THE LOWER TAIL PROBLEM FOR THE AREA OF A SYMMETRIC STABLE PROCESS

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Let $X = \{X_t^\alpha,\ t \geq 0\}$ a real symmetric stable Lévy process of index $\alpha \in (1,2]$, viz. a process with stationary and independent increments whose Fourier transformation is given by

$$\mathbb{E}\left[e^{i\lambda X_t}\right] \;=\; e^{-\kappa|\lambda|^\alpha} \quad \lambda \in \mathbb{R}, t \geq 0,$$

for some normalisation constant $\kappa > 0$. We are interested in the integrated process

$$A_t \;=\; \int_0^t X_s\, ds, \quad t \geq 0$$

and more specifically in the lower tails of its unilateral supremum, i.e. in the behavior of

$$\mathbb{P}\left[\sup_{t\in[0,1]} A_t < \varepsilon\right]$$

when $\varepsilon \to 0$. By $o(1)$ we will mean any real function which tends to 0 when $\varepsilon \to 0$. Our main result is the following

Theorem *When $\varepsilon \to 0$,*

$$\mathbb{P}\left[\sup_{t\in[0,1]} A_t < \varepsilon\right] \;=\; \varepsilon^{\beta/2+o(1)}$$

where $\beta = \beta(\alpha) := (\alpha-1)/(\alpha+1)$.

Sketch of the proof of the upper bound. Since $\alpha \in (1,2]$, it has been known since Boylan [3] that X possesses a jointly continuous local time process $L = \{L(t,x),\ t \geq 0,\ x \in \mathbb{R}\}$ in the sense that for any non-negative Borel function f,

$$\int_0^t f(X_s)\,\mathrm{d}s = \int_{\mathbb{R}} f(x)L(t,x)\,\mathrm{d}x.$$

Let

$$\tau_u(x) := \inf\left[t \geq 0 :\ L(t,x) > u\right], \qquad u \geq 0,$$

be the (càdlàg version of the) inverse local time at $x \in \mathbb{R}$, and set $\tau_u := \tau_u(0)$ for simplicity. It is easy to see from the scaling and the strong Markov properties of X that $\{\tau_t,\ t \geq 0\}$ is a stable subordinator with index $(\alpha-1)/\alpha$ and that the process $Y := \{Y_u = A_{\tau_u},\ u \geq 0\}$ is a symmetric stable Lévy process with index $\beta = (\alpha-1)/(\alpha+1)$. In particular, it follows from Proposition VIII.2 in [1] that

$$\mathbb{P}\left[\int_0^{\tau_u} X_s\,\mathrm{d}s \leq \varepsilon,\ \forall u \in [0,1]\right] \;\sim\; c_1\,\varepsilon^{\beta/2} \qquad \varepsilon \to 0, \tag{0.1}$$

for some constant $c_1 \in (0,\infty)$, which readily implies that

$$\mathbb{P}\left[\sup_{t\in[0,\tau_1]} A_t \le \varepsilon\right] \;\le\; \varepsilon^{\beta/2+o(1)}, \qquad \varepsilon \to 0. \tag{0.2}$$

This is not enough to obtain directly the upper bound because of the big values of τ_1. However, we can show that $\beta/2$ is actually the right upper exponent, in working under the law of the stable pseudo-bridge

$$X_t^{\#} := \frac{X_{t\tau_1}}{\tau_1^{1/\alpha}}, \qquad t \in [0,1], \tag{0.3}$$

through an absolute continuity relation with respect to the standard stable bridge, and partitionning the values of τ_1.

Sketch of the proof of the lower bound. In the case of Brownian motion $X = W$ $(\alpha = 2)$, it is easy to see that $\beta/2 = 1/6$ is also the right lower exponent, since by continuity W keeps the same sign during the excursion intervals $[\tau_{u-}, \tau_u]$, $u > 0$, so that

$$\sup_{0\le t\le\tau_1} \int_0^t W_s\,\mathrm{d}s \;=\; \sup_{0\le u\le 1} \int_0^{\tau_u} W_s\,\mathrm{d}s,$$

from which (0.1) entails

$$\mathbb{P}\left[\sup_{t\in[0,\tau_1]} A_t \le \varepsilon\right] \;\ge\; \varepsilon^{\beta/2+o(1)}, \qquad \varepsilon \to 0 \tag{0.4}$$

and, since the lower tails of τ_1 are exponentially small, we see that there is no hindrance in replacing τ_1 by 1 in (0.4). Let us notice that a much more precise result:

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/6}\,\mathbb{P}\left[\int_0^t W_s\,\mathrm{d}s \le \varepsilon,\ \forall t \in [0,1]\right] \;=\; \frac{3\Gamma(5/4)}{4\pi\sqrt{2\sqrt{2\pi}}}$$

with Γ the Gamma function, is actually already implicit in Mc Kean [9] - after simple computations using the last formula p. 229 and the closed formula 6 p. 231 therein. More recently, Sinai [13] proved that

$$\mathbb{P}\left[\sup_{t\in[0,T]} \int_0^T W_t\,dt < 1\right] \;\sim\; T^{-1/4}$$

when $T \to +\infty$, which is equivalent to McKean's result, save for the existence and computation of the constant. He also proved that the convergence speed $T^{-1/4}$ remains unchanged in replacing the fixed barrier 1 by a linear or quadratic barrier.

When $\alpha \in (1,2)$, the lower bound (0.4) is significantly more delicate to obtain because of the jumps: here X does not keep necessarily the same sign during the excursion interval (τ_{u-}, τ_u) anymore, so that we just have the inequality

$$\sup_{0\le t\le\tau_1} \int_0^t X_s\,\mathrm{d}s \;\ge\; \sup_{0\le u\le 1} \int_0^{\tau_u} X_s\,\mathrm{d}s,$$

and actually the difference may be quite large if X has big jumps during its excursion intervals. We overcome the difficulties in reducing the problem, by scaling, to

$$\mathbb{P}\left[A_t \le 1,\ t \in [0,\tau_N]\right] \ \ge\ N^{-1/2+o(1)}, \quad N \to +\infty,$$

which is then shown to hold true, first in proving the following reinforcement of (0.1):

$$\mathbb{P}\left[u^{1/\beta-\delta} \le A_{\tau_u} \le u^{1/\beta+\delta},\ \forall u \in [1,N]\right] \ \ge\ N^{-(1/2)+o(1)}, \quad N \to +\infty,$$

second in examining carefully the small probabilities that the area process makes a round trip in time $(\tau_{k+1} - \tau_k)$ between $x_k \ge k^{1/\beta-\delta}$, $y_k < 0$ and $z_k \ge (k+1)^{1/\beta-\delta}$, when $k \to +\infty$.

Some open questions

• Prove that the critical exponent is zero when $\alpha \le 1$ and then compute the exact speed of convergence.

• The lower tails of fractional Brownian motion were recently studied by Molchan et al. [10], who also gave in this conference a conjecture for the critical exponent of the integrated fractional Brownian motion. These processes are related to Riemann-Liouville process. For bilateral small deviations, the critical exponents are known to be equal for these two classes of processes [8]. Can we say the same things for unilateral small deviations?

• (A harder problem) Compute the critical exponent for n-times integrated symmetric stable process. This problem is probably already difficult when $n = \alpha = 2$ (the double integral of Brownian motion). As shown in a recent paper of Li & Shao [7], in the Brownian case the *asymptotics* of these critical exponents when $n \to +\infty$ are tightly related to those of $\mathbb{P}\left[N_n = 0\right]$, where N_n denotes the number of zeros of a real polynomial of degree n with i.i.d. Gaussian coefficients. The latter asymptotic is a long-time challenging problem in random polynomials - see [5] and the references therein.

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