ON ACCURACY OF APPROXIMATIONS FOR MULTIVARIATE SCALE MIXTURES IN STATISTICAL APPLICATIONS

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Introduction 1

Some Problems on Asymptotic Expansions

Edgeworth Expansion

$$
\begin{aligned}
\mathsf{P}(T \le x) &= G_f(x) \\
&+ \frac{1}{n}\sum_{j=0}^{k} b_j G_{f+2j}(x) + R_2(x) \\
&= G_f(x) - \frac{1}{n} a(x) g_f(x) + R_2(x)
\end{aligned}
$$

Cornish-Fisher Expansion

$$
\begin{aligned}
\mathsf{P}(T \le t(u)) &= G_f(u) = 1 - \alpha \\
t(u) &= u + \frac{1}{n} a(u) + \tilde{R}_2(u)
\end{aligned}
$$

Introduction 2

(i) **How to find asymptotic expansions**

(ii) **How to establish validity**

$$R_2(x) = \mathrm{O}(n^{-2})$$

(iii) **How to find error bounds**

(1) Order estimate; $R_2(x) = \mathrm{O}(n^{-2})$

$\exists n_0, c$ s.t. $|R_2(x)| \leq cn^{-2}$ for $n \geq n_0$
(n_0, c; unknown)

(2) Berry-Esseen type bounds
with known constant c

$|R_2(x)| \leq ch(\theta)n^{-2}$ for all n
($h(\dot{})$ – known function)

Introduction 3

Berry-Esseen Bound

Sum of i.i.d. variables

Let:

$$X_1, X_2, \cdots, X_n \sim i.i.d. X$$

$$\mathsf{E}(X) = 0 \quad \mathsf{E}(X^2) = 1 \quad \mathsf{E}(|X|^3) = \rho < \infty$$

Let $Y_n = n^{-1/2} \sum_{j=1}^{n} X_j$

Then for any x and $n = 1, 2, \cdots$

$$|\mathsf{P}(Y_n \leq x) - \Phi(x)| \leq \frac{c\rho}{\sqrt{n}}$$

c – absolute constant

$$c \leq 0.7655$$

Introduction 4

t-Statistic

Let:

$$X_1, X_2, \cdots, X_n \sim i.i.d. X$$

$$\mathsf{E}(X) = 0 \quad \mathsf{E}(X^2) = \sigma^2 \quad \mathsf{E}(|X|^3) = \rho < \infty$$

Let: $T_n = \frac{\sqrt{n}\bar{X}}{\hat{\sigma}}, \quad n \geq 2$

$$\hat{\sigma}^2 = \tfrac{1}{n} \textstyle\sum_{j=1}^{n} (X_j - \bar{X})^2$$

Then for any x and $n = 2, 3, \cdots$
there exists an absolute constant c such that

$$|\mathsf{P}(T_n \leq x) - \Phi(x)| \leq \frac{c\rho}{\sqrt{n}}$$

Bentkus and Gôtze (1996)

Statistic 1

One-Way MANOVA

Model

$$\boldsymbol{y}_j^{(i)} = \boldsymbol{\mu}^{(i)} + \boldsymbol{\varepsilon}_j^{(i)} \qquad \begin{array}{r}(j = 1, \cdots, N_i \\ i = 1, \cdots, q+1)\end{array}$$

$$\varepsilon_j^{(i)} : \text{Independent}$$

$$\mathrm{E}(\boldsymbol{\varepsilon}_j^{(i)}) = 0, \ \mathrm{Var}(\boldsymbol{\varepsilon}_j^{(i)}) = \Sigma$$

Hypothesis Testing

$$\begin{array}{ll} H_0 & : \boldsymbol{\mu}^{(1)} = \cdots = \boldsymbol{\mu}^{(q+1)} \\ \text{vs} & \\ H_1 & : \ \boldsymbol{\mu}^{(i)} \neq \boldsymbol{\mu}^{(j)} \text{ for some } i, \ j \end{array}$$

SS & SP Matrices

$$\begin{aligned} S_h &= \sum_{i=1}^{q+1} N_i(\bar{\boldsymbol{y}}^{(i)} - \bar{\boldsymbol{y}})(\bar{\boldsymbol{y}}^{(i)} - \bar{\boldsymbol{y}})' \\ S_e &= \sum_{i=1}^{q+1} \sum_{j=1}^{N_i} (\boldsymbol{y}_j^{(i)} - \bar{\boldsymbol{y}}^{(i)})(\boldsymbol{y}_j^{(i)} - \bar{\boldsymbol{y}}^{(i)})' \end{aligned}$$

Statistic 2

Multivariate Linear Model

Model

$$\begin{aligned} Y &= X\Theta + (\varepsilon_1, \ldots, \varepsilon_n)' \\ &\varepsilon_1, \ldots, \varepsilon_n \sim i.i.d. \end{aligned}$$

$$\mathsf{E}[\varepsilon_j] = 0, \quad \mathsf{Cov}(\varepsilon_j) = \Sigma$$

Hypothesis Testing

$$H_0 : C\Theta = 0$$

SS & SP Matrices

$$\begin{aligned} S_h &= \hat{\Theta}' C' \{C(X'X)^{-1} C'\}^{-1} C \hat{\Theta} \\ S_e &= Y'(I_n - X(X'X)^{-1} X')Y \end{aligned}$$

$$\hat{\Theta} = (X'X)^{-1} X'Y$$

Statistic 3

Test Statistics

(i) $T_{LR} = -n \log\{|S_e|/|S_e + S_h|\}$

(ii) $T_{LH} = T_0^2 = n \mathrm{tr} S_h S_e^{-1}$

$$S_e \sim W_p(n, \Sigma) \quad S_h \sim W_p(q, \Sigma)$$

$S_e \perp S_h$ independent; $\Sigma = I$

The case $p = 1$

(i) $T_{LR} = n \log\left(1 + \frac{1}{n} T_0^2\right)$

(ii) $T_{LH} = T_0^2 = \left(\frac{1}{n}\chi_n^2\right)^{-1} \chi_q^2$

Result 1

Asymptotic Approximations for $\mathsf{P}(T_{LR} \leq x)$ under Null Case

Asymptotic Expansion; Box (1949)

$$
\begin{aligned}
\mathsf{P}(T_{LR} \leq x) &= G_r(x) + \frac{1}{n}\sum_{j=0}^{1} b_j G_{r+2j}(x) \\
&\quad + \mathrm{O}(n^{-2}) \\
&= G_r(x) - \frac{x}{2n}(q-p-1) g_r(x) \\
&\quad + \mathrm{O}(n^{-2})
\end{aligned}
$$

$$
\begin{aligned}
r &= pq \\
b_0 &= r\,(q-p-1)/4 \\
b_1 &= -r\,(q-p-1)/4
\end{aligned}
$$

Result 2

Asymptotic Approximations for $\mathrm{P}(T_0^2 \leq x)$ under Null Case

Asymptotic Expansion; Siotani (1956)

$$
\begin{aligned}
\mathrm{P}(T_0^2 \leq x) &= G_r(x) + \frac{1}{n}\sum_{j=0}^{2} b_j G_{r+2j}(x) \\
&\quad + \mathrm{O}(n^{-2}) \\
&= G_r(x) \\
&\quad + \frac{x}{2n}\{q - p - 1 - (q + p + 1)x\} g_r(x) \\
&\quad + \mathrm{O}(n^{-2})
\end{aligned}
$$

$$
\begin{aligned}
b_0 &= r\,(q - p - 1)/4, \quad b_1 = -r\,q/2, \\
b_2 &= r\,(q + p + 1)/4, \quad r = pq
\end{aligned}
$$

Result 3

Error Bound for Asymptotic Expansions of $\mathsf{P}(T_0^2 \le x)$
UFS (2003)

$$|\mathsf{P}(T_0^2 \le x) - G_r(x)| \le \frac{1}{2n} p(p+1)\nu_{-1,2,p}$$

$$\begin{aligned}
\nu_{-1,2,p} &= \eta_{-1,2} + \frac{1}{2}(p-1)\xi_{-1,1}\eta_{-1,1} \\
g_r(x) &= \{2^{r/2}\Gamma(r/2)\}^{-1} \exp(-x/2)x^{r/2-1} \\
b_{-1,1}(x) &= -\frac{1}{2}(x-r) \\
b_{-1,2}(x) &= -\frac{1}{4}\{x^2 - 2rx + r(r-2)\} \\
\xi_{-1,1} &= \int_{-\infty}^{\infty} |b_{-1,1}(x)g(x)|dx \\
\xi_{-1,2} &= \frac{1}{2}\int_{-\infty}^{\infty} |b_{-1,2}(x)g(x)|dx \\
\eta_{-1,1} &= \xi_{-1,1} + 2 \\
\eta_{-1,2} &= \{\sqrt{\xi_{-1,2}} + \sqrt{2+\xi_{-1,1}}\}^2
\end{aligned}$$

Result 4

Values of $\nu_{-1,2,p}(\lambda)$ in gamma case

		p			
δ	λ	1	2	3	4
-1	0.5	4.574	5.175	5.776	6.377
	1.0	5.711	6.718	7.724	8.731
	1.5	6.691	8.044	9.397	10.750
	2.0	7.608	9.277	10.946	12.615
	2.5	8.469	10.434	12.399	14.364
	3.0	9.286	11.534	13.782	16.030
	3.5	10.069	12.589	15.109	17.630
	4.0	10.825	13.609	16.394	19.178

Result 5

Error Bound for Asymptotic Expansions of $\mathrm{P}(T_0^2 \le x)$

$$|\mathrm{P}(T_0^2 \le x) - G_r(x) - \frac{r}{4n}\{(q-p-1)G_r(x)$$

$$-2qG_{r+2}(x) + (q+p+1)G_{r+4}(x)\}|$$

$$\le \frac{r}{48n^2}(|h_1| + |h_2| + 48q)$$

$$+\frac{1}{2n^2}p(2p^2+5p+5)\min\{\eta_{-1,4,p}, \nu_{-1,4,p}\}$$

$$\eta_{\delta,4,p} = \left\{ V_{\delta,k,p}^{1/4} + \left(2 + p\sum_{j=1}^{3} V_{\delta,j,p}\right)^{1/4} \right\}^4$$

Result 6

Coefficients

$$
\begin{aligned}
V_{\delta,1,p} &= \xi_{\delta,1} \\
V_{\delta,2,p} &= \xi_{\delta,2} + \frac{1}{2}(p-1)\xi_{\delta,1}^{2} \\
V_{\delta,3,p} &= \xi_{\delta,3} + (p-1)\xi_{\delta,1}\xi_{\delta,2} + \frac{1}{6}(p-1)(p-2)\xi_{\delta,1}^{3} \\
V_{\delta,4,p} &= \xi_{\delta,4} + \frac{1}{2}\xi_{\delta,2}^{2} + (p-1)\xi_{\delta,1}\xi_{\delta,3} \\
&\quad + \frac{1}{2}(p-1)(p-2)\xi_{\delta,1}^{2}\xi_{\delta,2} \\
&\quad + \frac{1}{24}(p-1)(p-2)(p-3)\xi_{\delta,1}^{4}
\end{aligned}
$$

Result 7

Error Bound for Asymptotic Expansions of $\mathsf{P}(T_{LR} \leq x)$

FU (2003)

$$|\mathsf{P}(T_{LR} \leq x) - G_r(x)|$$

$$\leq \frac{1}{n}\{B_{-1,2,p} + C_{-1,2,p} + D_1\}$$

$$B_{-1,2,p} = \nu_{-1,2,p}\left\{1 + \frac{1}{12n}(p-1)(2p-7)\right\}$$

$$C_{-1,2,p} = \frac{r}{4}(p-1)$$

$$D_1 = \frac{r}{4}\Big[(4 + |q-2|)\left\{1 - n^{-1}q\right\}^{-(q+2)/2}$$

$$+(q+2)\left\{1 - n^{-1}(q+2)\right\}^{-(q+4)/2}\Big]$$

Result 8

Error Bound for Asymptotic Expansions of $\mathsf{P}(T_{LR} \le x)$

$$
\begin{aligned}
&|\mathsf{P}(T_{LR} \le x) - G_r(x) \\
&\qquad - \frac{r}{4n}(q-p-1)\{G_r(x) - G_{r+2}(x)\}| \\
&\qquad \le \frac{1}{n^2}\Big\{B_{-1,4,p} + C_{-1,4,p} \\
&\qquad\qquad + D_2 + D_3\}
\end{aligned}
$$

Outline 1

An Expression for $T_0^2 (= T_{LH})$

$$
\begin{aligned}
T_0^2 &= n\mathrm{tr}(U'U)S_e^{-1} \\
&= n\mathrm{tr}(H'U'UH)(H'S_eH)^{-1} \\
&\qquad U = (u_{ij}); \quad u_{ij} \sim i.i.d.N(0,1) \\
&= X_1 + \ldots + X_p
\end{aligned}
$$

$\boldsymbol{X} = (X_1, \ldots, X_p)'$

(i) $X_i = S_i Z_i, \quad i = 1, \ldots, p$

(ii) $Z_1, \ldots, Z_p \sim i.i.d. \chi_q^2$

(iii) $S_i = Y_i^{-1} (i = 1, \ldots, p)$
$Y_1 > \ldots > Y_p > 0$;
the characteristic roots of W
$nW \sim W_p(n, I_p)$

Outline 2

An Expression for T_{LR}

$$\begin{aligned} T_{LR} &= n \log \Lambda \\ &= n \log \prod_{j=1}^{p} V_j \\ &\qquad V_j \sim Be\left(\frac{1}{2}(n-j+1), \frac{1}{2}q\right) \\ &= \sum_{i=1}^{p} n \log(1 + X_i/n) \end{aligned}$$

$\boldsymbol{X} = (X_1, \ldots, X_p)'$

(i) $X_i = S_i Z_i, \quad i = 1, \ldots, p$

(ii) $Z_1, \ldots, Z_p \sim i.i.d. \chi_q^2$

(iii) $S_i = Y_i^{-1} (i = 1, \ldots, p)$
$nY_i \sim \chi^2_{m_i},\ m_i = n - (i-1)$

Outline 3

Multivariate Scale Mixture

$$\boldsymbol{X} = (X_1, \ldots, X_p)' \quad X_i = S_i Z_i, i = 1, \ldots, p$$

(i) $Z_1, \ldots, Z_p \sim i.i.d.$

(ii) $S_i > 0 \perp Z_i, i = 1, \ldots, p$

Approximation for the density $f(\boldsymbol{x})$ of $\boldsymbol{X}$;
$\widehat{f}(\boldsymbol{x}) = g_{\delta,k,p}(\boldsymbol{x})$

Error Bound:

$$\begin{aligned} &|\mathsf{P}(\boldsymbol{X} \in A) - \int_A g_{\delta,k,p}(\boldsymbol{x}) d\boldsymbol{x}| \\ &\qquad \le c_{\delta,k,p} \sum_{i=1}^{p} \mathsf{E}[|Y_i - 1|^k] \end{aligned}$$

$$Y_i = S_i^{-1} \quad k = 2, 4$$

Outline 4

Approximation Formulas

$$\mathsf{P}(T \in A) = Q_{\delta,k,p}(A) = \int_A g_{\delta,k,p}(\boldsymbol{x})d\boldsymbol{x}$$

$$Q_{-1,2,p}(A) = Q^{(0)}_{-1,2,p}(A) + Q^{(1)}_{-1,2,p}(A)$$

$$Q_{-1,4,p}(A) = Q_{-1,2,p}(A) + Q^{(2)}_{-1,4,p}(A) + Q^{(3)}_{-1,4,p}(A)$$

(1) The Case T_0^2:

$$\begin{aligned} A &= \{(x_1, \ldots, x_p) \in \mathbf{R}^p : \\ &\qquad x_1 + \ldots + x_p \le x\} \end{aligned}$$

(2) The Case T_{LR}:

$$\begin{aligned} A &= \tilde{A} \\ &= \{(x_1, \ldots, x_p) \in \mathbf{R}^p : \\ &\qquad \sum_{i=1}^{p} n \log(1 + x_i/n) \le x\} \end{aligned}$$

Outline 5

Reduction When A Is Symmetric

$$Q_{\delta,k,p}(A) = \int_A g_{\delta,k,p}(\boldsymbol{x})d\boldsymbol{x} = \sum_{j=0}^{k-1} Q^{(j)}(A)$$

$Q^{(0)}(A) = \int_A g_p(\boldsymbol{x})d\boldsymbol{x}$
For $j \geq 1$;

$$\begin{aligned} Q^{(j)}(A) &= \sum_{(j)} \frac{1}{j_1! \dots j_p!} \\ &\quad \times \int_A b_{\delta,j_1}(x_1) \dots b_{\delta,j_p}(x_p) g_p(\boldsymbol{x}) d\boldsymbol{x} \\ &\quad \times \mathsf{E}\left[(Y_1 - 1)^{j_1} \dots (Y_p - 1)^{j_p}\right]. \end{aligned}$$

$$\begin{aligned} Q^{(1)}(A) &= \int_A b_{\delta,1}(x_1) g_p(\boldsymbol{x}) d\boldsymbol{x} \times \mathrm{tr} V \\ Q^{(2)}(A) &= \frac{1}{2}\int_A b_{\delta,2}(x_1) g_p(\boldsymbol{x}) d\boldsymbol{x} \times \mathrm{tr} V^2 \\ &+ \int_A b_{\delta,1}(x_1) b_{\delta,1}(x_2) g_p(\boldsymbol{x}) d\boldsymbol{x} \\ &\quad \times \{(\mathrm{tr} V)^2 - \mathrm{tr} V^2\} \end{aligned}$$

$$V = S - I \quad S = \mathrm{diag}(S_1, \dots, S_p)$$

Univariate Scale Mixture 1

L_1-Norm Error Bound-Univariate

$$X = SZ \quad S > 0 \perp Z,\ S = Y^{\delta\rho}$$

Let g be density function of Z

Then density function of X:

$$f(x) = \mathsf{E}[Y^{-\delta\rho} g(xY^{-\delta\rho})]$$

Functions $b_{\delta,j}$:

$$\frac{\partial^j}{\partial y^j}\left(y^{-\delta\rho} g(xy^{-\delta\rho})\right)\Big|_{y=1} = b_{\delta,j}(x)g(x)$$

Approximation for $y^{-\delta\rho} g(xy^{-\delta\rho})$;

$$g_{\delta,k}(x,y) = g(x) + \sum_{j=1}^{k-1} \frac{1}{j!} b_{\delta,j}(x)g(x)(y-1)^j$$

Approximation for $f(x)$;

$$g_{\delta,k}(x) = \mathsf{E}[g_{\delta,k}(x,Y)]$$

Main Result

$$
\begin{aligned}
&|\mathsf{P}(X \in A) - \int_A g_{\delta,k}(x)dx| \\
&\quad \le \frac{1}{2}\eta_{\delta,k}\mathsf{E}[|Y-1|^k]
\end{aligned}
$$

$$
\eta_{\delta,k} = \left\{\xi_{\delta,k}^{1/k} + \left(2 + \sum_{j=1}^{k-1}\xi_{\delta,j}\right)^{1/k}\right\}^k
$$

$$
\begin{aligned}
\xi_{\delta,j} &= \frac{1}{j!}\left\|b_{\delta,j}(x)g(x)\right\|_1 \\
&= \frac{1}{j!}\int_{-\infty}^{\infty}|b_{\delta,j}(x)g(x)|dx
\end{aligned}
$$

Multivariate Scale Mixture

$$\boldsymbol{X} = (X_1, \ldots, X_p)' \quad X_i = S_i Z_i, i = 1, \ldots, p$$

(i) $Z_1, \ldots, Z_p \sim i.i.d.$
(ii) $S_i > 0 \perp Z_i, i = 1, \ldots, p$

Transformation;

$$Y_i = S_i^{\delta/\rho} \quad \delta = \pm 1 \quad \rho = 1, 1/2$$

The Conditional Density of $\boldsymbol{X}$
given $Y_i = y_i, i = 1, \ldots, p$

$$y_1^{-\delta\rho} g(x y_1^{-\delta\rho}) \ldots y_p^{-\delta\rho} g(x y_p^{-\delta\rho})$$

$$\begin{aligned}
&g_{\delta,k,p}(\boldsymbol{x}) \\
&= \mathsf{E}\left[g_p(\boldsymbol{x}) + \sum_{j=1}^{k-1} \frac{1}{j!} \left((Y_1 - 1)\frac{\partial}{\partial y_1} + \ldots + (Y_p - 1)\frac{\partial}{\partial y_p} \right)^j \right. \\
&\left. \times\, y_1^{-\delta\rho} g(x_1 y_1^{-\delta\rho}) \ldots y_p^{-\delta\rho} g(x_p y_p^{-\delta\rho}) \Big|_{y_1 = \ldots = y_p = 1} \right] \\
&= g_p(\boldsymbol{x}) + \sum_{j=1}^{k-1} \sum_{(j)} \frac{1}{j_1! \ldots j_p!} b_{\delta,j_1}(x_1) \ldots b_{\delta,j_p}(x_p) g_p(\boldsymbol{x}) \\
&\times \mathsf{E}\left[(Y_1 - 1)^{j_1} \ldots (Y_p - 1)^{j_p} \right]
\end{aligned}$$

Main Result (I)

$$
\begin{aligned}
&|\mathrm{P}(\boldsymbol{X} \in A) - \int_A g_{\delta,k,p}(\boldsymbol{x}) d\boldsymbol{x}| \\
&\qquad \leq \frac{1}{2}\eta_{\delta,k,p} \sum_{i=1}^{p} \mathrm{E}[|Y_i - 1|^k].
\end{aligned}
$$

$\eta_{\delta,1,p} = 2 + V_{\delta,1,p}$

For $k \geq 2$

$$
\eta_{\delta,k,p} = \left\{ V_{\delta,k,p}^{1/k} + \left(2 + p \sum_{j=1}^{k-1} V_{\delta,j,p} \right)^{1/k} \right\}^k,
$$

$$
V_{\delta,j,p} = \sum_{[j]} \frac{(p-1)!}{i_1! \ldots i_m!} \xi_{\delta,j_1} \ldots \xi_{\delta,j_p},
$$

$$
\begin{aligned}
V_{\delta,1,p} &= \xi_{\delta,1} \\
V_{\delta,2,p} &= \xi_{\delta,2} + \frac{1}{2}(p-1)\xi_{\delta,1}^2
\end{aligned}
$$

Main Result (II)

$$
\begin{aligned}
&|\mathsf{P}(\boldsymbol{X} \in A) - \int_A g_{\delta,k,p}(\boldsymbol{x}) d\boldsymbol{x}| \\
&\qquad \leq \frac{1}{2}\nu_{\delta,k,p} \sum_{i=1}^{p} \mathsf{E}[|Y_i - 1|^k].
\end{aligned}
$$

For $k \geq 2$;

$$\nu_{\delta,k,p} = p^{-1} \left(\eta_{\delta,k} + (p-1) \sum_{q=0}^{k-1} \nu_{\delta,k-q,p-1} \xi_{\delta,q} \right)$$

For $k \geq 1$;

$\nu_{\delta,1,p} = \eta_{\delta,1}, \quad \nu_{\delta,k,0} = 0$

$\nu_{\delta,k,1} = \eta_{\delta,k}$

$$
\begin{aligned}
\nu_{\delta,1,p} &= \eta_{\delta,1}, \\
\nu_{\delta,2,p} &= \eta_{\delta,2} + \frac{1}{2}(p-1)\xi_{\delta,1}\eta_{\delta,1}
\end{aligned}
$$

$$\eta_{\delta,1,p} = \nu_{\delta,1,p} \quad \eta_{\delta,2,p} \geq \nu_{\delta,2,p}$$