the zero mean Gaussian function $Z(x,y)$ with the covariance

$$G_Z((x,y),(u,v)) = G_X(x,u)G_Y(y,v).$$

We write $Z = X \otimes Y$ and we call Z *the tensor product* of X and Y using the term *field-product*.

Classical examples of such fields are the Brownian sheet $(W \otimes W)$, the Brownian pillow ($B \otimes B$) and the Kiefer field ($W \otimes B$).

The generalization to the multivariate case

$$\mathbb{X}(x) = \bigotimes_{j=1}^{d} X_j$$

is straightforward.

First results on *logarithmic* L_2-small deviations of such Gaussian fields belong to Csáki (1982) and Li (1992).

They showed that the "marginal" fields can have the *pure power* spectral asymptotics while the spectral asymptotics for the field-product can contain the *logarithmic factor.*

We consider more general case. Assume that the marginal eigenvalues $\lambda_n^{(j)}$ have *regular behaviour* :

$$\lambda_n^{(j)} \sim \frac{\varphi_j(n)}{n^{p_j}}, \quad n \to \infty, \tag{1}$$

where $p_j > 1$ and φ_j are some *slowly varying functions* (SVFs), in particular, the powers of logarithms.

This is common in papers on spectral asymptotics, see Birman and Solomyak (2001), Dereich (2003) and Luschgy and Pagès (2002, 2004).

We develop an original approach. It uses new theorems on spectral asymptotics for tensor products of self-adjoint operators in Hilbert space.

Recently Fill and Torcaso (2004) obtained *exact* asymptotics of L_2-small deviations for the Brownian sheet and its integrated counterparts.

Their approach uses the Mellin transform and is entirely different. It is not clear if their approach makes possible obtaining small deviation results for a vast class of Gaussian fields-products studied in the present talk, when the eigenvalues satisfy (1).

Moreover, our results are applicable for the weighted L_2-norm in the case of continuous weights on the unit cube in $\mathbb{R}^d$. This seems to be beyond the reach of the method using Mellin transform.

2. Slowly varying functions

Recall that a positive function $\varphi(\tau)$, $\tau > 0$, is called a *slowly varying* function (SVF) at infinity if for any $c > 0$

$$\varphi(c\tau)/\varphi(\tau) \to 1 \qquad \text{as} \quad \tau \to +\infty.$$

We define the *Mellin convolution* of two SVFs φ and ψ as

$$(\varphi * \psi)(\tau) = \int_1^{\tau} \varphi(\sigma)\psi(\tau/\sigma)\frac{d\sigma}{\sigma} = h_{\varphi,\psi}(\tau) + h_{\psi,\varphi}(\tau),$$

where

$$h_{\varphi,\psi}(\tau) = \int_1^{\sqrt{\tau}} \varphi(\sigma)\psi(\tau/\sigma)\frac{d\sigma}{\sigma}.$$

We have proved some auxiliary lemmas concerning the properties of Mellin convolutions. In particular, $h_{\varphi,\psi}$ is also SVF.

3. Spectral asymptotics for tensor products of compact self-adjoint operators

Let $\mathcal{T} = \mathcal{T}^* \geq 0$ be a compact self-adjoint operator in a Hilbert space H; let $\lambda_n = \lambda_n(\mathcal{T})$ be its positive eigenvalues according to their multiplicity.

In spectral theory it is common to consider the counting function

$$\mathcal{N}(t) = \mathcal{N}(t, \mathcal{T}) = \#\{n : \lambda_n(\mathcal{T}) > t\}.$$

It is easily seen that

1. The relation $\mathcal{N}(t, \mathcal{T}) = O(t^{-1/p})$, $t \to +0$, is equivalent to $\lambda_n(\mathcal{T}) = O(n^{-p})$, $n \to \infty$.

2. The asymptotics

$$\mathcal{N}(t, \mathcal{T}) \sim \frac{\varphi(1/t)}{t^{1/p}}, \quad t \to +0,$$

(here $p > 0$ while φ is a SVF) is equivalent to the spectral asymptotics

$$\lambda_n(\mathcal{T}) \sim \frac{\psi(n)}{n^p}, \quad n \to \infty,$$

where ψ is also a SVF.

First suppose that the counting functions have distinct asymptotic exponents near zero.

Theorem 1. *Let operators $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ be such compact self-adjoint nonnegative operators in a Hilbert space that as $t \to +0$*

$$\mathcal{N}(t) \sim \frac{\varphi(1/t)}{t^{1/p}}, \quad \widetilde{\mathcal{N}}(t) = O(t^{-1/\widetilde{p}}), \; \widetilde{p} > p.$$

Then the operator $\mathcal{T} \otimes \widetilde{\mathcal{T}}$ satisfies the relation

$$\mathcal{N}_{\otimes}(t) \sim \frac{\varphi(1/t)}{t^{1/p}} \sum_n \lambda_n^{1/p}(\widetilde{\mathcal{T}}), \quad t \to +0.$$

In the case of equal asymptotic exponents for counting functions, the result is more refined.

Theorem 2. *Let $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ be such that*

$$\mathcal{N}(t) \sim a \cdot \frac{\varphi(1/t)}{t^{1/p}}, \; \widetilde{\mathcal{N}}(t) \sim \widetilde{a} \cdot \frac{\widetilde{\varphi}(1/t)}{t^{1/p}}, \; t \to +0.$$

1. *If*
$$\int_1^\infty \varphi(\tau)\frac{d\tau}{\tau} = \int_1^\infty \widetilde{\varphi}(\tau)\frac{d\tau}{\tau} = \infty,$$
then
$$\mathcal{N}_\otimes(t) \sim \frac{\chi(1/t)}{t^{1/p}}, \qquad t \to 0, \tag{2}$$
where
$$\chi(s) = \frac{a\widetilde{a}}{p}(\varphi * \widetilde{\varphi})(s).$$
2. *If*
$$\int_1^\infty \varphi(\tau)\frac{d\tau}{\tau} < \infty, \qquad \int_1^\infty \widetilde{\varphi}(\tau)\frac{d\tau}{\tau} = \infty,$$
and some mild regularity conditions on φ *are fulfilled then (2) holds true with*
$$\chi(s) = \widetilde{a}\widetilde{\varphi}(s)\sum_n \lambda_n^{1/p} + \frac{a\widetilde{a}}{p}h_{\widetilde{\varphi},\varphi}(s).$$
3. *If*
$$\int_1^\infty \varphi(\tau)\frac{d\tau}{\tau} < \infty, \qquad \int_1^\infty \widetilde{\varphi}(\tau)\frac{d\tau}{\tau} < \infty,$$
and the same regularity conditions on $\varphi, \widetilde{\varphi}$ *are fulfilled, then (2) holds true with*
$$\chi(s) = a\varphi(s)\sum_n \widetilde{\lambda}_n^{1/p} + \widetilde{a}\widetilde{\varphi}(s)\sum_n \lambda_n^{1/p}.$$

Example 1. Let, as $t \to +0$,

$$\mathcal{N}(t) \sim a \cdot \frac{\ln^{\varkappa_1}(1/t)}{t^{1/p}}, \quad \widetilde{\mathcal{N}}(t) \sim \widetilde{a} \cdot \frac{\ln^{\varkappa_2}(1/t)}{t^{1/p}}.$$

Then the asymptotics (2) holds true with

$$\chi(s) =$$

$$= \begin{cases} \frac{a\widetilde{a}}{p} \cdot \mathbf{B}(\varkappa_1 + 1, \varkappa_2 + 1) \cdot \ln^{\varkappa_1+\varkappa_2+1}(s), & \\ & \varkappa_1, \varkappa_2 > -1; \\ \frac{a\widetilde{a}}{p} \cdot \ln(\ln(s)) \cdot \ln^{\varkappa_2}(s), & \\ & \varkappa_1 = -1, \varkappa_2 > -1; \\ \frac{2a\widetilde{a}}{p} \cdot \ln(\ln(s)) \ln^{-1}(s), & \\ & \varkappa_1 = \varkappa_2 = -1; \\ \widetilde{a} \sum\limits_n \lambda_n^{1/p} \cdot \ln^{\varkappa_2}(s), & \\ & \varkappa_1 < -1, \varkappa_1 < \varkappa_2; \\ (a \sum\limits_n \widetilde{\lambda}_n^{1/p} + \widetilde{a} \sum\limits_n \lambda_n^{1/p}) \cdot \ln^{\varkappa_1}(s), & \\ & \varkappa_1 = \varkappa_2 < -1, \end{cases}$$

where $\mathbf{B}$ is the Euler beta-function.

The particular case $\varkappa_1 = 0$, $\varkappa_2 = 0$ is known from Papageorgiou and Wasilkowski (1990), see also Luschgy and Pagès (2004).

4. Small ball asymptotics and the rate of eigenvalues decrease

Consider the case when

$$\lambda_n \sim C\,\frac{\psi(n)}{n^p}\,, \quad n \to \infty, \tag{3}$$

where $p > 1$ while ψ is an SVF.

The case $\lambda_n = Cn^{-p}$, $p > 1$, is now classical one and is well-known.

Put $\Phi(n) = \frac{\psi(n)}{n^p}$ and consider for large u the equation

$$r = \frac{2^{1/p}\pi}{2p\sin\frac{\pi}{p}} \cdot \frac{\Phi^{-1}(1/u)}{u},$$

which yields

$$u(r) \sim r^{-\frac{p}{p-1}}\zeta(1/r), \quad r \to 0, \tag{4}$$

where ζ is some SVF.

Theorem 3. *Let the eigenvalues λ_n have the asymptotics (3). Then, as $\varepsilon \to 0$,*

$$\ln \mathbf{P}\{||X||_\rho \le \varepsilon\} \sim -(p-1)C^{\frac{1}{p-1}}\varepsilon^{-\frac{2}{p-1}}\zeta(\varepsilon^{-2}),$$

where the SVF ζ is defined in (4).

Example 2. *Let the eigenvalues* λ_n *satisfy*

$$\lambda_n \sim C\frac{\ln^{\sigma}(n+1)}{n^p}, \quad \sigma \in \mathbb{R},\ p>1,\ C>0.$$

Then

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{2}{p-1}} \ln^{-\frac{\sigma}{p-1}}(1/\varepsilon) \cdot \ln \mathbf{P}\{||X||_\rho \le \varepsilon\} =$$

$$= -C^{\frac{1}{p-1}} \left(\frac{p-1}{2}\right)^{1-\frac{\sigma}{p-1}} \left(\frac{\pi}{p \sin\frac{\pi}{p}}\right)^{\frac{p}{p-1}}.$$

Let examine briefly the special case $p = 1$. In this case we *cannot obtain* explicitly the logarithmic asymptotics. We give only one example of weaker (*double-logarithmic*) asymptotics.

Theorem 4. *Suppose that as* $n \to \infty$

$$\lambda_n \sim \frac{C}{n \ln^{\sigma}(n+1)}, \quad \sigma > 1,\ C > 0.$$

Then

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{2}{\sigma-1}} \cdot \ln\left|\ln \mathbf{P}\{||X||_\rho \le \varepsilon\}\right| = \left(\frac{C}{\sigma-1}\right)^{\frac{1}{\sigma-1}}.$$

5. Fields-products with equal marginal exponents

We deal with the fields of the form

$$X(x) = \bigotimes_{j=1}^{d} X_j(x_j),\ x = (x_1, \dots, x_d) \in [0,1]^d,$$

where X_j are independent zero mean Gaussian fields with *equal* marginal exponents p_j in the asymptotic representation (1). Also we consider weight functions $\rho \in \mathcal{C}([0,1]^d)$.

Consider the fractional Brownian sheet $\mathbb{W}_d^H(x)$. Its marginal processes (factors) X_j are the fractional Brownian motions W_j^H ($H \in]0,1[$ is the Hurst parameter), and the covariance function is

$$G_{\mathbb{W}_d^H}(x,y) = \frac{1}{2^d} \prod_{j=1}^{d} (x_j^{2H} + y_j^{2H} - |x_j - y_j|^{2H}).$$

Theorem 5. *Let $0 < H < 1$ and let $\rho \in \mathcal{C}([0,1]^d)$. Denote $J_h = \int_{[0,1]^d} \rho(x)^{1/h} dx$. Then*

$$\lim_{\varepsilon \to 0} \varepsilon^{1/H} \ln^{-(d-1)\frac{2H+1}{2H}}(1/\varepsilon) \cdot \ln \mathbf{P}\{||\mathbb{W}_d^H||_\rho \le \varepsilon\} =$$

$$= -\frac{J_{2H+1} H^{2-d}}{(d-1)!\pi^{d-1}(2H+1)\sin\left(\frac{\pi}{2H+1}\right)} \cdot$$

$$\left(\frac{J_{2H+1}\big(\Gamma(2H+1)\sin(\pi H)\big)^d}{(d-1)!(\pi H)^{d-1}(2H+1)\sin\left(\frac{\pi}{2H+1}\right)} \right)^{\frac{1}{2H}}.$$

Remark. This statement in the case $\rho \equiv 1$ was obtained by Luschgy and Pagès (2004).

Similar result holds for the *isotropically integrated* fractional Brownian sheet

$$(\mathbb{W}_d^H)_m(x) = \bigotimes_{j=1}^{d} (W^H)_m(x_j),$$

where

$$(W^H)_m(t) \equiv (W^H)_m^{[\beta_1, \ldots, \beta_m]}(t) =$$
$$= (-1)^{\beta_1 + \ldots + \beta_m} \underbrace{\int_{\beta_m}^{t} \ldots \int_{\beta_1}^{t_1}}_{m} W^H(s)\, ds\, dt_1 \ldots$$

(here any β_k equals either zero or one, $t \in [0, 1]$; for various j the multi-indices $[\beta_1, \ldots, \beta_m]$ can differ), for the fractional Ornstein – Uhlenbeck sheet and for the isotropically integrated fractional OU sheet.

6. Fields-products with distinct marginal exponents

Now let the marginal processes of fields-products have *different* power rates of eigenvalues decrease. We call X_j a *slow factor* if it has the least exponent p_j in (1) and call it a *fast factor* otherwise.

From Theorem 1 it follows that the rate of spectral asymptotics for the field-product is determined only by slow factors but the constant requires precise information on the fast factors spectra.

Theorem 6. *Let $\lambda_n^{(X)}$ and $\lambda_n^{(Y)}$ be the eigenvalues corresponding to the zero mean Gaussian processes X and Y on $[0,1]$. Suppose that*

$$\lambda_n^{(X)} \sim C\frac{\psi(n)}{n^p},\ \lambda_n^{(Y)} = O(n^{-\widetilde{p}}), n \to \infty,$$

with $\widetilde{p} > p$. Then, as $\varepsilon \to 0$,

$$\ln \mathbf{P}\{||\sqrt{\rho} X \otimes Y|| \le \varepsilon\} \sim$$

$$\sim \left(\sum_n (\lambda_n^{(Y)})^{1/p}\right)^{\frac{p}{p-1}} \ln \mathbf{P}\{||X||_\rho \le \varepsilon\}. \quad (5)$$

The result for the tensor product with arbitrary number of fast factors looks similarly.

Unfortunately, the trace-type sums in (5) usually cannot be written out *explicitly*.

(The eigenvalues of Anderson – Darling process , see Anderson and Darling(1952),

$$B(t)/\sqrt{t(1-t)}, t \in [0,1],$$

are $\lambda_n = (n^2 + n)^{-1}$, $n \geq 1$. Even in this simple example the trace-type sums have no explicit expressions).

Consider some cases when these sums can be expressed in terms of Riemann ζ-function. For any Gaussian process $X(t)$, $t \in [0,1]$, we introduce the *centered process* $\overline{X}(t) = X(t) - \int_0^1 X(s)\,ds$ and the *on-line centered process*

$$\widehat{X}(t) = X(t) - t^{-1}\int_0^t X(s)ds.$$

Theorem 7. *Let the eigenvalues of X satisfy (1). If $p \in]1,2[$, then, as $\varepsilon \to 0$*

$$\ln \mathbf{P}\{||\sqrt{\rho}X \otimes W|| \le \varepsilon\}$$
$$\sim (2^{\frac{2}{p}} - 1)^{\frac{p}{p-1}} \pi^{-\frac{2}{p-1}} \zeta^{\frac{p}{p-1}}(2/p) \ln \mathbf{P}\{||X||_\rho \le \varepsilon\};$$

$$\ln \mathbf{P}\{||\sqrt{\rho}X \otimes \overline{W}|| \le \varepsilon\}$$
$$\sim \pi^{-\frac{2}{p-1}} \zeta^{\frac{p}{p-1}}(2/p) \ln \mathbf{P}\{||X||_\rho \le \varepsilon\};$$

$$\ln \mathbf{P}\{||\sqrt{\rho}X \otimes \widehat{W}|| \le \varepsilon\}$$
$$\sim \pi^{-\frac{2}{p-1}} \zeta^{\frac{p}{p-1}}(2/p) \ln \mathbf{P}\{||X||_\rho \le \varepsilon\}.$$

Similar results hold when replacing W by B and by some integrated processes linked to W and B.

Example 3. Consider the field $W^H \otimes B \otimes (\overline{W})_1$, $0 < H \le 1/2$. If $H < 1/2$, then, as $\varepsilon \to 0$,

$$\varepsilon^{1/H} \ln \mathbf{P}\{||\sqrt{\rho} W^H \otimes B \otimes (\overline{W})_1|| \le \varepsilon\} \sim$$
$$\sim -\pi^{-\frac{3}{H}} \zeta^{\frac{2H+1}{2H}} \left(\frac{4}{2H+1}\right) \zeta^{\frac{2H+1}{2H}} \left(\frac{2}{2H+1}\right) \cdot$$
$$\cdot \; \frac{J_{2H+1} H}{(2H+1)\sin\left(\frac{\pi}{2H+1}\right)} \left(\frac{J_{2H+1}\Gamma(2H+1)\sin(\pi H)}{(2H+1)\sin\left(\frac{\pi}{2H+1}\right)} \right)^{\frac{1}{2H}}.$$

If $H = 1/2$, then, as $\varepsilon \to 0$,

$$\ln \mathbf{P}\{||\sqrt{\rho} W \otimes B \otimes (\overline{W})_1|| \le \varepsilon\} \sim$$
$$\sim \pi^{-4} \zeta^2(2) \ln \mathbf{P}\{||\sqrt{\rho} W \otimes B|| \le \varepsilon\} \sim$$
$$\sim -\frac{\left(\int_0^1 \rho(t)^{1/2} dt\right)^2}{72\pi^2} \varepsilon^{-2} \ln^2(1/\varepsilon).$$

Even for $\rho \equiv 1$ when our field has the exactly known spectrum

$$\lambda_{jk\ell} = \pi^{-8}(j - 1/2)^{-2} k^{-2} \ell^{-4}, \quad j, k, \ell \ge 1,$$

the result is new and is not covered by the paper of Li (1992).

7. Additional results

Another type of Gaussian fields we consider is the class of *additive processes*. For shortness we call it *fields-sums*. Let $X_j(x_j)$, $1 \leq j \leq d$, $x_j \in [0,1]^{\ell_j}$, be independent random functions. The field-sum $\mathbb{X}(x)$ is defined as

$$\mathbb{X}(x) = \sum_{j=1}^{d} X_j(x_j), x = (x_1, ..., x_d) \in [0,1]^D,$$

where $D = \sum_{j=1}^{d} \ell_j$, with the covariance

$$G_{\mathbb{X}}(x,y) = \sum_{j=1}^{d} G_{X_j}(x_j, y_j).$$

Such fields appear, e.g., in the theory of intersections of Brownian processes. Chen and Li (2003) proved some small deviation results with respect to supremum norm. The results in the L_2-norm we obtained appear for the first time.

We study the sum of tensor products of compact operators and prove various theorems on the asymptotic behavior of counting function for the sum

$$\mathcal{N}(t, \mathcal{S}_1 + \mathcal{S}_2)$$

in terms of asymptotic behavior of counting functions $\mathcal{N}(t, \mathcal{S}_1)$ and $\mathcal{N}(t, \mathcal{S}_2)$. Then we pass to the small deviations of fields-sums.

Example 4. *Let $H \in (0, 1/2)$. Then, as $\varepsilon \to 0$*

$$\ln \mathbf{P}\left\{\left\|B^H \otimes W + \overline{B} \otimes W^H\right\| \le \varepsilon\right\} \sim$$
$$\sim C(H)\, \zeta^{\frac{2H+1}{2H}} \left(\tfrac{2}{2H+1}\right) \ln \mathbf{P}\left\{\left\|W^H\right\| \le \varepsilon\right\}.$$

The results of previous sections can be also generalized to the vector-valued zero mean Gaussian fields $X(x) \in \mathbb{R}^\ell$, $x \in [0,1]^d$.

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