EXACT SMALL DEVIATION ASYMPTOTICS IN L_2-NORM FOR SOME WEIGHTED GAUSSIAN PROCESSES

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Let $X(t)$, $0 \leqslant t \leqslant 1$, be a zero mean Gaussian process and let ψ be a non-negative function on [0,1]. Denote

$$\|X\|_{2,\psi} = \left(\int_0^1 X^2(t)\psi(t)dt\right)^{1/2}.$$

The problem is to define the exact behavior of $\mathbf{P}\left\{\|X\|_{2,\psi} \leqslant \varepsilon\right\}$ as $\varepsilon \to 0$.

The case of general weight ψ can be reduced to the case $\psi \equiv 1$ by replacing X by the Gaussian process $X\sqrt{\psi}$. However, it is more convenient to consider the general case.

Theoretically the problem of small deviation asymptotics was solved by Sytaya (1974), but in an implicit way.

Simplification of the expression for the small ball probability for various classes of processes was made in the works of
Zolotarev (1979),
Dudley, Hoffmann-Jørgensen, Shepp (1979),
Ibragimov (1979),
Csáki (1982),
Li (1992),
Dunker, Lifshits, Linde (1998),
Chen and Li (2001),
Gao, Hannig, Lee, Torcaso (2003), and others.

Nazarov and Nikitin (2003) developed a new approach. Their method enabled to obtain the small deviation asymptotics in L_2-norm for Gaussian process X under condition that covariance G_X is the Green function for the self-adjoint differential operator from a rather wide class.

We extend this result to the case of weighted processes.

Theorem. Let the covariance $G_X(t,s)$ of zero mean Gaussian process $X(t)$, $0 \leqslant t \leqslant 1$, be the Green function for the self-adjoint positively definite operator L of order 2ℓ

$$Ly \equiv (-1)^\ell y^{(2\ell)} + \left(p_{\ell-1}y^{(\ell-1)}\right)^{(\ell-1)} + \ldots + p_0 y,$$
$$p_0, \ldots, p_{\ell-2} \in L_1(0,1), \quad p_{\ell-1} \in L_\infty(0,1),$$

with "separated" boundary conditions: ℓ boundary conditions (denote by $k_1, \ldots, k_\ell$ their orders) contain the values of derivatives of y only at the endpoint zero while ℓ ones (of order $k'_1, \ldots, k'_\ell$) contain them only at the endpoint one. Let

$$0 \leqslant k_1 < \ldots < k_\ell \leqslant 2\ell - 1,$$
$$0 \leqslant k'_1 < \ldots < k'_\ell \leqslant 2\ell - 1,$$
$$\varkappa \equiv \sum_{j=1}^{\ell} (k_j + k'_j) < 2\ell^2.$$

Let the weight function $\psi \in W_\infty^\ell(0,1)$ and $\psi(x) > 0$, $x \in [0,1]$.

Then, as $\varepsilon \to 0$,

$$\mathbf{P}\left\{\|X\|_{2,\psi} \leqslant \varepsilon\right\} \sim$$
$$\sim \mathcal{C}\,\varepsilon^{\gamma} \exp\left(-\frac{2\ell-1}{2}\left(\frac{\vartheta_\ell}{2\ell\sin\frac{\pi}{2\ell}}\right)^{\frac{2\ell}{2\ell-1}} \varepsilon^{-\frac{2}{2\ell-1}}\right).$$

Here

$$\gamma = -\ell + \frac{\varkappa+1}{2\ell-1}, \qquad \vartheta_\ell = \int_0^1 \psi^{\frac{1}{2\ell}}(x)\,dx,$$
$$\mathcal{C} = C_{\mathrm{dist}}\,\frac{(2\pi)^{\frac{\ell}{2}}(\frac{\pi}{\vartheta_\ell})^{\ell\gamma}(\sin\frac{\pi}{2\ell})^{\frac{1+\gamma}{2}}}{(2\ell-1)^{\frac{1}{2}}(\frac{\pi}{2\ell})^{1+\frac{\gamma}{2}}\Gamma^{\ell}(\ell-\frac{\varkappa}{2\ell})},$$

while C_{dist} is the so-called *distortion* constant

$$C_{\mathrm{dist}} = \prod_{n=1}^{\infty} \frac{\mu_n^{1/2}}{\left(\pi/\vartheta_\ell\cdot\left[n+\ell-1-\frac{\varkappa}{2\ell}\right]\right)^{\ell}},$$

and μ_n are the eigenvalues of BVP

$$Ly = \mu\psi y \quad + \quad \text{boundary conditions.}$$

Processes with eigenfunctions connected with special functions

Though we have written "explicit" expression for C_{dist}, it is not easy to evaluate this constant in general case. However, when the eigenfunctions can be expressed in terms of elementary or special functions, there exist explicit formulas for the distortion constants.

Well-known examples for $\psi \equiv 1$: $X = B$ (Brownian bridge); $X = W$ (Wiener process). For these processes $C_{\text{dist}} = 1$.

More examples for $\psi \equiv 1$: $X = U$ (Ornstein – Uhlenbeck process, OU); m-times integrated processes W_m, B_m, U_m. In these examples the eigenfunctions are connected with trigonometric functions.

$X = B$, $\psi(t) = \frac{1}{t(1-t)}$ (Anderson – Darling process): the eigenfunctions are expressed via Jacobi polynomials.

$X = W$, $X = B$; $\psi(t) = t^{\beta}$, $\beta > -2$ or $\psi(t) = e^{qt}$: the eigenfunctions are expressed via Bessel functions.

The distortion constants for these processes are evaluated in Nazarov (J. Math. Sci., 2003). Partial results were obtained also by Deheuvels and Martynov (Progr. Probab., 2003) and Gao, Hannig, Lee, Torcaso (EJP, 2003).

We calculate C_{dist} for a number of weighted processes.

Let $u \leqslant 1$. We denote by $W_{(u)}(t)$ the zero mean Gaussian process $W(t) - utW(1)$, $0 \leqslant t \leqslant 1$. Its covariance is $G_{W_{(u)}}(t,s) = s \wedge t - (2u - u^2)st$.

NB: $W_{(1)} = B$, $W_{(0)} = W$.

Denote by $U_{(\alpha)}$ the stationary OU process, i.e., the centered Gaussian process with covariance $G_{U_{(\alpha)}}(t,s) = e^{-\alpha|t-s|}\Big/(2\alpha)$.

Denote by $\mathring{U}_{(\alpha)}$ the OU process starting at 0, i.e., the centered Gaussian process with covariance $G_{\mathring{U}_{(\alpha)}}(t,s) = \left(e^{-\alpha|t-s|} - e^{-\alpha(t+s)}\right)\Big/(2\alpha)$.

NB: $\mathring{U}_{(0)} = W$.

Denote by $\widehat{W}$ the on-line centered Wiener process, $\widehat{W}(t) = W(t) - \frac{1}{t}\int_0^t W(s)\,ds$.

Table 1. Some weighted processes with eigenfunctions expressed via trigonometric functions

X	$\psi(t)$
$W_{(u)}$, $u < 1$	$(t + a)^{-2}$, $a > 0$
$W_{(1)} = B$	$(t + a)^{-2}$, $a > 0$
$W_{(u)}$, $u < 1$	$(a^2 + t^2)^{-2}$, $a \neq 0$
$W_{(1)} = B$	$(a^2 + t^2)^{-2}$, $a \neq 0$
$W_{(u)}$, $u < 1$	$(a^2 - t^2)^{-2}$, $\|a\| > 1$
$W_{(1)} = B$	$(a^2 - t^2)^{-2}$, $\|a\| > 1$

Table 2. Some weighted processes with eigenfunctions connected with Bessel functions

X	$\psi(t)$
$W_{(u)}$, $u < 1$	$(t + a)^{\beta}$, $a > 0$, $\beta \neq -2$
$W_{(1)} = B$	$(t + a)^{\beta}$, $a > 0$, $\beta \neq -2$
$\mathring{U}_{(\alpha)}$, $\alpha \in \mathbb{R}$	e^{qt}, $q \in \mathbb{R}$
$U_{(\alpha)}$, $\alpha > 0$	e^{qt}, $q \in \mathbb{R}$
$\widehat{W}$	t^{β}, $\beta > -2$

Two particular examples

Let $X = \mathring{U}_{(\alpha)}$ and $\psi(t) = e^{2qt}$. If $q \neq 0$, then, as $\varepsilon \to 0$,

$$\mathbf{P}\left\{\|\mathring{U}_{(\alpha)}\|_{2,\psi} \leqslant \varepsilon\right\} \sim \\ \sim \frac{e^{\alpha/2}}{e^{q/4}} \frac{4q}{\sqrt{\pi}(e^q - 1)} \varepsilon \exp\left(-\frac{(e^q-1)^2}{8q^2}\varepsilon^{-2}\right).$$

Let $X = W_{(u)}$ and $\psi(t) = (t+a)^{-2}$. If $u < 1$ and $a > 0$, then, as $\varepsilon \to 0$,

$$\mathbf{P}\left\{\|W_{(u)}\|_{2,\psi} \leqslant \varepsilon\right\} \sim \\ \sim \frac{4a^{-\frac{1}{4}}(a+1)^{\frac{1}{4}}}{(1-u)\pi^{\frac{1}{2}} \ln\frac{a+1}{a}} \varepsilon \exp\left(-\frac{\left(\ln\frac{a+1}{a}\right)^2}{8}\varepsilon^{-2}\right).$$