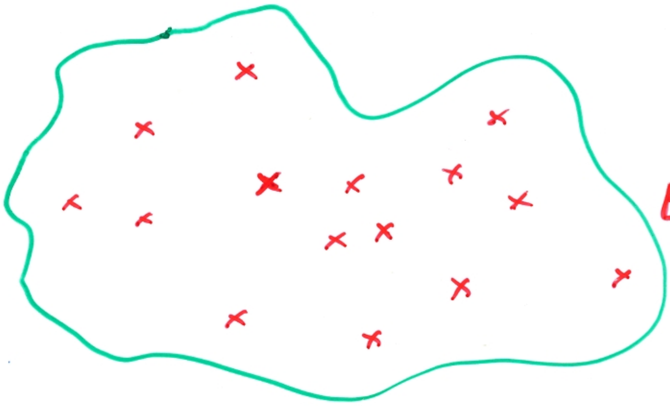
High resolution quantization and entropy coding for fractional Brownian motion

Michael Scheutzow, TU Berlin

(joint work with Steffen Dereich, TU Berlin)

(E,d) metric space, μ p.m. on (E,d), $N \in \mathbb{N}$



E

$$D^{(q)}(\log N \mid p) = \inf_{|A| \le N} \left(\int_E (d(x,A))^p \, d\mu(x) \right)^{1/p}$$

quantization error

$$= \inf_{\substack{\pi : E \to E \\ |\pi(E)| \le N}} \| d(Y, \pi(Y)) \|_p$$

$\mathcal{L}(Y) = \mu$

π : "(coding) strategy"

$r = \log N$ "rate" (= number of "nats")

<u>Ref.</u>: Graf-Luschgy, Dereich, Luschgy-Pagès, ...

$$D^{(e)}(r\,|\,p) = \inf_{\substack{\pi : E \to E \\ \mathbb{H}(\pi(Y)) \leq r}} \| d(Y, \pi(Y)) \|_p \qquad (p \in (0,\infty])$$

$$\mathbb{H}(Z) = -\sum_z p_z \log p_z \qquad \text{entropy of } Z$$

Clearly,

$$D^{(e)}(r\,|\,p) \leq D^{(q)}(r\,|\,p)$$

how close ?

Small ball function $\leftrightarrow$ quantization

$(E, \|\cdot\|)$ separable Banach space, μ centered Gaussian m.,

$\mu \neq \delta_0$

$$\phi(\varepsilon) := -\log \mu(B(0,\varepsilon)), \quad \varepsilon > 0$$

small ball function

e.g $\phi(\varepsilon) \sim \frac{c_H}{\varepsilon^{1/H}}$ for FBM with Hurst index $H \in (0,1)$

$E = L^2[0,1], C[0,1]$

Proposition (DFMS, 03)

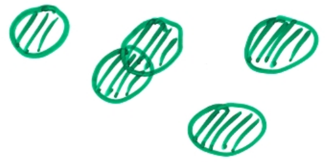
a) For all $p, \gamma > 0$

$$D^{(q)}(r|p) \geq (1-e^{-\gamma})^{1/p}\, \phi^{-1}(r+\gamma)$$

b) If ϕ is r.v. at 0

$$D^{(q)}(r|p) \gtrsim \phi^{-1}(r)$$

Proof:



$\leftarrow \lfloor e^r \rfloor$ balls of radius ε_r with centers $y_1, y_2, \dots$

$$\mu(\quad) \leq \sum_{i=1}^{\lfloor e^r \rfloor} \mu(B(y_i, \varepsilon_r)) \leq e^r \mu(B(0,\varepsilon_r))$$

Anderson's inequality

$$\Rightarrow D^{(q)}(r|p) \geq \Big(\varepsilon_r^p \big(1 - e^r \underbrace{\mu(B(0,\varepsilon_r))}_{= e^{-\phi(\varepsilon_r)}}\big)\Big)^{1/p}$$

$$= \phi^{-1}(r+\gamma)\,(1-e^{-\gamma})^{1/p}$$

$\varepsilon_r = \phi^{-1}(r+\gamma)$

$\square$

Now:

$E = C[0,1]$, $d(f,g) = \|f-g\| := \sup_{t\in[0,1]} |f(t)-g(t)|$

X FBM on $[0,1]$

Known (DFMS, Dereich's thesis): $D^{(q)}(r|p) \approx D^{(e)}(r|p) \approx r^{-H}$ $(\approx \phi^{-1}(r))$
$p \in (0,\infty)$

Theorem: There exists $\kappa \in (0,\infty)$ s.t. for all $p_1 \in (0,\infty]$, $p_2 \in (0,\infty)$

$$\lim_{r\to\infty} r^H D^{(e)}(r|p_1) = \lim_{r\to\infty} r^H D^{(q)}(r|p_2) = \kappa .$$

Lemma 1: $D^{(e)}(r|\infty) \approx r^{-H}$

Idea of proof: $\underset{\text{original}}{X}, \underbrace{X_1, X_2, \dots}_{\text{"random codebook"}}$ i.i.d. copies of FBM on $[0,1]$

$$T^{(r)}(X(\omega)) := \inf\{n \in \mathbb{N} : \|X(\omega) - X_n(\omega)\| \leq r^{-H}\}$$

$$\pi^{(r)}(X(\omega)) := X_{T^{(r)}(\omega)}(\omega)$$

Clearly $\|\pi^{(r)}(X) - X\| \leq r^{-H}$ a.s.

Given $X(\omega)$, $T^{(r)}$ has geometric law w.p. $P(\|X-X_1\| \leq r^{-H} | X)$

$$\Rightarrow \mathbb{E} \log T^{(r)} = \mathbb{E}(\mathbb{E}(\log T^{(r)}|X)) \underset{\text{Jensen}}{\leq} \mathbb{E}\Big(\underbrace{\log \frac{1}{P(\|X-X_1\| \leq r^{-H}|X)}}_{\overset{\text{Dereich, Lifshits}}{\approx}\ \phi^{-1}(r^{-H}) \approx r}\Big) \lesssim c \cdot r$$

There ex. deterministic $x_1, x_2, \dots \in C[0,1]$ s.t. $\mathbb{E} \log \tilde{T}^{(r)} \lesssim c \cdot r$

$$\Rightarrow \mathbb{H}(\tilde{\pi}^{(r)}(X)) \leq \mathbb{H}(\tilde{T}^{(r)}(X)) \underset{\text{easy!}}{\leq} \log \frac{\pi^2}{6} + 2\,\mathbb{E} \log \tilde{T}^{(r)} \lesssim 2cr$$

<u>Lemma 2:</u> $\varkappa := \lim_{r\to\infty} r^H D^{(e)}(r|\infty) \in (0,\infty)$ exists.

Idea of proof: Lemma 1 + self-similarity of FBM

<u>Lemma 3:</u> There exist strategies $\pi^{(r)}$ s.t.

$\|X - \pi^{(r)}(X)\| \leq \frac{\varkappa}{r^H}$ a.s. and prob. weights $p_\cdot^{(r)}$ on the range of $\pi^{(r)}$ s.t. $-\log p^{(r)}_{\pi^{(r)}(X)} \lesssim r$ in prob.

<u>Consequence of Lemma 3:</u>

$$\mathbb{P}\left(-\log p^{(r)}_{\pi^{(r)}(X)} \leq (1+\varepsilon)r\right) \geq 1-\varepsilon \quad \text{for } r \geq r(\varepsilon)$$

$$\| $$

$$\mathbb{P}\left(p^{(r)}_{\pi^{(r)}(X)} \geq e^{-(1+\varepsilon)r}\right)$$

$\Rightarrow$ there exists a "typical" set A_r, $|A_r| \leq e^{(1+\varepsilon)r}$ s.t.

$$\mathbb{P}(\pi^{(r)}(X) \in A_r) \geq 1-\varepsilon \quad \text{for} \quad r \geq r(\varepsilon)$$

Define quantitation function $\tilde{\pi}^{(r)}: E \to E$:

$$\tilde{\pi}^{(r)}(f) = \begin{cases} \pi^{(r)}(f) & \text{if } \pi^{(r)}(f) \in A_r \\ 0 & \text{otherwise} \end{cases}$$

$\rightsquigarrow$ $r^H D^{(q)}(r|p) \xrightarrow[r\to\infty]{} \varkappa$ for all $p \in (0,\infty)$

"General" Lemma 4: Assume $f:[0,\infty)\to(0,\infty)$ is decreasing and convex and satisfies

$$\limsup_{r\to\infty} \frac{-r\frac{d^+}{dr}f(r)}{f(r)} < \infty .$$

e.g. $f(r)=r^{\alpha}$ $(\alpha<0)$

$f(r)=e^{-cr}$ $(c>0)$

Suppose that, for some $0<p_1<p_2$

$$D^{(g)}(r+\log 2\,|\,p_1) \sim D^{(g)}(r\,|\,p_2) \gtrsim f(r) .$$

Then for any $p>0$

$$D^{(e)}(r|p) \gtrsim f(r).$$

NB: "$\log 2$" cannot be dropped!

Lemma 4 $\Rightarrow$ Theorem $(f(r)=K\cdot r^{-H})$