Shifted small ball problem for jump processes, in particular, for symmetric stable processes

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Consider a symmetric α-stable process X_α of index $\alpha \in (1,2)$, namely, the Lévy process determined by the centered generating triplet $(0, 0, \Lambda_\alpha(d\ell))$, where the Lévy measure is given by

$$\Lambda_\alpha(d\ell) = \frac{1}{|\ell|^{1+\alpha}} d\ell.$$

By X_α^γ denote the part of the process X_α with jumps eliminated out of the level γ, i.e., this is the Lévy process determined by the centered generating triplet $(0, 0, 1_{|\ell|\le\gamma}\Lambda_\alpha(d\ell))$. Clearly that for any $\rho > 0$ the following equality in law holds

$$\frac{X_\alpha^\gamma(\rho\,\cdot)}{\rho^{1/\alpha}} \overset{d}{=} X_\alpha^{\gamma\cdot\rho^{-1/\alpha}}(\cdot).$$

By $\|\cdot\|$ denote sup-norm on $B(0,1)$ and consider an absolutely continuous function f on $[0,1]$ such that $\int_0^1 f'^2 dt < \infty$ and there exists a modification of f' of finite total variation ($Var f' < \infty$).

Theorem 1. *If the parameters λ, r satisfy the conditions*

(1) $\lambda \to \infty$, $r/\lambda \to 0$,

(2) $\widetilde{\gamma}\lambda^{1/(\alpha-1)} \to 0$

we have

$$\begin{aligned} \mathbf{P}\{\|X_\alpha^{\widetilde{\gamma}} - \lambda f\| < r\} \le \exp\Big\{ &-\frac{2-\alpha}{4}\lambda^2\widetilde{\gamma}^{-(2-\alpha)}\int_0^1 f'^2 dt + \\ &+\frac{2-\alpha}{2} r\lambda\widetilde{\gamma}^{-(2-\alpha)}(|f'(1)| + \mathrm{Var} f')(1+o(1)) + \\ &-\frac{(2-\alpha)^4}{6\cdot 16(4-\alpha)}\cdot\lambda^4\widetilde{\gamma}^{3\alpha-4}\left(\int_0^1 f'^4(t)\,dt + o(1)\right) - \\ &-\frac{\pi^2}{4(2-\alpha)r^2}\widetilde{\gamma}^{2-\alpha}(1+o(1))\Big\}, \end{aligned}$$

$$
\begin{aligned}
\mathbf{P}\{\|X_\alpha^{\widetilde{\gamma}} - \lambda f\| < r\} \geq \exp\Big\{ &- \frac{2-\alpha}{4}\lambda^2 \widetilde{\gamma}^{\alpha-2} \int_0^1 f'^2 dt + \\
&+ (2\delta - 1)\frac{2-\alpha}{2} r\lambda\widetilde{\gamma}^{\alpha-2} \left(|f'(1)| + \mathrm{Var} f'\right)(1+o(1)) + \\
&- \frac{(2-\alpha)^4}{6\cdot 16(4-\alpha)} \cdot \lambda^4 \widetilde{\gamma}^{3\alpha-4} \left(\int_0^1 f'^4(t)\, dt + o(1)\right) - \\
&- \frac{\pi^2}{4(2-\alpha)(1-\delta)^2 r^2} \widetilde{\gamma}^{2-\alpha}(1+o(1)) \Big\},
\end{aligned}
$$

where $\delta \in (0,1)$.

We are also interested in the law of iterated logarithm for the introduced processes. The LIL for X_α itself doesn't exist because of infinite jumps. Nevertheless, Griffin (1985) has established a LIL result for symmetric α-stable processes without large jumps.

Fact (LIL for truncated symmetric α-stable process) *Given* $\gamma = (n/\log\log n)^{1/\alpha}$, *the following holds*

$$
\limsup_{n\to\infty} \frac{|X_\alpha^\gamma(n)|}{(n/\log\log n)^{1/\alpha} \log\log n} = c \quad a.s.
$$

where $0 < c < \infty$.

The exact form of the corresponding functional LIL is not known. But using the previous small deviation result we can obtain some information. Namely, applying Theorem 1 with parameters

$$
\lambda = \frac{2}{\sqrt{2-\alpha}} (\log\log n)^{1-1/\alpha-\varepsilon(2-\alpha)},
$$

$$
\widetilde{\gamma} = (\log\log n)^{-1/\alpha - 2\varepsilon},
$$

where $0 < \varepsilon < 1/6$, we obtain the following result.

Theorem 2. *Given* $\gamma = n^{1/\alpha}(\log\log n)^{-1/\alpha-2\varepsilon}$, *we have*

if $\int_0^1 f'^2 dt < 1$, *then*

$$
\liminf_{n\to\infty} \left\| \frac{X_\alpha^\gamma(n\cdot)}{C_1 (n/\log\log n)^{1/\alpha} (\log\log n)^{1-\varepsilon(2-\alpha)}} - f \right\| (\log\log n) = C_2 \quad a.s.,
$$

where $C_2 = 2\pi^{-1}\sqrt{(2-\alpha)(1-\int_0^1 f'^2 dt)}$;

if $\int_0^1 f'^2 dt = 1$, *then*

$$1 \leq \liminf_{n\to\infty} \| \frac{X_\alpha^\gamma(n\cdot)}{C_1(n/\log\log n)^{1/\alpha}(\log\log n)^{1-\varepsilon(2-\alpha)}} - f \| (\log\log n)^{2\varepsilon\alpha} C_3 \leq 3 \quad a.s.$$

$C_1 = 2/\sqrt{2-\alpha}$, $C_3 = (2-\alpha)^{3/2}(6(4-\alpha)(|f'(1)| + \mathrm{Var} f'))^{-1}$.

We have also obtained that if $\gamma = n^{1/\alpha}(\log\log n)^{-1/\alpha - 2\varepsilon}$, then

$$\limsup_n \frac{\|X_\alpha^\gamma(n\cdot)\|}{(n/\log\log n)^{1/\alpha}(\log\log n)^{1-\varepsilon(2-\alpha)}} = c,$$

where $0 \leq c < \infty$.

References

[1] Ph.S. Griffin, *Laws of the iterated logarithm for symmetric stable processes*, Z. Wahrscheinlichkeitstheorie verw. Gebiete (1985), no. 68, 271-285.

[2] E. Yu. Shmileva, *Small ball probability for centered Poisson process of high intensity*, Zapiski Nauchniix Seminarov POMI (2003), vol. 298, 280-304 (in Russian).