# A non-finitely based systems of polynomial identities in associative nil-algebras

E. V. Aladova (Moscow, Russia)

Let $\mathbb{F}$ be a field, let $A$ be a free associative algebra (without 1) over $\mathbb{F}$ on free generators $x_1, x_2, \ldots$ and let $G$ be an associative $\mathbb{F}$-algebra. Let $f(x_1, \ldots, x_n) \in A$. We say that $f(x_1, \ldots, x_n) = 0$ is a *polynomial identity* (or an *identity*) in $G$ if $f(g_1, \ldots, g_n) = 0$ for all $g_1, \ldots, g_n \in G$. Two systems of polynomial identities are *equivalent* if every associative $\mathbb{F}$-algebra satisfying all the identities of the first system satisfies all the identities of the second system and vice versa. If a system of polynomial identities is equivalent to some finite system we say that the system is *finitely based* or *has a finite basis.* If $\mathbb{F}$ is a field of characteristic 0 then every system of polynomial identities in associative $\mathbb{F}$-algebras has a finite basis (Kemer, 1987). On the other hand, if $\mathbb{F}$ is a field of a prime characteristic then there exist non-finitely based systems of polynomial identities (Belov, 1999; Grishin, 1999; Shchigolev, 1999).

We study the following

**Problem.** *Find the smallest positive integer $n = n(\mathbb{F})$ such that the identity $x^n = 0$ can be included in the non-finitely based system of polynomial identities in associative algebras over a field $\mathbb{F}$ of a prime characteristic.*

Note that, if a system of polynomial identities in associative algebras over a field $\mathbb{F}$ of a characteristic $p$ contains the identity $x^n = 0$ with $n < p$ then the system is finitely based. Indeed, according to Nagata-Higman-Dubnov-Ivanov theorem the identity $x^n = 0$ $(n < p)$ implies over $\mathbb{F}$ the identity of nilpotency $x_1 x_2 \ldots x_k = 0$ for some $k = k(n) \in \mathbb{N}$ and every system containing the latter identity is finitely based.

In 1999 Grishin constructed a system of polynomial identities in nil-algebras over a field of characteristic 2 which contains the identity $x^{32} = 0$. Similar system with the identity $x^6 = 0$ was constructed by Gupta and Krasilnikov in 2002. Over a field of a prime characteristic $p \geqslant 3$ Shchigolev (2002) constructed a system of polynomial identities containing the identity $x^{2p^3+p^2+1} = 0$. In 2003 authors constructed a system of polynomial identities in nil-algebras over a field of characteristic $p \geqslant 3$ which contains the identity $x^{6p} = 0$. The aim of the present talk is to improve the our late result.

Let $\mathbb{F}$ be a field of characteristic $p \geqslant 3$, let $[x,y] = xy - yx$, $f(x_1,x_2) = x_1^{p-1}x_2^{p-1}[x_1,x_2]$ and let $w_n = [[x_1,x_2],x_3]f(x_3,y_3)\cdots$
$\cdots f(x_n,y_n)[[y_1,y_2],y_3]([[x_3,x_1],x_2][[y_3,y_1],y_2])^{p-1}$. Our main result is as follows.

**Theorem** (E.V.Aladova, A.N.Krasilnikov). *Over a field $\mathbb{F}$ of characteristic $p \geqslant 3$ the system of polynomial identities*

$$\{x^{2p} = 0\} \cup \{w_n = 0 \mid n = 3,4,\dots\}$$

*is not equivalent to any finite system of identities in associative $\mathbb{F}$-algebras.*

## On the proper class generated by supplement submodules

Rafail Alizade (Izmir, Turkey)

Let $R$ be an associative ring with identity. A submodule $N$ of an $R$-module $M$ has a supplement in $M$ if $N + K = M$ for some submodule $K$ of $M$ and $K$ is minimal with respect to this equality. We will consider the class of all short exact sequences $0 \to N \xrightarrow{f} M \to L \to 0$ where $\mathrm{Im} f$ has a supplement in $M$ (see [3]) and its relationship with the class *Suppl* of short exact sequences $0 \to K \xrightarrow{f} M \to L \to 0$ where $\mathrm{Im} f$ is a supplement of some submodule $N$ of $M$ (see [1], [2]).

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## Triply factorized groups and nearrings

Bernhard Amberg (Mainz, Germany)

In the theory of factorized groups $G = AB$ with subgroups $A$ and $B$ one often has to deal with groups of the following form $G = AB = AM = BM$, where the normal subgroup $M$ of $G$ satisfies the intersection property $A \cap M = B \cap M = 1$. Such triply factorized groups - even with nonabelian $M$ - may be constructed by using (local) nearrings.

A nearring $R$ is an algebraic structure with two operations "$+$"and "."such that $R$ is a not necessary abelian group under the operation "$+$"and $R$ is a semigroup under the operation "."so that a one-sided distributive law holds. A nearring $R$ with a unit element 1 is called local if the set of all invertible elements forms a subgroup of the additive group of $R$.

If $L$ is the subgroup of all non-invertible elements of the local nearring $R$, then the set $1 + L$ is a subgroup of the multiplicative group of $R$ acting on $L$, such that the semidirect product of $(1 + L)$ with $L$ is a triply factorized group $G$, where $M$ is isomorphic to $L$ and $A$ and $B$ are both isomorphic to $1 + L$.

Conversely, it can be shown that for every triply factorized group $G$ with $A \cap B = 1$ there exists a sub-nearring of the nearring of all mappings from $M$ into itself, such that the given triply factorized group can be obtained via this construction.

## Derived equivalences and flips on CW-complexes

M. A. Antipov (St. Petersburg, Russia)

We introduce Brauer complex of symmetric special biserial algebra (SB-algebra). We reformulate in terms of Brauer complex the so far known invariants of stable and derived equivalence of symmetric SB-algebras. In particular, the genus of Brauer complex turns out to be invariant under derived equivalence. We study transformations of Brauer complexes which preserve class of derived equivalence. Additionally, we establish a new invariant of derived equivalence of symmetric SB-algebras. As a consequence, we obtain a classification of symmetric SB-algebras with Brauer complex of genus 0 and a partial classification of symmetric SB-algebras of genus 1.

# Projective embeddings of homogeneous spaces with small boundary

I. V. Arzhantsev (Moscow, Russia)

Let $G$ be a connected algebraic group over an algebraically closed field of characteristic zero, and $H$ a closed subgroup of $G$. An *embedding* $G/H \hookrightarrow X$ of the homogeneous space $G/H$ is an irreducible normal algebraic $G$-variety $X$ with an open $G$-equivariant embedding of $G/H$. We say that an embedding $G/H \hookrightarrow X$ has *small boundary* if $\mathrm{codim}_X(X \setminus G/H) \geqslant 2$. For example, the diagonal $\mathrm{SL}(n+1)$-action on $\mathbb{P}^n \times \mathbb{P}^n$ provides a non-trivial embedding of the open orbit, where the boundary has codimension $n$.

In this talk we shall study emdeddings with small boundary in the case when $X$ is a projective variety. A criterion of existence of such an embedding for $G/H$ will be given. Moreover, a given homogeneous space $G/H$ admits only finitely many projective embeddings with small boundary. This result is based on realization of embeddings in the framework of Geometric Invariant Theory.

Under some mild technical conditions we prove in [1] that projective embeddings with small boundary and equivariant morphisms between them are parametrized by the so-called GIT-fan in the space generated by the character lattice of $H$. An explicit description of all projective embeddings with small boundary for some concrete homogeneous spaces $G/H$ with $G = \mathrm{SL}(n)$ and $\mathrm{Sp}(2n)$ will be given.

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# Estimates for weight multiplicities in representations of classical algebraic groups

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Lower estimates for the maximal weight multiplicities in irreducible representations of the classical algebraic groups in positive characteristic $p$ are found under some minor restrictions on $p$. Modular representations of classical groups with all weight spaces of dimension 1 were classified in [1, 2]. That result was used in the description of maximal subgroups of classical algebraic groups in [1]. In this paper we indicate irreducible representations with relatively small weight multiplicities with respect to the group rank. For groups of type $A_n$ the estimates obtained imply that if $n$ is large enough with respect to $p$, then the maximal weight multiplicity in a "typical"irreducible representation grows with growth of $n$ (for fixed $p$). Exceptional classes of representations are indicated. For other classical groups the situation is much easier. If $n > 8$, $p > 2$ for groups of types $B_n$ and $D_n$ and $p > 7$ for type $C_n$, then either the maximal weight multiplicity for an irreducible representation $\varphi$ of $G$ is at least $n-8$, or all its weight multiplicities are equal to 1. For the natural embeddings of classical groups, inductive systems of representations with totally bounded weight multiplicities are classified. For indecomposable inductive systems an analogue of the well-known Steinberg tensor product theorem is proved.

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## On abelian groups which centers of the endomorphism rings are isomorphic

T. S. Barry, A. M. Sebeldin, and A. L. Sylla (Conakry, Guinea)

The abelian groups with isomorphic centers of the endomorphism rings are considered (see [1],[2]). This problem is resolved for some class of abelian groups.

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## Some linear groups of degree $2n$ containing a representation of the special linear group of degree $n$

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Let $R$ be an associative ring with the identity 1, $n$ an integer, $n \geqslant 2$. A transvection in the general linear group $GL_n(R)$ is any element in $GL_n(R)$ conjugate in $GL_n(R)$ to the matrix $\mathrm{diag}\Big(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, 1, \dots, 1\Big)$. By the special linear group $SL_n(R)$ we mean the subgroup of $GL_n(R)$ generated by all transvections in $GL_n(R)$. Suppose $R$ is a ring with involution $\sigma$. If $\Phi$ is a $\sigma$-skew-Hermitian form in $n$ variables over $R$ and if $U_n(R, \Phi, \sigma)$ is the unitary group of $\Phi$, then $SU_n(R, \Phi, \sigma)$ is the normal subgroup of $U_n(R, \Phi, \sigma)$ generated by the set of all transvections in $U_n(R, \Phi, \sigma)$. Assume $R$ is commutative, $n$ is even. If $\Psi$ is an arbitrary alternating matrix of degree $n$ over $R$, then the symplectic group $Sp_n(R)$ is the group of all $g \in GL_n(R)$ such that ${}^t g \Psi g = \Psi$, where ${}^t g$ is the transpose of $g$.

Let $P/k$ be an algebraic extension of fields of degree $l \geqslant 2$. There is a regular embedding $P^* \to \mathrm{Aut}_k P, a \to f_a$ of the multiplicative group $P^*$ of $P$ into the group of all $k$-linear automorphisms of $P$ considered as a $k$-vector space, where $f_a(x) = ax, x \in P$. So, if $G \leqslant GL_n(P)$, and if we pick a $k$-basis $a_1, a_2, \ldots, a_l$ in $P$, then $G$ can be treated as being a subgroup of the group $GL_{nl}(k)$. Specifically, if $G = X_n(P)$ is a classical linear group over $P$, then the subgroup of $GL_{nl}(k)$ identified with $X_n(P)$ is denoted by $X_n(k(a_1, a_2, \ldots, a_l)/k)$. Thus we are led to the problem about the description of subgroups of $GL_{nl}(k)$ that contain some classical linear group $X_n(P)$ over $P$ realized as the group $X_n(k(a_1, \ldots, a_l)/k) \leqslant GL_{nl}(k)$. When $X$ is either one of the groups $SL, Sp, SU$, or the commutant $\Omega_n(P, f)$ of the orthogonal group of a symmetric bilinear form $f$, this problem is examined by Li Shang Zhi [6], [7] and R. H. Dye [4], [5]. Another approach in this area has been developed by F. G. Timmesfield [8].

Here we restrict ourselves to the case when $P$ is a quadratic extension of $k$ but study overgroups of $X_n(P)$ in the group $GL_{2n}(K)$, where $K$ is an algebraic field extension of $k$. So, assume that the polynomial $x^2 - b$, where $x$ is a variable and $b \in k$, defines $P$ as a quadratic extension of $k$. Let $\sqrt{b}$ be a root of $x^2 - b$ and pick the set $\{1, \sqrt{b}\}$ as a basis of $P$ over $k$. We write $SL_n(k(\sqrt{b})/k) = SL_n(k(1, \sqrt{b})/k)$. Now let $K$ be a field which is an algebraic extension of $k$. The goal of this talk is to study subgroups of $GL_{2n}(K)$ that contain $SL_n(k(\sqrt{b})/k)$.

**Theorem.** *Let $K$ be a field which is algebraic extension of a subfield $k$. Suppose* char $k \neq 2$, *$k$ is infinite, and $k$ contains an element $b$ such that $b$ is not a square in $K$. Let $n$ be an integer, $n \geqslant 2$. If $SL_n(k(\sqrt{b})/k) \leqslant X \leqslant GL_{2n}(K)$, then $X$ contains a normal subgroup $G$ for which one of the following assertions holds:*

1) *$n = 2, G \cong SU_2(L, \Phi, \sigma)$, where $L$ is a quaternion division ring such that $L$ contains a subfield which is isomorphic to the field $k(\sqrt{b})$, the center of $L$ contains $k$, $L$ is an algebra with involution $\sigma$ either of orthogonal or of unitary type, $\Phi$ is a non-degenerate $\sigma$-skew-Hermitian form with Witt index* 1.
2) *$n \geqslant 2, G \cong SL_n(L)$, where $L$ is either a quaternion division ring as in item* 1)*, or $L$ is a subfield of the field $K(\sqrt{b})$ containing the subfield $k(\sqrt{b})$.*
3) *$n \geqslant 2, G$ is conjugate in $GL_{2n}(K)$ to the group $SL_{2n}(L)$, where $L$ is a subfield of $K$ containing $k$.*
4) *$n = 2, G$ is conjugate in $GL_4(K)$ to the group $Sp_4(L)$, where $L$ is a subfield of $K$ containing $k$.*

5) *$n = 2, G$ is conjugate in $GL_4(K)$ to the group $SU_4(L, \Phi, \sigma)$, where $L$ is a subfield of $K$ containing $k$, $\sigma$ is an involution defined on $L$, $\Phi$ is a non-degenerate $\sigma$-skew-Hermitian form in four variables over $L$ with Witt index* 2.

The concept of a transvection parameter set introduced and used in [1], [2], [3] is meaningful to the proof of Theorem.

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# Cohomological birational invariants of algebraic tori

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Let $k$ be a field, $T$ an algebraic $k$-torus, $G$ the Galois group of the minimal splitting field of $T$. Then the module $\hat{T}$ of rational characters of $T$ is a torsion-free $\mathbb{Z}[G]$-module of finite rank. Any $G$-module $\hat{T}$ admits the following exact sequence

$$0 \longrightarrow \hat{T} \longrightarrow \hat{S} \longrightarrow \hat{N} \longrightarrow 0 \tag{1}$$

here $\hat{S}$ is a permutation $G$-module, $\hat{N}$ is a flasque $G$-module, i.e. $H^{-1}(\pi, \hat{N}) = 0$ for any subgroup $\pi$ of $G$. The resolvent (1) plays a significant role in the birational geometry of algebraic torus, because the class of equivalence $[\hat{N}]$ of a module $\hat{N}$ is the main birational invariant of $T$. It was introduced by Prof. V.E.Voskresenskii, who called the sequence (1) the *canonical resolvent*. The class $[\hat{N}]$ is called the Picard class since one of the representative of this class is geometrical Picard module $PicX$ of a smooth projective model $X$ of $T$. In fact, one can construct (1) for any $G$-module $\hat{T}$ using standard algorithms but rk $\hat{N}$ is large even if rk $\hat{T}$ is small. We calculate induced birational invariants of $T$ so called *cohomological invariants* $\{H^1(\pi, \hat{N})\}_{\pi<G}$. It is the result of Prof. A.A. Klyachko [1]

$$H^1(\pi, \hat{N}) = Ker(H^2(G, \hat{T}) \to \prod_{g \in G} H^2(<g>, \hat{T}))$$

which allows to calculate these invariants without calculation of $\hat{N}$ in terms of the group $H^2$. We develop his idea in the case of four Klein's group $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and obtain the following formula

$$H^1(\pi, \hat{N}) = \frac{Ker(a+e)(b+e)}{Ker(a+e) + Ker(b+e) + ker(c+e)} \tag{2}$$

here $a$, $b$ are generators of $G$ considering as operators on $\hat{T}$ and $c = a \cdot b$.

Using this formula and the classification of all indecomposable $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-modules by Nazarova [2] we calculate the cohomological birational invariants of algebraic tori with biquadratic splitting field of $\dim\hat{T} \leqslant 8$. The following $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-modules have nontrivial cohomological birational invariants (we use the notifications introduced in [2]).

(1) $\hat{T} = C^6(E^3_{11}, E^2_{22})$, $H^1(G, \hat{T}) = \mathbb{Z}_2$
(2) $\hat{T} = C^6(E^3_{12}, E^2_{21})$, $H^1(G, \hat{T}) = \mathbb{Z}_2$
(3) $\hat{T} = C^6(E^1_{21}, E^4_{12})$, $H^1(G, \hat{T}) = \mathbb{Z}_2$
(4) $\hat{T} = C^6(E^4_{11}, E^1_{22})$, $H^1(G, \hat{T}) = \mathbb{Z}_2$
(5) $\hat{T} = C^8(E^5_{11})$, $H^1(G, \hat{T}) = \mathbb{Z}_2$
(6) $\hat{T} = C^8(E^5_{12})$, $H^1(G, \hat{T}) = \mathbb{Z}_2$
(7) $\hat{T} = C^8(E^5_{21})$, $H^1(G, \hat{T}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$
(8) $\hat{T} = C^8(E^5_{22})$, $H^1(G, \hat{T}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$
(9) $\hat{T} = \Phi(f(t))$, $f(t) = t^2 + t + 1$, $H^1(G, \hat{T}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$

We also find the series of such nonrational algebraic tori.

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## On trivial zeros of $p$-adic $L$-functions

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In [3], Perrin-Riou generalized Iwasawa construction of the $p$-adic zeta function to any motif $M$ over $\mathbb{Q}$ having good reduction at $p$. She conjectured that this Iwasawa $L$-function $L_{\mathrm{Iw}}(M,s)$ coincides with the $p$-adic $L$-function of $M$ up to a unity in the Iwasawa algebra $\Lambda$. This conjecture can be seen as a vaste generalisation of the Iwasawa main conjecture. She proved that if the Euler factor $E_p(M,s)$ of $M$ at $p$ is not 0 at $s=0,1$ then the special value of $L_{\mathrm{Iw}}(M,s)$ at $s=0$ satisfies the Bloch-Kato conjecture up to a $p$-adic unity.

The phenomena of trivial zeros appears if the Euler factor $E_p(M,s)$ vanishes at 0 or 1. In this case the order of zero of the $p$-adic $L$-function can be greater that the order of zero of the complex $L$-function $L(M,s)$.

If $M$ is ordinary at $p$, Greenberg [1] defined an important invariant $\ell_p(M)$, the so-called $\ell$-invariant, and conjectured that it appears as additional factor in the usual formula relating special values of complex and $p$-adic $L$-functions:

$$\frac{d^e}{ds^e}\,L_p(M,0)\;=\;\ell_p(M)\,\frac{L(M,0)}{\Omega_M}.$$

In this talk we show how using the theory of $(\varphi,\Gamma)$-modules [2], generalise Greenberg's construction of the $\ell$-invariant to the semi-stable case and prove that it appears as additional factor in the Bloch-Kato type formula for the special value of $L_{\mathrm{Iw}}(M,s)$ at $s=0$.


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# On endomorphism rings of a class of torsion-free abelian groups

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**Definition 1.** A torsion-free abelian group $X$ belongs to the class $\mathcal{D}$ if it contains a completely decomposable subgroup $R(X) = \bigoplus\limits_{\tau \in T_{cr}(R(X))} A_\tau$ such that the following conditions are satisfied:

(1) $T_{cr}(R(X))$ is a finite or countable set of pairwise incomparable types;
(2) $A_\tau$ is a pure subgroup of $X$ of finite rank for each $\tau \in T_{cr}(R(X))$;
(3) $X/R(X) = \bigoplus\limits_{p \in P_X} T_p^X$ for a finite or countable set of primes $P_X$ and finite cyclic $p$-groups $T_p^X$;
(4) for every $p \in P_X$ the set $\{q \in P_X : [T_p^X] \cap [T_q^X] \neq \emptyset\}$ is finite; here $[T_p^X]$ is the minimal subset $T_p \subset T_{cr}(R(X))$ satisfying $T_p^X \subseteq ((\bigoplus_{\tau \in T_p} A_\tau)_*^X + R(X))/R(X)$.

If the set $T_{cr}(R(X))$ is finite then $X \in \mathcal{D}$ is a crq-group of finite rank (that is an almost completely decomposable group with cyclic regulator quotient $X/R(X)$). If $T_{cr}(R(X))$ is countable then we say that $X$ is a *local crq-group* of countable rank since all its fully invariant pure subgroups of finite rank are crq-groups. In both cases the endomorphism ring End$X$ of a group $X$ can be considered as a subring of the direct product $\prod_{\tau \in T_{cr}(R(X))} \mathrm{End} A_\tau \cong \mathrm{End} R(X)$, then any element of End$X$ is recorded as $F^0 = (\ldots, F_\tau^0, \ldots)_{\tau \in T_{cr}(R(X))}$ with $F_\tau^0 \in \mathrm{End} A_\tau$. For each $\tau$ denote the image of End$X$ in End$A_\tau$ under the canonical projection as follows, $E_\tau(X) = \{F_\tau :$ there exists $F^0 = (\ldots, F_\tau^0, \ldots)_{\tau \in T_{cr}(R(X))} \in$ End$X$ with $F_\tau = F_\tau^0\}$. Automorphism group of End$X$ is described by

**Theorem 1.** *Let $X \in \mathcal{D}$. Then*

$$\mathrm{Aut}(\mathrm{End} X) \cong \prod_{\tau \in T_{cr}(R(X))} \mathrm{Aut}(E_\tau(X))$$

*and any automorphism of* End$X$ *uniquely extends to an automorphism of the ring* End$R(X)$.

Generalizing the near isomorphism equivalence to torsion-free abelian groups of infinite ranks we introduce

**Definition 2.** *Let $X$ and $Y$ be torsion-free abelian groups. Then $X$ and $Y$ are called* **nearly isomorphic**, $X \cong_{nr} Y$, *if for every prime $p$ there exist monomorphisms $\Phi_p : X \to Y$ and $\Psi_p : Y \to X$ such that*

(1) *$Y/X\Phi_p$ and $X/Y\Psi_p$ are torsion groups with $p$-components $(Y/X\Phi_p)_p = 0 = (X/Y\Psi_p)_p$;*
(2) *for every finite rank pure subgroups $X' \subseteq X$ and $Y' \subseteq Y$ the quotients $(X'\Phi_p)_*^Y/X'\Phi_p$ and $(Y'\Psi_p)_*^X/Y'\Psi_p$ are finite.*

This equivalence, which is the usual near isomorphism in case of finite rank groups, preserves decomposability properties of local crq-groups similarly to [2, Corollary 12.9(b)]. It leads to the following extension of the main result of [2].

**Theorem 2.** *Let $X, Y \in \mathcal{D}$. Then $X \cong_{nr} Y$ if and only if* End$X \cong$ End$Y$.

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# Relative Gröbner-Shirshov Bases for Algebras and Groups


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We introduce the notion of relative Gröbner-Shirshov bases for algebras and groups. The relative composition lemma and relative (composition-) diamond lemma are established. In particular, we show that the relative normal forms of certain groups arise from the Malcev's embedding problem can be expressible as the irreducible normal forms of these groups with respect to their relative Gröbner-Shirshov bases. Other examples of such groups are given by showing that any group $G$ in a Tits system $(G, B, N, S)$ has a relative $(B-)$ Gröbner-Shirshov basis such that the irreducible words are the Bruhat words of $G$.

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## Orthogonal decompositions of the Lie algebra $sl(n)$ and harmonic analysis on graphs

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### 1. Introduction

We study decompositions of the Lie algebra $sl(n)$ into the direct sum of Cartan subalgebras orthogonal with respect to the Killing form. The famous Winnie-the-Pooh conjecture states that such decompositions exist only when $n$ is a power of a prime. A relation of this problem to representation theory of Hecke algebras of some (multi-connected) graphs $\Gamma$ was found by the first author about 20 years ago (cf. [1]). Here we factorize Hecke algebra and its representations through some smaller algebras $B(\Gamma)$. These are (reduced) Temperley-Lieb algebras related to graphs, which firstly appeared in studying the percolation problem in physics ([2]).

We found that it is useful to regard graphs as topological spaces, because representations of $B(\Gamma)$ are parameterized by local systems on these spaces. This is the place where harmonic analysis on graphs comes into the play. It turns out that representations related to orthogonal decompositions are those for which the Laplace operator of the graph which acts on sections of the local system is annihilated by a polynomial of low degree.

The problem of finding local systems on graphs with low degree minimal polynomial appeared to be very interesting and hard. Coefficients of the characteristic polynomial of the Laplacian are functions on the moduli space of local systems.We prove that if the minimal polynomial has degree less than the minimal length of cycles in the graph, then the local system is a critical point of all coefficients of the characteristic polynomial of the Laplacian.

## 2. Orthogonal decompositions and algebra $B(\Gamma)$.

Let $k$ be an algebraically closed field of characteristic zero. Let $\mathcal{L}$ be a simple Lie algebra over $k$ and $K$ the Killing form on $\mathcal{L}$.

A pair $(\mathcal{H}_1, \mathcal{H}_2)$ of Cartan subalgebras is called *orthogonal* if $K(h_1, h_2) = 0$, for all $h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2$.

A decomposition of the Lie algebra $\mathcal{L}$ into the direct sum of Cartan subalgebras $\mathcal{L} = \oplus_{i=1}^{h+1} \mathcal{H}_i$ is called *orthogonal decomposition (OD)* if $K(\mathcal{H}_i, \mathcal{H}_j) = 0$ for $i \neq j$.

**Winnie-the-Pooh Conjecture 1.** (see [1]) A simple Lie algebra $\mathcal{L} = sl(n)$ has an orthogonal decomposition if and only if $n = p^m - 1$ .

We will study OD by means of representation theory of some algebras related to graphs. Let $\Gamma$ be a simply laced connected graph. We define (reduced) Temperley-Lieb algebra $B(\Gamma)$ as a unital algebra over $k[r, r^{-1}]$ as follows. Generators $x_v$ of $B(\Gamma)$ are numbered by all vertices $v$ of $\Gamma$. They subject to relations:

1. $x_v^2 = x_v$ for any $v$ in $\Gamma$,
2. $x_v x_w x_v = r x_v$, $x_w x_v x_w = r x_w$, if $v, w$ are adjacent in $\Gamma$,
3. $x_v x_w = x_w x_v = 0$, if there is no edge connecting $v$ and $w$ in $\Gamma$.

Note that the standard Temperley-Lieb algebra $TL(\Gamma)$ is defined similarly with the last relation replaced by $x_v x_w = x_w x_v$ when there is no edge connecting $v$ and $w$. Algebra $B(\Gamma)$ is a quotient of $TL(\Gamma)$.

Any automorphism of the graph $\Gamma$ induces an automorphism of the algebra $B(\Gamma)$. For a fixed $r \in k^*$ we denote by $B_r(\Gamma)$ the algebra over $k$ obtained from $B(\Gamma)$ by specialization. The representation $\psi$ of $B_r(\Gamma)$ is the *hight 1* representation if rk $\psi(x_v) = 1$ for some (then for all) $v$. A *genuine* representation $\psi$ of $B_r(\Gamma)$ is a representation for which $\psi(x_v) \neq \psi(x_w)$ for $v \neq w$. The relation with Winnie-the-Pooh conjecture is given via the following graphs. Let $\Gamma_m(n)$ be the graph with $m$ rows and $n$ vertices in each row, such that any two vertices from different rows are connected by an edge and any two vertices lying in the same row are disconnected.

**Theorem 2.** 1. Orthogonal decompositions of $sl(n)$ taken up to automorphism of the Lie algebra are in bijective correspondence with $Aut(\Gamma_{n+1}(n))$ - orbits of genuine height 1 n-dimensional modules over $B_{1/n}(\Gamma_{n+1}(n))$.

2. Orthogonal pairs of Cartan subalgebras of $sl(n)$ are in bijective correspondence with $Aut(\Gamma_2(n))$ - orbits of genuine height 1 n-dimensional modules over $B_{1/n}(\Gamma_2(n))$.

## 3. The path algebras and Laplacians of graphs.

Consider the graph $\Gamma$ as a topological space. Let $k[\Gamma]$ be the path algebra of $\Gamma$. Generators of this algebra correspond to homotopy classes of paths connecting vertices in $\Gamma$. Multiplication comes from juxtaposition of paths.

We denote by $e_v$ the element in $k[\Gamma]$ corresponding to the constant path in the vertex $v$. Any oriented edge $(ij)$ in $\Gamma$ can be interpreted as element $l_{ij}$ in $k[\Gamma]$. The element in $k[\Gamma]$

$$\Delta = \sum l_{ij},$$

where the sum is taken over all oriented edges in $\Gamma$, will be called the *Laplacian* of the graph. Fix a vertex $v_0$ in $\Gamma$. Denote by $\pi(\Gamma) = \pi(\Gamma, v_0)$ the fundamental group of $\Gamma$, i.e. homotopy classes of loops in $\Gamma$ with origin in $v_0$. Let $A-\text{mod}$ denote the category of finite dimensional modules over algebra $A$. The categories $k[\Gamma]-\text{mod}$ and $\pi(\Gamma)-\text{mod}$ are easily seen to be equivalent.

Fix an $r \in k^*$. Let $t$ be an element in $k^*$ such that $t^2 = r$. There is a homomorphism $\varphi : B_r(\Gamma) \to k[\Gamma]$ given by the formula

$$(*) \qquad x_v \mapsto e_v(1 + t\Delta).$$

This homomorphism is the clue to relating the representations of $B_r(\Gamma)$ to the harmonic analysis on graphs. It induces natural functors between categories $\mathrm{k}[\Gamma]-\text{mod}$ and $\mathrm{B_r}(\Gamma)-\text{mod}$. When composed with the equivalence between $k[\Gamma]-\text{mod}$ and $\pi(\Gamma)-\text{mod}$ we obtain adjoint functors: $\varphi_* : \pi(\Gamma)-\text{mod} \to \mathrm{B_r}(\Gamma)-\text{mod}$ and $\varphi^* : \mathrm{B_r}(\Gamma)-\text{mod} \to \pi(\Gamma)-\text{mod}$. The functor $\varphi^*$ transforms genuine height 1 $B_r(\Gamma)$-modules into 1-dimensional representations of $\pi(\Gamma)$. The functor $\varphi_*$ transforms 1-dimensional representations into genuine height 1 $B_r(\Gamma)$-modules. But this module can differ from the original one. Any irreducible genuine height 1 module can be obtained from a 1-dimensional representation $M$ of $\pi(\Gamma)$ by factorizing $\varphi_* M$ by the maximal trivial submodule. This gives a parametrization of the moduli space of irreducible genuine height 1 representations by the characters of the fundamental group of the graph. Let us go back to the graphs $\Gamma_m(n)$. By theorem 2 orthogonal decompositions/pairs correspond to irreducible genuine height 1 representations which are of dimension $n$. Our problem now is to determine which characters of the fundamental group of the graph correspond to these representations. We can describe them in terms of the action of the Laplacian on the corresponding representation of $k[\Gamma]$. This representation is the space of functions on the vertices of the graph 'twisted' by the character of the fundamental group. They have the geometric meaning of the horizontal sections of the local system on the graph defined by the character.

One can check that the character of the fundamental group of the graph $\Gamma_m(n)$ defines a representation of $B_{1/n}(\Gamma_m(n))$ of dimension $n$ if and only if $\Delta$ in the corresponding representation of $k[\Gamma_m(n)]$ satisfies the relation:

$$(\dagger) \qquad (\Delta + \sqrt{n})(\Delta - (m-1)\sqrt{n}) = 0.$$

**Theorem 3.** The character of the fundamental group of the graph $\Gamma_{n+1}(n)$ defines an orthogonal decompositions of the Lie algebra $sl(n)$ iff and only if the corresponding representation of $k[\Gamma_{n+1}(n)]$ the Laplacian satisfies the relation $(\Delta + \sqrt{n})(\Delta - n\sqrt{n}) = 0$. Similarly, orthogonal pairs in $sl(n)$ correspond to representations of $k[\Gamma_2(n)]$, where the Laplacian satisfies the relation $\Delta^2 - n = 0$.

Using $\varphi$ from $(*)$ we obtain the following bound on the homological dimension of the category $\mathrm{B_r}(\Gamma) - \mathrm{mod}$.

**Theorem 4.** Homological dimension of the category $\mathrm{B_r}(\Gamma) - \mathrm{mod}$ is less than or equal to 2, for any $r \in k^*$.

Let $\Gamma$ be a graph, $\Delta$ the Laplacian of $\Gamma$. Every character induces a representation of $k[\Gamma]$. Hence, the trace $f_s = \mathrm{Tr}\Delta^s$ is, for any $s$, a function on the variety $X(\Gamma)$ of $\pi(\Gamma)$-characters.

**Theorem 5.** Let $h$ be the minimal length of non-trivial reduced cycle in graph $\Gamma$, $p$ the polynomial of degree strictly less than $h$. Suppose that a character $x \in X(\Gamma)$ of $\pi(\Gamma)$ corresponds to a representation of $k[\Gamma]$ which satisfies the relation $p(\Delta) = 0$. Then $df_s = 0$ at point $x$, for any $s$.

Since the minimal length of cycles on graphs $\Gamma_{n+1}(n)$ and $\Gamma_2(n)$ are, respectively, 3 and 4, we obtain that orthogonal decompositions and orthogonal pairs are described by critical points of trace functions $f_s$ for corresponding graphs.

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# Weights for cohomology: weight structures, filtrations, spectral sequences, and weight complexes (for motives and spectra)

M. V. Bondarko (St. Petersburg, Russia)

Our basic notion is those of a *weight structure* for a triangulated $\underline{C}$. A weight structure defines Postnikov towers of objects of $\underline{C}$; these towers are canonical and functorial "up to cohomology zero". For $Hw$ being the *heart* of the weight structure there exists a canonical conservative *weakly exact* functor $t$ from $\underline{C}$ to a certain *weak category of complexes* $K_w(Hw)$. For any (co)homological functor $H : \underline{C} \to A$ for an abelian $A$ we construct a *weight spectral sequence* $T : H(X^i[j]) \implies H(X[i+j])$ where $(X^i) = t(X)$; it is canonical and functorial starting from $E_2$. This spectral sequences specializes to the usual weight spectral sequences for "classical"realizations of (Voevodsky's) motives. Besides, $K_0(\underline{C}) \cong K_0(Hw)$ in the bounded case if $Hw$ is idempotent complete. Under certain restrictions, a similar equality is valid for $K_0(\operatorname{End}\underline{C})$.

These result give us a better understanding of Voevodsky's motives and also of the stable homotopy category $SH$. In particular, we calculate very explicitly the groups $K_0(SH_{fin})$ and $K_0(\operatorname{End} SH_{fin})$ (and also certain $K_0(\operatorname{End}^n SH_{fin})$ for $n \in n$). In this case we also have $K_w(Hw) = K(Hw) \cong K(Ab_{fin.fr})$. Besides we obtain a certain "weight filtration"on homotopy groups of spectra (and the corresponding "weight"spectral sequence).

The definition of a weight structure for a triangulated category $\underline{C}$ is almost dual to those of a $t$-structure; yet some properties of these definitions are surprisingly distinct. Under certain conditions for a weight structure $w$ one can construct a certain $t$-structure which is *adjacent* to $w$. Vice versa, for a $t$-structure one can often construct adjacent weight structures (such that either $\underline{C}^{w\leqslant 0} = \underline{C}^{t\leqslant 0}$ or $\underline{C}^{w\geqslant 0} = \underline{C}^{t\geqslant 0}$). In particular, this is the case for the Voevodsky's category $DM_{-}^{eff}$ (one obtains certain *Chow* weight and $t$-structures) and for the stable homotopy category. The hearts of *adjacent* structures are dual in a very interesting sense.

## Generating of prime numbers based on the multidimensional matrices

M. Bulat, A. Zgureanu, I. Ciobanu, and L. Bivol (Chisinau, Moldova)

There are given $n$ sets $X_1 = \{x_{11}, ..., x_{1m_1}\}$, $X_2 = \{x_{21}, ..., x_{2m_2}\}$, $\ldots$,$X_n = \{x_{n_1}, ..., x_{nm_n}\}$ under which, using elements of set $\Omega = = \{\omega_1, ..., \omega_t\}$, $g$ relations $R_{X_{i_1}X_{i_2}\ldots X_{i_d}}$ are defined, where $2 \leqslant d \leqslant n$, and indices $i_1, i_2, .., i_d$=1,2,...,$n$. These relations form a set $R_1$. According to [1,2], using a transformation $\Phi(\vec{R_1})$ we can build a $n$-dimensional matrix $A = \Phi(\vec{R_1})$. If $\Omega = \{0, 1, 2, \ldots, m-1\}$, $g = n$, $d = 2$, $m_1 = m_2 = \cdots = m_n = t = m$ then the elements of this matrix are nonnegative integer numbers and prime numbers are replaced on the lines which depend on the value of the parameter m. Using the properties of this matrix a generator of prime number **BUZGUCIBI PRIM 1** was elaborated. This generator builds prime numbers of kind $p = b \cdot m + r$, where $m = p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdot \cdots \cdot p_s^{\lambda_s}, b = q_1^{\beta_1} \cdot q_2^{\beta_2} \cdot \cdots \cdot q_k^{\beta_k}$, $r = 1$ and $p_i, q_j$ are prime numbers. The generator consists of three modules. The first module gives prime numbers $p_i$ and their exponents $\lambda_i$ provided that $p$ contains a given number $c$ of decimal digits. The numbers $p_i$ are chosen from a set $P$ of known prime numbers and exponents $\lambda_i$ are functions of nonnegative integer argument $n_i$ $(\lambda_i = f_i(n_i))$. The second module gives numbers $q_j$ and their exponents $\beta_j$ provided that $m$ is defined by the first module. On the one of lines of matrix $A$ this module choses a number $p$ with founded values of $m$ and $b$. Thus, we obtain a factory number $p - 1 = q_1^{\beta_1} \cdot \cdots \cdot q_k^{\beta_k} \cdot p_1^{\lambda_1} \cdot \cdots \cdot p_s^{\lambda_s}$ which allows to apply easily a Lucas-Lehmer criterium of primality of number $p$. This work is done by the third module. To obtain a number $p$ it is sufficient to introduce a single

parameter $c$ of decimal digits which must to contain number $p$. A generator can build also a set $P^*$ of prime numbers with the given decimal digits. For this purpose we must indicate a number of elements of set $P^*$. If necessary, we can chose a value $s$, change elements of set $P^*$, change functions $f_i(n_i)$. A generator allows to build the prime numbers which contain ten of thousands of decimal digits. Each module can be used to solve the problems for which they are elaborated. We bring some information about this generator. To build the sets $P^*$ each of them consisting of 10 numbers of length 200, 500, and 1000 of decimal digits, a computer (processor Athlon 3200) needs respectively about 1.7, 18.8 and 92.4 sec. Having a value of m, the second module (working separately) has constructed this prime number: $p = (2^2 \cdot 7^2 \cdot 31) \cdot (797512048017273358771\times$
$\times 793207819539153918901 \cdot 790748260408799953261\times$
$\times 787673811495857496211 \cdot 787058921713269004801\times$
$\times 775990905626676159421 \cdot 765537779322671805451\times$
$\times 760618661061963874171 \cdot 758773991714198399941\times$
$\times 755699542801255942891 \cdot 754469763236078960071\times$
$\times 736637959541012709181)^{130} + 1$ which contains 32589 decimal digits.

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# The spectra of the finite simple classical groups

A. A. Buturlakin (Novosibirsk, Russia)

The spectrum $w(G)$ of a finite group $G$ is the set of all of its element orders. Many problems concerning finite simple groups lead us to the situation when we need to know whether the given natural number is in $w(G)$ or not. This problem can be easily solved in the case of alternating groups. The spectra of sporadic groups is known. The conjugacy classes of each finite simple exceptional group of Lie type are described, thus the spectra is also known. But there is no convenient description of the spectra for the remaining class of the classical simple groups of Lie type.

In the current talk we present the description of the spectra of finite classical groups of Lie type and as a conclusion we receive an arithmetical criterion for determining whether the given natural number in in the spectrum of a given classical simple group.

Let $G$ be a finite group of Lie type over a field of characteristic $p$. The spectrum $w(G)$ of $G$ divides in natural way into three subsets: the set $w_p(G)$ of orders of $p$-elements, the set $w_{p'}(G)$ of orders of $p'$-elements, and the set $w_m(G)$ of orders of the rest elements. According to this division, the problem of describing the spectrum of a finite group of Lie type can be considered as three independent problems. A description of $w_p(G)$ can be found in [1]. In [2] the author and M.A. Grechkoseeva determined the cyclic structure of maximal tori in all finite classical simple groups, and therefore found $w_{p'}(G)$. Thus to finish the description we must describe $w_m(G)$ for all classical simple groups. Such descriptions were obtained and as a example we give the theorem concerning linear groups.

The set $\omega(G)$ is ordered by divisibility relation and we denote by $\mu(G)$ the set of its elements that are maximal under this relation. Obviously, $\omega(G)$ is uniquely determined by $\mu(G)$. Denote by $\mu_m(G)$ the set $\mu(G)\cap w_m(G)$; denote by $\mathrm{lcm}\{m_1, m_2, \ldots, m_s\}$ the least common multiple of natural numbers $m_1, m_2, \ldots, m_s$.

**Theorem**. *Let $G = L_n(q)$, $n \geqslant 2$, and $q$ be a power of a prime $p$. Put $d = (n, q-1)$. Suppose that for every $k > 1$ such that $n_0 = p^{k-1} + 1 < n$ and every partition $n - n_0 = n_1 + n_2 + \cdots + n_s$, the set $\eta(G)$ contains the number $p^k(q^{n_1} - 1)/d$ if $s = 1$, and the number $p^k \,\mathrm{lcm}\{q^{n_1} - 1, q^{n_2} - 1, \ldots, q^{n_s} - 1\}$ if $s > 1$, and the set $\eta(G)$ has no other elements. Then $\mu_m(G) \subseteq \eta(G) \subseteq \omega_m(G)$.*

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# Supplemented Modules over non-Noetherian Domains

Engin Büyükaşık, Yilmaz Mehmet Demirci (Urla, Izmir, Turkey)

Supplemented modules over Dedekind domains are characterized by H. Zöschinger [3, Theorem 3.1]. Rudlof[2] extended these results over commutative Noetherian rings. We use Zöschinger' s ideas to obtain characterization of these modules over some non-Noetherian commutative domains, in particular over h-local domains. It is shown that:

**Theorem 1** (Theorem)**.** *Over a non-local h-local domain a module $M$ is supplemented if and only if it is torsion and every primary-component is supplemented.*

This result is an extension of Zöschinger' s result over Dedekind domains.

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# Counting Hopf-Galois Structures on Field Extensions

Nigel Byott (Exeter, United Kingdom)

Hopf-Galois theory provides a generalisation of classical Galois theory of field extensions. In particular, there may be many Hopf algebras which act on a given field extension, giving rise to different Hopf-Galois structures. This talk will describe all Hopf-Galois structures on a class of extensions of degree $2^n$, $n \geqslant 3$, which includes all such extensions which are cyclic or radical. This complements work of T. Kohl on simple radical extensions of degree $p^n$ for an odd prime $p$. In contrast to the odd $p$ case, however, for $p = 2$ the Hopf algebras concerned need not be commutative; indeed, they can be forms of the group algebra of a cyclic, dihedral or quaternion group.

## Recent applications of Hamburger-Noether expansions

Antonio Campillo (Valladolid, Spain)

Hamburger-Noether expansions are useful for studying singularities of algebraic curves in any characteristic. In particular, they provide parameterizations in terms of parameters which are rational functions. This fact provides a way to do symbolic computations for parameterizations and equations at the same time, in such a way that expansions are replaced by symbolic expressions. It avoids difficulties of dealing with power series, and gives rise to efficient calculations. For instance, adjoint divisors and Weierstarss semigroups of points of projective plane curves can be naturally computed in that way. Also, deformations of Hamburger-Noether expansions become significant in deformation theory.

In recent joint work with G. M. Greuel and Ch. Lossen a theory of equisingular deformations for plane curve singularities in arbitrary characteristic is established. The functor of equisingular deformations of a parameterization is not isomorphic to the functor of deformations of the curve itself. Non trivial equisingular deformations of the parameterization which are trivial as deformations of the equations play a key role in the context. Such objects do not exist in characteristic zero, but they do exist in positive characteristic. The same behaviour occurs for equitrivial deformations.

In particular, we show how the tangent space T to the functor of equisingular (resp. equitrivial) deformations of the parameterization is the obstruction of having a theory of equisingular deformations free of pathologies. In general, in positive characteristic, one may have deformations of the equation which become induced ones from equisingular deformations of the parameterization after some finite base change. Such deformations are called weakly equingular. Among them, the strongly equisingular deformations are those of them which do not need such finite base change.

Each (non equisingular) deformation has a well defined weakly equisingular stratum. Such stratum becomes smooth if the original deformation is a versal one of a plane curve singularitiy. Using those strata we show how one has T= (0) if and only if all weakly equisingular (resp. trivial) deformations are strong ones. Finally, using calculations with Hamburger-Noether symbolic expressions, we show that the condition T= (0) is true for generic moduli.

## Invariants of symmetric bundles

Ph. Cassou-Noguès (Bordeaux, France)

This is a joint work with M. J. Taylor and B. Erez. We establish comparison results between the Hasse-Witt invariants $w_t(E)$ of a symmetric bundle $E$ over a scheme and the invariants of one of its twists $E_\alpha$. For general twists we describe the difference between $w_t(E)$ and $w_t(E_\alpha)$ up to terms of degree 3. Next we consider a special kind of twist, which has been studied by A. Fröhlich. This arises from twisting by a cocycle obtained from an orthogonal representation. A simple important example of this twisting procedure is the bilinear trace form of an étale algebra, which is obtained by twisting the standard/sum of squares form by the orthogonal representation attached to the algebra. We show how to explicitly describe the twist for representations arising from very general tame actions. This involves the "square root of the inverse different" which Serre, Esnault, Kahn, Viehweg and ourselves had studied before. For torsors we show that, in our geometric set-up, Jardine's generalization of Fröhlich's formula holds. Namely let $(X, G)$ be a torsor with quotient $Y$, let $E$ be a symmetric bundle over $Y$, let $\rho : G \to \mathbf{O}(E)$ be an orthogonal representation and let $E_{\rho,X}$ be the corresponding twist of $E$, then we verify up to degree 3 that the formula $w_t(E_{\rho,X})sp_t(\rho) = w_t(E)w_t(\rho)$ holds. Here $sp_t(\rho)$ and $w_t(\rho)$ are respectively the spinor invariant and the Stiefel-Whitney class of $\rho$. The case of genuinely tamely ramified actions is geometrically more involved and leads us to introduce an invariant of ramification, which in a sense gives a decomposition in terms of representations of the inertia groups of the invariant introduced by Serre for curves. The comparison result in the tamely ramified case proceeds by reduction to the case of a torsor. Finally we will indicate how some of the previous results can be generalized to the case where $G$ is a non constant group scheme.

## Rings in which elements are uniquely the sum of an idempotent and a unit that commute

Jianlong Chen, Zhou Wang, and Yiqiang Zhou (Nanjing, P.R.China)

A ring is called uniquely clean if every element is uniquely the sum of an idempotent and a unit. The rings described by the title include uniquely clean rings, and they arise as triangular matrix rings over commutative uniquely clean rings. Various basic properties of these rings are proved and many examples are given.

## On Divisible and Torsionfree Modules

Nanqing Ding (Nanjing, China)

This talk is a report on joint work with Lixin Mao.

Let $R$ be a ring. A left $R$-module $M$ is said to be *divisible* (or *$P$-injective*) if $\mathrm{Ext}^1(R/Ra, M) = 0$ for all $a \in R$. A right $R$-module $N$ is called *torsionfree* if $\mathrm{Tor}_1(N, R/Ra) = 0$ for all $a \in R$. The definitions of divisible and torsionfree modules coincide with the classical ones in case $R$ is a commutative domain. It is clear that a right $R$-module $N$ is torsionfree if and only if the character module $N^+$ is divisible by the standard isomorphism $\mathrm{Ext}^1(R/Ra, N^+) \cong \mathrm{Tor}_1(N, R/Ra)^+$ for every $a \in R$.

In this paper, a ring $R$ is called left $P$-coherent in case each principal left ideal of $R$ is finitely presented. A left $R$-module $M$ (resp. right $R$-module $N$) is called $D$-injective (resp. $D$-flat) if $\mathrm{Ext}^1(G, M) = 0$ (resp. $\mathrm{Tor}_1(N, G) = 0$) for every divisible left $R$-module $G$. It is shown that every left $R$-module over a left $P$-coherent ring $R$ has a divisible cover; a left $R$-module $M$ is $D$-injective if and only if $M$ is the kernel of a divisible precover $A \to B$ with $A$ injective; a finitely presented right $R$-module $L$ over a left $P$-coherent ring $R$ is $D$-flat if and only if $L$ is the cokernel of a torsionfree preenvelope $K \to F$ with $F$ flat. We also study the divisible and torsionfree dimensions of modules and rings. As applications, some new characterizations of von Neumann regular rings and $PP$ rings are given.

## Derived tame and derived wild algebras

Y. A. Drozd (Kiev, Ukraina)

Let $\Lambda$ be a finite dimensional algebra over an algebraically closed field $\Bbbk$. We consider the *bounded derived category* of the category of finitely generated $\Lambda$-modules, which can be identified with the category of finite complexes of finitely generated projective $\Lambda$-modules modulo homotopy. Similarly to [2] we define derived tame and derived wild algebras. Nonformally, $\Lambda$ is called *derived tame* if, for any fixed dimensions of the components of such a complex, the non-isomorphic indecomposble complexes form at most one-parameter families; it is called *derived wild* if the classification problem for such complexes contains the classification problem for all representations of any finitely generated $\Bbbk$-algebra. (The formal definitions can be found in [1, 3, 5].) The following main result has been proved by V. Bekkert and the author [1, 3, 5].

**Theorem.** *Every finite dimensional algebra over an algebraically closed field is either derived tame or derived wild (never both).*

We also introduce the *parameter number* defining a complex with the prescribed dimensions of the components and prove the following result [3, 4, 5].

**Theorem.** *The parameter number is upper semicontinuous in flat families of algebras.*

**Corollary 1.** A flat deformation of a derived tame algebra is derived tame. Equivalently, a flat degeneration of a derived wild algebra is derived wild.

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# Algebras with skew-symmetric identity of degree 3

A. S. Dzhumadil'daev (Almaty, Kazachstan)

Algebras with one of the following identities are considered:

$$[[t_1,t_2],t_3]+[[t_2,t_3],t_1]+[[t_3,t_1],t_2]=0, \text{(Lie-Admissible)}$$

$$[t_1,t_2]t_3+[t_2,t_3]t_1+[t_3,t_1]t_2=0, \text{ (0-Lie-Admissible (shortly 0-Alia))}$$

$$\{[t_1,t_2],t_3\}+\{[t_2,t_3],t_1\}+\{[t_3,t_1],t_2\}=0,$$

$$\text{(1-Lie-admissible (shortly 1-Alia))}$$

where $[t_1,t_2]=t_1t_2-t_2t_1$ and $\{t_1,t_2\}=t_1t_2+t_2t_1$. For algebra $A=(A,\circ)$ with multiplication $\circ$ denote by $A^{(q)}$ an algebra with vector space $A$ and multiplication $a\circ_q b=a\circ b+q\,b\circ a$.

**Theorem** *Any algebra with a skew-symmetric identity of degree* 3 *is (anti)-isomorphic to one of the following algebras:*

*Lie-admissible algebra*
0-*Alia algebra*
1-*Alia algebra*
*algebra of the form* $A^{(q)}$ *for some left-Alia algebra* $A$ *and* $q\in K$, *such that* $q^2\neq 0,1$.

Any right (left) Alia algebra is anti-isomorphic to its opposite algebra, left (right) Alia Algebra.

For anti-commutative algebra $(A,\circ)$ call a bilinear map $\psi: A\times A\to A$ *commutative cocycle*, if

$$\psi(a\circ b,c)+\psi(b\circ c,a)+\psi(c\circ a,b)=0,$$

$$\psi(a,b) = \psi(b,a),$$

for any $a, b, c \in A$.

Algebra with identities

$$[a,b]\circ c + [b,c]\circ a + [c,a]\circ b = 0$$

$$a\circ [b,c] + b\circ [c,a] + c\circ [a,b] = 0$$

is called two-sided Alia.

**Theorem 2**. *For any anti-commutative algebra $(A,\circ)$ with commutative cocycle $\psi$ an algebra $(A,\circ_\psi)$, where $a\circ_\psi b = a\circ b + \psi(a,b)$, is 1-Alia. Conversely, any 1-Alia algebra is isomorphic to algebra of a form $(A,\circ_\psi)$ for some anti-commutative algebra $A$ and some commutative cocycle $\psi$. Moreover, if $(A,\circ)$ is Lie algebra with commutative cocycle $\psi$, then $(A,\circ_\psi)$ is two-sided Alia and, conversely, any two-sided Alia algebra is isomorphic to algebra of a form $(A,\circ_\psi)$ for some Lie algebra $A$ and commutative cocycle $\psi$.*

**Theorem 3**. *Let $L$ be classical Lie algebra over a field of characteristic $p \neq 2$. Then it has non-trivial commutative cocycles only in the following cases $L = sl_2$ or $p = 3$.*

Standard construction of $q$-Alia algebras. Let $(U,\cdot)$ be associative commutative algebra with linear maps $f, g : U \to U$. Denote by $\mathcal{A}_q(U,\cdot,f,g)$ an algebra defined on a vector space $U$ by the rule

$$a\circ b = a\cdot f(b) + g(a\cdot b) - q\, f(a)\cdot b.$$

Then $\mathcal{A}_q(U,\cdot,f,g)$ is $q$-Alia.

**Example 1**. $(\mathbf{C}[x],\star)$ under multiplication $a\star b = \partial(a)\partial^2(b)$ is 1-Alia and simple.

**Example 2**. $(\mathbf{C}[x],\star)$, where $a\star b = \partial^3(a)b + 4\partial^2(a)\partial(b) + 5\partial(a)\partial^2(b) + 2a\,\partial^3(b)$, is 0-Alia and simple. It is exceptional 0-Alia algebra.

**Example 3**. Let $(\lambda_{i,j})$ be symmetric matrix. Then $(\mathbf{C}[x_1,\ldots,x_n],\star)$, where $a\star b = \sum_{\lambda_{i,j}}(\partial_i(a)\partial_j(b) + \partial_i\partial_j(a)b/2)$ is 0-Alia. It is simple iff the matrix $(\lambda_{i,j})$ is non-degenerate.

**Example 4**. Let $m$ be positive integer and $A = (\mathbf{C}[x],\star)$ an algebra with multiplication $a\star b = a\partial^m(b) - q\partial^m(a)b + q\,\partial^m(ab)$ Then $A$ is $q$-Alia and simple.

## On automorphic functions on arithmetic schemes

I. B. Fesenko (Nottingham, United Kingdom)

The classical works of Tate and of Iwasawa in which they used analytic dualities to prove the functional equation and meromorphic continuation of twisted by Galois character zeta functions of global fields was extended in the seventies, as part of activity in the Langlands programme, to the noncommutative one dimensional theory of zeta integrals over algebraic groups on global fields (Godement, Jacques, Langlands). A two dimensional version of the works of Tate and Iwasawa was developed in 2001-2005, it deals with zeta functions of regular models of elliptic curves over global fields using new translation invariant measures and integration on two dimensional local and adelic spaces associated to the models. In my talk I will report on a very recent work 2006-2007, part of which was stimulated by conversations with D. Kazhdan, H. Kim and D. Gaitsgory, on the extension of the two dimensional work to algebraic groups. In the case of arithmetic surfaces instead of one single adelic space two very different adelic spaces come into play, one for integral structures of rank 1 (1-cycles, more algebraic geometry aspects) and one for integral structures of rank 2 (0-cycles, arithmetic aspects). A very general problem is to describe a new kind of empowered arithmetic geometry on arithmetic surfaces in which the structures of rank 1 and of rank 2 are appropriately blended. I will describe one blending which leads to the definition of an object functions on which are automorphic functions on the surface, and I will explain several possible applications of this new development.

## Versal deformations of algebraic structures

Alice Fialowski (Budapest, Hungary)

Versal deformations of different objects is a basic concept in studying their properties. It decribes completely the local behaviour in the variety of a given type objects. Namely, it characterizes its all nonequivalent deformations and is unique at the infinitesimal level. In addition, versal deformations provide us with a natural division of the moduli space of given structures into families, and give a geometric picture of the moduli space. I will demonstrate the concepts in the case of Lie algebras.

## On generalized prime essential rings and special and nonspecial radicals

Halina France-Jackson (Port Elizabeth, South Africa)

In this paper all rings are associative and all classes of rings are closed under isomorphisms and contain the one-element ring 0. The fundamental definitions and properties of radicals can be found in [1] and [4]. If $\mu$ is a hereditary class of rings, $\mathcal{U}(\mu)$ denotes the upper radical generated by $\mu$ , that is, the class of all rings which have no nonzero homomorphic images in $\mu$. For any class $\mu$ of rings an ideal $I$ of a ring $R$ is called an $\mu$-ideal if the factor ring $R/I$ is in $\mu$. For a radical $\rho$ the class of all $\rho$-semisimple rings is denoted by $\mathcal{S}(\rho)$. $\pi$ denotes the class of all prime rings and $\beta$ the prime radical. The notation $I \lhd R$ means that $I$ is a two-sided ideal of a ring $R$. For $I \lhd R \in \mathcal{S}(\beta)$, $\{r \in R : rI = 0\} = \{r \in R : Ir = 0\}$ is an ideal of $R$ which we shall denote by $I^*$. An ideal $I$ of a ring $R$ is called essential in $R$ if $I \cap J \neq 0$ for any nonzero two-sided ideal $J$ of $R$. If $R \in \mathcal{S}(\beta)$ this is equivalent to $I^* = 0$. 0 is an inessential ideal. Hereditary and essentially closed class of prime rings (respectively semiprime rings) is called a special class (respectively weakly special class) and the upper radical generated by a special class (respectively weakly special class) is called a special radical (respecively supernilpotent radical). Unless otherwise stated, throughout this paper the letter $\alpha$ denotes a supernilpotent radical and $\sigma$ denotes a special class of rings.

A ring $R$ is called $(\alpha, \sigma)$-essential if $R$ is $\alpha$-semisimple and each $\sigma$-ideal $P$ of $R$ is essential in $R$. $(\alpha, \sigma)$-essential rings form a generalization of prime essential rings, that is, semiprime rings whose every $\pi$-ideal is essential. Prime essential rings were first introduced by Rowen [7] in his study of semiprime rings and their subdirect decompositions and they have been a subject of investigations of many authors (see [2], [3], [6] and [9]) since.

In this paper we show that many important results concerning prime essential rings are also valid for $(\alpha, \sigma)$-essential rings and demonstrate how $(\alpha, \sigma)$-essential rings can be used to determine whether a supernilpotent radical is special. Using $(\alpha, \sigma)$-essential rings, we generalize Ryabukhin's example of supernilpotent nonspecial radical by constructing infinitely many supernilpotent nonspecial radicals $\alpha$ such that $\mathcal{S}(\alpha) \cap \pi = \{0\}$ and show that such radicals form a sublattice of the lattice of all supernilpotent radicals.

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# Normalizers of subsystem subgroups in finite groups of Lie type

A. A. Galt (Novosibirsk, Russia)

Finite simple groups of Lie type form the main part of known finite simple groups. One of the most important subgroups in finite groups of Lie type are reductive subgroups of maximal rank. These subgroups appear as Levi factors of parabolic subgroups and centralizers of semisimple elements, and also as subgroups containing maximal tori. Moreover, reductive subgroups of maximal rank are the principal subgroups in the inductive investigation of subgroup structure in finite groups of Lie type. However, some important questions concerning internal structure of reductive subgroups of maximal rank remain unsolved.

Let $G$ be a group, $A$, $B$, $H$ are subgroups of $G$ and $B$ be a normal subgroup of $A$ ($B \trianglelefteq A$). Denote $N_H(A/B) = N_H(A) \cap N_H(B)$. If $x \in N_H(A/B)$, then $x$ induces an automorphism $Ba \mapsto Bx^{-1}ax$ of the quotient group $A/B$. Thus, there is a homomorphism of $N_H(A/B)$ into $\mathrm{Aut}(A/B)$. The image of $N_H(A/B)$ under this homomorphism is denoted by $\mathrm{Aut}_H(A/B)$ and is called a *group of induced automorphisms.*

If $R$ is a reductive subgroup of maximal rank in a finite group of Lie type $G$, then $R = T(G_1 \circ \ldots \circ G_m)$, where $T$ is a maximal torus of $G$, which is contained in $R$, and $G_1 \circ \ldots \circ G_m$ is a central product of finite groups of Lie type defined over some extension of the base field of $G$ and their ranks are less than the rank of $G$. Subgroups $G_1, \ldots, G_m$, arising in this way, are called *subsystem subgroups.* In our work we obtain the structure of $N_R(G_i)$ in terms of $\mathrm{Aut}_R(G_i)$.

## Reconstructing schemes from abelian categories

Grigory Garkusha (Swansea, United Kingdom)

Let $\mathrm{Mod}\, R$ (respectively $\mathrm{QGr}\, A$) denote the category of $R$-modules (respectively graded $A$-modules modulo torsion modules) with $R$ (respectively $A = \oplus_{n \geqslant 0} A_n$) a commutative ring (respectively a commutative graded ring).

**Theorem 2** (Classification, Garkusha-Prest, 2006)**.** *Let $R$ (respectively $A$) be a commutative ring (respectively commutative graded ring which is finitely generated as an $A_0$-algebra). Then the maps*

$$V \mapsto \mathcal{S} = \{M \in \mathrm{Mod}\, R \mid \mathrm{supp}_R(M) \subseteq V\}, \quad \mathcal{S} \mapsto V = \bigcup_{M \in \mathcal{S}} \mathrm{supp}_R(M)$$

*and*

$$V \mapsto \mathcal{S} = \{M \in \mathrm{QGr}\, A \mid \mathrm{supp}_A(M) \subseteq V\}, \quad \mathcal{S} \mapsto V = \bigcup_{M \in \mathcal{S}} \mathrm{supp}_A(M)$$

*induce bijections between*

(1) *the set of all subsets $V \subseteq \mathsf{Spec}\, R$ ( respectively $V \subseteq \mathsf{Proj}\, A$) of the form $V = \bigcup_{i \in \Omega} Y_i$ with $\mathsf{Spec}\, R \setminus Y_i$ (respectively $\mathsf{Proj}\, A \setminus Y_i$) quasi-compact and open for all $i \in \Omega$,*

(2) *the set of all torsion classes of finite type in $\mathrm{Mod}\, R$ (respectively tensor torsion classes of finite type in $\mathrm{QGr}\, A$).*

The following result says that there is a 1-1 correspondence between the finite localizations in $\operatorname{Mod} R$ and the triangulated localizations of perfect complexes $\mathcal{D}_{\mathrm{per}}(R)$.

**Theorem 3** (Garkusha-Prest, 2006)**.** *Let $R$ be a commutative ring. The map*

$$\mathcal{S} \mapsto \mathcal{T} = \{X \in \mathcal{D}_{\mathrm{per}}(R) \mid H_n(X) \in \mathcal{S} \textit{ for all } n \in \mathbf{Z}\}$$

*induces a bijection between*

*the set of all torsion theories of finite type in* $\operatorname{Mod} R$*,*
*the set of all thick subcategories of* $\mathcal{D}_{\mathrm{per}}(R)$*.*

We consider the lattices $L_{\mathrm{Serre}}(\operatorname{Mod} R)$ and $L_{\mathrm{Serre}}(\operatorname{QGr} A)$ of (tensor) torsion classes of finite type in $\operatorname{Mod} R$ and $\operatorname{QGr} A$, as well as their prime ideal spectra $\mathsf{Spec}(\operatorname{Mod} R)$ and $\mathsf{Spec}(\operatorname{QGr} A)$. These spaces come naturally equipped with sheaves of rings $\mathcal{O}_{\operatorname{Mod} R}$ and $\mathcal{O}_{\operatorname{QGr} A}$. The following result says that the schemes $(\mathsf{Spec}\, R, \mathcal{O}_R)$ and $(\mathsf{Proj}\, A, \mathcal{O}_{\mathsf{Proj}\, A})$ are isomorphic to $(\mathsf{Spec}(\operatorname{Mod} R), \mathcal{O}_{\operatorname{Mod} R})$ and $(\mathsf{Spec}(\operatorname{QGr} A), \mathcal{O}_{\operatorname{QGr} A})$ respectively.

**Theorem** (Reconstruction, Garkusha-Prest, 2006)**.** *Let $R$ (respectively $A$) be a commutative ring (respectively commutative graded ring which is finitely generated as an $A_0$-algebra). Then there are natural isomorphisms of ringed spaces*

$$(\mathsf{Spec}\, R, \mathcal{O}_R) \xrightarrow{\sim} (\mathsf{Spec}(\operatorname{Mod} R), \mathcal{O}_{\operatorname{Mod} R})$$

*and*

$$(\mathsf{Proj}\, A, \mathcal{O}_{\mathsf{Proj}\, A}) \xrightarrow{\sim} (\mathsf{Spec}(\operatorname{QGr} A), \mathcal{O}_{\operatorname{QGr} A}).$$

A scheme $X$ is *locally coherent* if it can be covered by open affine subsets $\mathsf{Spec}\, R_i$, where each $R_i$ is a coherent ring. $X$ is *coherent* if it is locally coherent, quasi-compact and quasi-separated. By $\operatorname{coh}(X)$ we denote the category of coherent sheaves on $X$.

**Proposition 4.** *Let $X$ be a quasi-compact, quasi-separated scheme. Then $X$ is coherent if and only if the category of coherent sheaves* $\operatorname{coh}(X)$ *is abelian.*

The next result was proven by Gabriel for noetherian schemes.

**Theorem 5** (Classification)**.** *Let $X$ be a coherent scheme. Then the maps*

$$V \mapsto \mathcal{S} = \{\mathcal{F} \in \operatorname{coh}(X) \mid \operatorname{supp}_X(\mathcal{F}) \subseteq V\}$$

*and*

$$\mathcal{S} \mapsto V = \bigcup_{\mathcal{F}\in\mathcal{S}} \operatorname{supp}_X(\mathcal{F})$$

*induce bijections between*

*the set of all subsets of the form* $V = \bigcup_{i\in\Omega} V_i$ *with quasi-compact open complement* $X \setminus V_i$ *for all* $i \in \Omega$*,*
*the set of all tensor Serre subcategories in* $\operatorname{coh}(X)$*.*

**Theorem 6.** *Suppose* $X$ *is a coherent scheme. The map*

$$\mathcal{S} \mapsto \mathcal{T} = \{\mathcal{X} \in \mathcal{D}_{\text{per}}(X) \mid H_n(\mathcal{X}) \in \mathcal{S} \text{ for all } n \in \mathbf{Z}\}$$

*induces a bijection between*

*the set of all tensor Serre subcategories in* $\operatorname{coh}(X)$*,*
*the set of all tensor thick subcategories of* $\mathcal{D}_{\text{per}}(X)$*.*

**Theorem 7** (Reconstruction)**.** *Suppose* $X$ *is a coherent scheme. There is an isomorphism of ringed spaces*

$$(X, \mathcal{O}_X) \xrightarrow{\sim} (\mathsf{Spec}(\operatorname{coh}(X)), \mathcal{O}_{\operatorname{coh}(X)}),$$

*where* $(\mathsf{Spec}(\operatorname{coh}(X)), \mathcal{O}_{\operatorname{coh}(X)})$ *is a ringed space associated to the lattice* $L_{\text{Serre}}(\operatorname{coh}(X))$ *of tensor Serre subcategories in* $\operatorname{coh}(X)$*.*

## Cohomology of algebras with small complexity

A. I. Generalov (St. Petersburg, Russia)

The complexity of an algebra $R$ is a rate of growth of dimensions of modules in minimal projective resolutions of simple $R$-modules. We announce recent results on calculation of cohomology algebras of some families of algebras with “small” complexity, namely,

1) Yoneda algebras for several families of dihedral, or semidihedral type (in classification of K. Erdmann [1]);

2) Hochschild cohomology algebra for certain families of dihedral, or quaternion type.

In this calculation we use direct methods of construction of resolutions of suitable modules. Such methods were earlier developed in [2, 3].

Using similar technique we describe also Hochschild cohomology algebra for the integer group ring $\mathbb{Z}D_{4m}$ of the dihedral group $D_{4m}$ of order $4m$.

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# On Hochschild cohomology of algebras of quaternion type with two vertices

A. I. Generalov, A. A. Ivanov, and S. O. Ivanov (St. Petersburg, Russia)

Let $K$ be an algebraically closed field, and let $Q(2B)_1(k,s,a,c)$ be a family of $K$-algebras of quaternion type from classification of K. Erdmann [1]. We describe the Hochschild cohomology algebra $\mathrm{HH}^*(R)$ of an algebra $R$ in this family over the field $K$ with characteristic 2. We distinguish several cases (depending on such conditions as $k$ and $s$ are even or odd, $c$ is zero or nonzero). In the each of these cases we pick out generators for the algebra $\mathrm{HH}^*(R)$ and write down relations which are satisfied by the corresponding generators. Using results in [2] we extend the description of the algebra $\mathrm{HH}^*(R)$ to all algebras of quaternion type with two simple modules.

The calculation of the algebra $\mathrm{HH}^*(R)$ is based on the construction of the (4-periodic) bimodule minimal projective resolution of the algebra $R = Q(2B)_1(k,s,a,c)$ (over the field $K$ with arbitrary characteristic). This resolution was discovered independently of [3].


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# On characterization of finite simple linear groups by spectrum

M. A. Grechkoseeva (Novosibirsk, Russia)

The spectrum $\omega(G)$ of a finite group $G$ is the set of element orders of $G$. A finite group is called recognizable by spectrum if every finite group $H$ with $\omega(H) = \omega(G)$ is isomorphic to $G$. If denote by $h(G)$ the number of pairwise non-isomorphic finite groups $H$ with $\omega(H) = \omega(G)$, then $G$ is recognizable if and only if $h(G) = 1$. If $1 < h(G) < \infty$ then $G$ is said to be almost recognizable, and if $h(G) = \infty$ that $G$ is said to be non-recognizable. We say that the recognizability question is solved for a group $G$ if the value of $h(G)$ is known.

Since a finite group with nontrivial normal soluble subgroup is non-recognizable, each recognizable or almost recognizable group is an extension of the direct product $M$ of several nonabelian simple groups by some subgroup of $\mathrm{Out}(M)$. For this reason of prime interest is the recognition problem for simple and almost simple groups. The first examples of recognizable by spectrum simple groups, namely, the alternating group $A_5$ and the projective special linear group $L_2(7)$, appeared in works by Shi Wujie in the middle of 1980th after the classification of finite simple groups had been announced to be complete. These works initiated numerous investigations on the subject. Here we consider the recognition problem for simple linear groups over fields of characteristic 2.

At present the recognizability question is solved for $L_2(2^m)$ [1], $L_3(2^m)$ [2], $L_n(2^m)$, $n = 2^k > 16$ [3], and $L_n(2)$ [4]. It turns out all these groups are recognizable by spectrum. We prove that finite simple linear groups over fields of characteristic 2 are recognizable or almost recognizable providing their dimension is sufficiently large.

**Theorem.** *Let $L = L_n(q)$ where $q$ is even and $n > 24$. Then $h(L) < \infty$. Moreover, if $n$ and $q-1$ are coprime then $h(L) = 1$.*

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## Non-additive geometry

M. Shai Haran (Haifa, Israel)

We develop a language that makes the analogy between geometry and arithmetic more transparent. In this language there exists a base field $F$, "the field with one element". There is a fully faithful functor from commutative rings to "F-rings". There is a notion of the $F$-ring of "integers"of a real or complex prime of a number field $K$ (analogous to the $p$-adic integers), and there is a compactification of the spectrum of the ring of integers of $K$. There is a notion of tensor product of $F$-rings giving rise to arithmetical surface relevant for the study of arithmetical problems whose geometric analogues are well known (such as ABC and the Riemann hypothesis).

## Subgroups of unitriangular groups of infinite matrices

Waldemar Hołubowski (Gliwice, Poland)

In our talk we show, that for any associative ring $R$, the subgroup $UT_r(\infty, R)$ of row finite matrices in $UT(\infty, R)$ – the group of all infinite dimensional (indexed by $\mathbb{N}$) upper unitriangular matrices over $R$, is generated

by strings (block-diagonal matrices with finite blocks along the main diagonal). We note that similar result is true for the group of infinite invertible banded matrices for any associative ring $R$ [1]. This allows us to define a large family of subgroups of $UT_r(\infty, R)$ associated to some growth functions. The smallest subgroup in this family, called the group of banded matrices, is generated by 1-banded simultaneous elementary transvections (a slight generalization of the usual notion of elementary transvections). We introduce a notion of net subgroups and characterize the normal net subgroups of $UT(\infty, R)$ [2].

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## Artinian rings with supersolvable adjoint group

R. Iu. Ievstafiev (Kiev, Ukraina)

Let $R$ be an associative ring, not necessarily with a unit element. The set of all elements of R forms a monoid under the circle operation $a \circ b = a + b + ab$ for all $a$ and $b$ from $R$. The group of all invertible elements of this monoid is called the *adjoint group* of $R$ and is denoted by $R^\circ$. Obviously, if $R$ has a unity 1, then $1 + R^\circ$ coincides with the multiplicative group $R^*$ of $R$ and the mapping $r \mapsto 1 + r$ with $r \in R^\circ$ is an isomorphism from $R^\circ$ onto $R^*$.

Recall that every associative ring $R$ can also be viewed as a Lie ring under the Lie multiplication $[a, b] = ab - ba$ for all $a, b \in R$. For additive subgroups $V$ and $W$ of $R$, we denote by $[V, W]$ the additive subgroup of $R$ generated by all Lie commutators $[v, w]$ with $v \in V$ and $w \in W$. The subgroup $V$ is a *Lie ideal* of $R$ if $[V, R] \subseteq V$.

The derived chain of a Lie ring $R$ is defined inductively as $\delta_0(R) = R$ and $\delta_{n+1}(R) = [\delta_n(R), \delta_n(R)]$ for each integer $n \geqslant 0$. The ring $R$ is called *Lie solvable* if $\delta_m(R) = 0$ for some $m \geqslant 1$. Recall also that solvable groups are defined in a corresponding way, where the usual group commutator replaces the Lie commutator. A group is *supersolvable* if it contains a finite invariant series with cyclic sections. Similarly to this definition, we say that a ring is *Lie supersolvable* if it has a finite chain of Lie ideals with cyclic sections.

We shall say that a ring is *artinian* if it satisfies the minimal condition for its right or left ideals. Denote by $J(R)$ and $Z(R)$ the Jacobson radical and the center of a ring $R$, respectively. Following Jacobson [1], a ring

$R$ is called *radical* if $R = R^\circ$, which means that $R$ coincides with its Jacobson radical. A ring $R$ with a unit element is *local* if $R$ modulo its Jacobson radical is a division ring.

It was shown by Eldridge [2] that an artinian ring whose adjoint group is finitely generated and solvable must be finite. Therefore in the case of artinian rings with the supersolvable adjoint group we may only consider finite rings. By an immediate consequence from [3, Corollary B], every finite ring $R$ with the supersolvable adjoint group can be written in the form $R = S \oplus T$, where either $S = 0$ or $S = \mathrm{M}_2(\mathbb{F}_2) \oplus \ldots \oplus \mathrm{M}_2(\mathbb{F}_2)$ and $T$ is commutative modulo $J(R)$. The following theorem supplements this result.

**Theorem A.** *Let $R$ be a finite ring. If the adjoint group $R^\circ$ is supersolvable and the factor ring $R/J(R)$ is commutative, then $R$ is a direct sum of ideals $R_i$ each of which is a ring of one from the following types:*

*(1) $R_i$ is a nilpotent ring;*

*(2) the factor ring $R_i/J(R_i)$ is a direct sum of fields of the same prime order;*

*(3) $R_i$ is a local ring such that $R_i = Z(R_i) + J(R_i)$ and the factor ring $R_i/J(R_i)$ is a field that differs from its simple subfield.*

It was proved by Zalesskii and Smirnov [4] that if a ring $R$ of characteristic 0 with a unit element is Lie solvable, then its multiplicative group $R^*$ is solvable. However, for arbitrary rings, the corresponding result does not hold. For instance, the ring $R$ of all $(2\times 2)$-matrices over the field $F(x)$, where $F(x)$ is the field of rational functions over the Galois field $\mathbb{F}_2$, satisfies the equality $[\delta_2(R), R] = 0$, whereas the multiplicative group $R^*$ contains a non-abelian free subgroup and so is non-solvable.

On the other hand, Amberg and Sysak [5] have shown that every radical ring with the solvable adjoint group must be Lie solvable. Furthermore, the adjoint group of any Lie metabelian ring is metabelian by a result of Krasil'nikov [6]. Now we can pose the following question. What correlation exists between the Lie supersolvability of an associative ring and the supersolvability of its adjoint group? In particular, the next example shows that the adjoint group of a Lie supersolvable ring is not neccesarily supersolvable.

**Example.** *There exists a finite local Lie supersolvable ring with the non-supersolvable adjoint group.*

At the same time we have the following result.

**Theorem B.** *Let $R$ be a finite ring. If the adjoint group $R^\circ$ is supersolvable, then $R$ is Lie supersolvable.*

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## Subregular subsets and subregular characters of the unitriangular group

M. V. Ignatev (Samara, Russia)

Let $k = \mathbb{F}_q$, $\operatorname{char} k = p$. Let $G = G_n(k)$ be the group of all strictly lower-triangular matrices with 1's on the main diagonal; we assume that $p \geqslant n$. The orbit method establishes one-to-one correspondence between irreducible complex characters and coadjoint orbits of $G$ [3]. Orbits of maximum dimension $\mu(n)$ are described in [4], correspondent characters are described in [1]. *Subregular* orbits (i.e., orbits of dimension $= \mu(n)-2$) and correspondent characters also play an important role. Subregular orbits are described in [2]. Here we give an explicit description of subregular characters.

By $\Phi = \Phi(n)$ denote the set of all pairs $(i,j)$, $1 \leqslant j < i \leqslant n$. Let $n_1 = [(n-1)/2]$. Let $f = (\xi_{ij})$ be the *canonical form* on a subregular orbit, $1 \leqslant d = j_0 < n_1$ (see [2, Theorem 3.3]), and $\chi = \chi_f$ be the correspondent character. Denote $\operatorname{Supp}(f) = \{(i,j) \in \Phi \mid f(e_{ij}) \neq 0\}$, $\alpha = (n-d+1, n-d), \gamma = (n-d+1, d)$. We say that $D \subset \Phi$ is *$d$-subregular* if $D \subset \operatorname{Supp}(f) \cup \alpha \cup \gamma$.

Given any $d$-subregular subset $D$ and any map $\varphi\colon D \to \mathbb{F}_q^*\colon (i,j) \mapsto \varphi_{ij}$, we define $e_D(\varphi) = \sum_{(i,j)\in D} \varphi_{ij} e_{ij} \in \mathfrak{g}$, $x_D(\varphi) = 1_n + e_D(\varphi) \in G$, $\mathcal{K}_D(\varphi) = \{g \cdot x_D(\varphi) \cdot g^{-1}, g \in G\}$ and

$$\mathcal{K}_f = \bigcup_{\substack{D\in S_d\\ \varphi\in M_f(D)}} \mathcal{K}_D(\varphi),$$

where $S_d$ is the set of all $d$-subregular subsets of $\Phi$ and $M_f(D) = \{\varphi\colon D \to \mathbb{F}_q^* \mid \xi_{d,n-d}\varphi_{d+1,d} = \xi_{d+1,n-d+1}\varphi_{n-d+1,n-d}\}$. Note that all $\mathcal{K}_D(\varphi)$ can be described in terms of coefficients of minors of the characteristic matrix.

Let

$R(D) = \{(i,j) \in \Phi \mid (i,k),(k,j) \notin D \text{ for all } i<k<j\}$,

$\Phi_d = \{(i,j) \in \Phi(n) \mid i > n-j+1, j \notin \{d, n-d\}, i \notin \{n-d+1, n-d\}\}$,

$\Phi_{\text{reg}} = \{(i,j) \in \Phi(n) \mid i > n-j+1\}$,

$$m_D = \begin{cases} |R(D)\cap\Phi_{\text{reg}}| - 1, & \text{if } \alpha \notin D,\\ |R(D)\cap\Phi_d| + n - 2d - 1, & \text{if } \alpha \in D.\end{cases}$$

Fix any non-trivial character $\theta$ of $\mathbb{F}_q$ and put $\theta_f\colon \mathfrak{g} \to \mathbb{C}\colon x \mapsto \theta(f(x))$.

**Theorem**. Let $g \in G$. If $g \notin \mathcal{K}_f$ then $\chi(g) = 0$. If $g \in \mathcal{K}_D(\varphi) \subset \mathcal{K}_f$ then $\chi(g) = q^{m_D} \cdot \theta_f(e_D(\varphi))$.

This theorem can be proved by Mackey method of semi-direct decomposition of $G$ and by induction on $n$. The case $d = n_1$ is studied similarly.

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## A Characterizaton of $^2D_n(q)$ by order of normalizer of Sylow subgroups

A. Iranmanesh and N. Ahanjideh (Tehran, Iran)

In 1992, *Bi* showed that *projective special linear group* $L_2(q)$ can be characterized by the order of normalizer of its Sylow subgroups. This type of characterization is done for the following groups: *Projective special linear group* $L_n(q)$ *such that* $n \geqslant 3$, *projective symplectic group* $S_4(q)$, *alternating groups*, *Janko groups*, *Mathieu groups* and $U_n(q)$.
Let $(V, f)$ be a orthogonal space, where $V = V_{2n}(q)$, f is a nondegenerate orthogonal form and there are maximum $(n-1)$ distinct hyperplanes in V. Define $SO^-_{2n}(q) = \{A \in SL_{2n}(q) | f(Av, Aw) = f(v, w)$ for all $v$, $w \in V\}$ and $^2D_n(q) = \Omega^-_{2n}(q)/Z$, where $\Omega^-_{2n}(q) = (SO^-_{2n}(q))'$ and Z is the center of $\Omega^-_{2n}(q)$. In this paper, we characterize the simple group $^2D_n(q)$ by the order of normalizer of its Sylow subgroups. In fact we proved the following Theorem:
**Theorem**. Let G be a finite group. If $|N_G(R)| = |N_{^2D_n(q)}(\bar{R})|$ for every prime r, with $n \geqslant 2$, then $G \cong {}^2D_n(q)$, where $R \in Syl_r(G)$ and $\bar{R} \in Syl_r(^2D_n(q))$.

## Noether's problem for some non-abelian p-groups

Ming-chang Kang (Taipei, Taiwan)

Let $K$ be any field and $G$ be a finite group. Let $G$ act on the rational function field $K(x_g : g \in G)$ by $K$-automorphisms and $h \cdot x_g = x_{hg}$ for any $g, h \in G$. Define $K(G) = K(x_g : g \in G)^G$ to be the fixed field of $K(x_g : g \in G)$ under the action of $G$. Noether's problem asks under what situations the field $K(G)$ is rational (= purely transcendental ) over $K$.

Noether's problem is related to the inverse Galois problem which asks whether there is a Galois extension field $L$ over $K$ with $Gal(L/K) \simeq G$, provided that $K$ is a prescribed algebraic number field and $G$ is a prescribed finite group. In fact, if $K(G)$ is rational over $K$, it follows that $K(G)$ is retract rational (equivalently, there exists a generic Galois $G$-extension over $K$), which will guarantee the existence of the inverse Galois problem for $K$ and $G$.

In this talk we will discuss Noether's problem, both rationality and retract rationality, for some non-abelian p-groups.

## Twisted K-theory

Max Karoubi (Paris, France)

Twisted $K$-theory has its origins in the author's PhD thesis (http://www.numdam.org/item?id=ASENS_1968_4_1_2_161_0) and in the paper with P. Donovan <http://www.numdam.org/item?id=PMIHES_1970__38__5_0>

The objective of this lecture is to revisit the subject in the light of new developments inspired by Mathematical Physics. See for instance E. Witten (hep-th/9810188), J. Rosenberg (http://anziamj.austms.org.au/JAMSA/V47/Part3/Rosenberg.html), C. Laurent-Gentoux, J.-L. Tu, P. Xu (ArXiv math/0306138) and M.F. Atiyah, G. Segal (ArXiv math/0407054), among many authors.

## Natural differential operations on manifolds and algebraic groups

P. I. Katsylo, D. A. Timashev (Moscow, Russia)

Let $M$ be a differentiable $n$-dimensional manifold over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\mathcal{V}, \mathcal{W} \to M$ be two tensor bundles over $M$. Let $\Gamma(\mathcal{V})$ denote the space of global sections of a bundle $\mathcal{V} \to M$.

**Definition.** A *natural differential operation* (*NDO*) of order $k$ from $\mathcal{V}$ to $\mathcal{W}$ is a map $D : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{W})$ such that:

(1) In local coordinates, $(Dv)^\lambda = \delta^\lambda(\{x^i, v^\mu, \partial_1^{l_1} \cdots \partial_n^{l_n} v^\mu\})$, where $v \in \Gamma(\mathcal{V})$, $\lambda, \mu$ are multi-indices of tensor coordinates, $x^i$ ($i = 1, \dots, n$) are coordinates on $M$, and $\partial_i = \partial/\partial x^i$, $1 \leqslant l_1 + \cdots + l_n \leqslant k$;
(2) $\delta^\lambda$ are polynomials in $v^\mu, \partial_1^{l_1} \cdots \partial_n^{l_n} v^\mu$;
(3) $\delta^\lambda$ do not change under any coordinate transform.

It is easy to see that condition (3) implies that the coefficients of the polynomials $\delta^\lambda$ are constant numbers which do not depend on $x^i$'s. It also implies that the local formulæ (1) may be used to define the NDO $D$ between any tensor bundles of the same type as $\mathcal{V}, \mathcal{W}$ on any $n$-manifold.

More generally, NDO's may be defined on an open subbundle of $\mathcal{V}$ consisting of "nondegenerate" tensors (in a certain sense). Then it is natural to replace (2) by a condition that $\delta^\lambda$ are rational functions whose denominators depend only on $v^\mu$'s.

NDO's are important in differential geometry since they have an intrinsic geometric or "physical" meaning, being independent of the choice of local coordinates.

**Examples:**

(1) NDO's of order 0 are just *tensor operations* (contraction, permutation of indices, tensoring by the Kronecker delta, and their combinations).
(2) The exterior differential $d$ is an NDO of order 1 from $\bigwedge^p T^*$ to $\bigwedge^{p+1} T^*$, where $T$ denotes the tangent bundle.
(3) The commutator of vector fields is an NDO of order 1 from $T \times T$ to $T$.
(4) The Riemann curvature tensor is an NDO of order 2 from the bundle of Riemannian metrics $(S^2T^*)^+$ to $T \otimes (T^*)^{\otimes 3}$.

We propose an algebraic approach to the study of natural differential operations. It is based on an observation that an NDO $D : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{W})$ of order $k$ defines a polynomial (or rational) map $\delta : V^{(k)} := V \otimes J_n^{(k)} \to W$, where $V, W$ are typical fibers of $\mathcal{V}, \mathcal{W}$ (i.e., certain spaces of tensors over $\mathbb{K}^n$) and $J_n^{(k)} = \mathbb{K}[x^1, \ldots, x^n]/(x^1, \ldots, x^n)^{k+1}$ is the truncated polynomial algebra of $k$-jets of functions. Furthermore, coordinate transforms induce an action of $GL_n^{(k+1)}$ on $V^{(k)}$, where $GL_n^{(k)} = \mathrm{Aut} J_n^{(k)}$ is the group of $k$-jets of diffeomorphisms of $\mathbb{K}^n$ fixing 0. The differential operation $D$ is natural iff $\delta$ is $GL_n^{(k+1)}$-equivariant, where $GL_n^{(k+1)}$ acts on $W$ through its quotient group $GL_n^{(1)} = GL_n$.

Note that there is a Levi decomposition $GL_n^{(k)} = GL_n \leftthreetimes NGL_n^{(k)}$, where $NGL_n^{(k)}$ consists of jets of diffeomorphisms with identical differential. Thus $\delta$ must be $GL_n$-equivariant and $NGL_n^{(k+1)}$-invariant. We conclude that the study (e.g., classification) of NDO's reduces to a problem in invariant theory of (non-reductive) algebraic groups.

Let us formulate our main results. The first one is a finiteness theorem:

**Theorem.** *Fix a tensor bundle* $\mathcal{V}$ *and consider NDO's* $D : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{W})$ *of order* $\leqslant k$ *and of degree* $\leqslant d$ *in partial derivatives* $\partial_1^{l_1} \cdots \partial_n^{l_n} v^\mu$. *Then there exists finitely many tensor bundles* $\mathcal{W}_1, \ldots, \mathcal{W}_N$ *and NDO's* $D_p : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{W}_p)$ *such that every* $D$ *is representable as* $Dv = = \sum_{p,q} \Phi_{pq}(D_p v \otimes v^{\otimes q})$, $\forall v \in \Gamma(\mathcal{V})$, *where* $\Phi_{pq}$ *are linear tensor operations.*

Our next result is a short conceptual proof for the classification of linear NDO's due to Schouten, Rudakov, Chuu-Lian Terng, and Kirillov:

**Theorem.** *Every linear NDO of order* $> 0$ *is a composition of the exterior differential with tensor operations.*

To formulate the 2nd classification theorem, we need to extend the definition of NDO to varieties with an additional structure, by which we mean a tensor field $\omega$ satisfying some natural differential equation and/or nondegeneracy condition (e.g., a symplectic form). We extend Definition 3 by assuming that $\delta^\lambda$ also depend on the components of $\omega$ and their partial derivatives.

**Theorem.** *Every linear NDO of order* $> 0$ *on symplectic manifolds is a composition of the exterior differential* $d$ *or symplectic Laplacian* $dd^*$ *with tensor operations.*

Let us explain the idea of our proofs. Without loss of generality we may assume that $V, W$ are irreducible $GL_n$-modules. Given an equivariant linear map $\delta : V^{(k)} \to W$, we prove that $S^k(\mathbb{K}^n)^* \otimes V \subset \operatorname{Ker} \delta$ by showing that $S^k(\mathbb{K}^n)^* \otimes V$ is spanned by $\xi \cdot v$, where $\xi$ is in the Lie algebra of $NGL_n^{(k)}$ and $v \in V^{(k)}$. Thus we reduce the order of the linear operation to 1 (resp. 2) and even to 0, unless $D = d$ or $dd^*$.

Our final result concerns deformation quantization. Recall:

**Definition.** Let $M$ be a Poisson variety with the Poisson bracket $\{f, g\} = \beta(df, dg)$, where $\beta \in \Gamma(\bigwedge^2 T)$ is the Poisson bivector, and $\mathcal{F}$ be the sheaf of differentiable functions. A *deformation quantization* on $M$ is an associative product $\star$ on the sheaf $\mathcal{F}[[\bar{h}]]$ of formal power series with differentiable coefficients that is $\mathbb{K}[[\bar{h}]]$-linear with respect to infinite formal sums and is defined on $\mathcal{F}$ by a formula

$$(\ddagger) \qquad f \star g = fg + \bar{h}\{f, g\} + \cdots + \bar{h}^m \beta_m(f, g) + \cdots ,$$

where $\beta_m$ $(m \geqslant 2)$ are bilinear differential operators.

There are several constructions of deformation quantization on symplectic and Poisson manifolds, and each of them requires some additional structure like symplectic connection etc. We justify this principle by proving the following:

**Theorem.** *There exist no NDO's $\beta_m$, $m = 2, \dots,$ on a Poisson manifold such that Formula (‡) defines an associative product on $\mathcal{F}[[\bar{h}]]$. There even does not exist NDO $\beta_2$ such that $\star$ in (‡) is associative* $\bmod \bar{h}^3$.

## PI Index of Cartesian Product Graphs

M. H. Khalifeh and H. Yousefi-Azari (Tehran, Iran), A. R. Ashrafi (Kashan, Iran)

Let $G$ be a connected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. As usual, the distance between the vertices $u$ and $v$ of $G$ is denoted by $d(u, v)$ and it is defined as the number of edges in a minimal path connecting the vertices $u$ and $v$.

A topological index is a real number related to a graph. It must be a structural invariant, i.e., it preserves by every graph automorphisms. The Wiener index W is the first topological index to be used in mathematics. Usage of topological indices began in 1947 when Harold Wiener developed the most widely known topological descriptor, the Wiener index, and used it to determine physical properties of types of alkanes known as paraffins. In a graph theoretical language, the Wiener index is equal to the count of all shortest distances in a graph.

Let $G$ be a graph and $e = uv$ an edge of $G$. $n_{eu}(e|G)$ denotes the number of edges lying closer to the vertex $u$ than the vertex $v$, and $n_{ev}(e|G)$ is the number of edges lying closer to the vertex $v$ than the vertex $u$. The Padmakar-Ivan (PI) index of a graph $G$ is defined as $\mathrm{PI}(G) = \sum_{e\in E(G)}$ $[n_{eu}(e|G){+}n_{ev}(e|G)]$. In this definition, edges equidistant from both ends of the edge $e = uv$ are not counted. We call This index, edge PI index and denote by $PI_e(G)$. We also define vertex PI index of $G$, $PI_v(G)$, as the sum of $[m_{eu}(e|G) + m_{ev}(e|G)]$ over all edges of $G$, where $m_{eu}(e|G)$ is the number of vertices lying closer to the vertex $u$ than the vertex $v$

and $m_{ev}(e|G)$ is the number of vertices lying closer to the vertex $v$ than the vertex $u$.

The Cartesian product $G \times H$ of graphs $G$ and $H$ has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G \times H$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $x = y$. If $G_1, G_2, \cdots, G_n$ are graphs then we denote $G_1 \times \cdots \times G_n$ by $\bigotimes_{i=1}^n G_i$. The main result of this paper is as follows:

**Theorem 1.** Let $G_1, G_2, \cdots, G_n$ be connected graphs. Then $PI_v(\bigotimes_{i=1}^n G_i) = \sum_{i=1}^{n}(\prod_{j=1, j\neq i}^{n} |V(G_i)|^2) PI_v(G_i)$, and

$$PI_e(\bigotimes_{i=1}^n G_i) = \sum_{i=1}^{n}(\prod_{j=1, j\neq i}^{n} |V(G_j)|^2) PI_e(G_i) + \sum_{i=1}^{n} PI_v(G_i) \sum_{j=1, j\neq i}^{n} |V(G_j)||E(G_j)| \prod_{k=1, k\neq i,j}^{n} |V(G_k)|^2.$$

**Corollary.** If $G$ is a connected graph then $PI_e(G^n) = |V(G)|^{2(n-1)}$ $(PI_e(G) + n(n-1)\frac{|E(G)|}{|V(G)|} PI_v(G))$ and $PI_v(G^n) = n|V(G)|^{2(n-1)} PI_v(G)$.

**Lemma 1.** Let $G$ be a graph. Then $PI_v(G) \leqslant |E(G)||V(G)|$ with equality if and only if $G$ is bipartite.

The previous lemma shows that for a tree $T$ with exactly n vertices, $PI_v(T) = n(n-1)$.

**Corollary** (Klavzar 2006). If $G$ is a bipartite connected graph then $PI_e(G^n) = n|V(G)|^{2(n-1)} PI(G) + n(n-1)|E(G)|^2 |V(G)|^{2(n-1)}$.

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# On the number of maximal 2-signalizers in finite simple symplectic and orthogonal groups

A. S. Kondratiev (Ekaterinburg, Russia)

The notion of a signalizer introduced by J. Thompson in [1] plays an important rôle in the finite group theory, in particular, in the method of signalizer functor.

If $G$ is a finite group, $p$ is a prime and $P$ is a Sylow $p$-subgroup of $G$, then any $P$-invariant $p'$-subgroup of $G$ is called *$P$-signalizer* or simply *$p$-signalizer*. The case of 2-signalizers is of the greatest interest. In connection with announcing of the classification of finite simple groups, D. Gorenstein in [2, Section 4.15] posed the question of the study of properties of signalizers in known finite simple groups.

Because $N_G(P)$ acts on the set $F_G(P)$ of all maximal $P$-signalizers of the group $G$, the following problems naturally arise:

(1) *What is the cardinality of the set $F_G(P)$?*

(2) *How many $N_G(P)$-orbits has the set $F_G(P)$?*

(3) *What $N_G(P)$-orbits in $F_G(P)$ are conjugated in $G$ (or $Aut(G)$)?*

(4) *What are the isomorphism types of elements of $F_G(P)$?*

In [3] the author jointly with V. D. Mazurov obtained the complete classification of maximal 2-signalizers in finite simple groups up to conjugacy, in particular, the solution of the problem (4). In addition, the problems (1), (2), (3) are solved for maximal 2-signalizers in all finite simple groups except symplectic and orthogonal groups. In the given work, we solve these problems for remaining cases. The following theorem is proved.

**Theorem.** *If $G$ is a symplectic or orthogonal finite simple group and $P$ is a Sylow 2-subgroup of $G$, then $F_G(P) = 1$.*

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## Affine representations of three dimensional algebraic tori

Yu. Yu. Krutikov (Samara, Russia)

Let $k$ be an arbitrary field, $k_s$ its separable closure, $G = Gal(k_s/k)$ the absolute Galois group of $k$. Recall, that an algebraic $k$-torus is an affine group scheme of the form

$$T = Spec(k_s[\widehat{T}])^G,$$

here $\widehat{T}$ is a torsion-free $G$-module of finite $\mathbb{Z}$-rank such that $\widehat{T} \cong X(T)$, where $X(T)$ is a module of rational characters of $T$. As linear algebraic group $T$ admits a regular embedding $\varphi : T \hookrightarrow GL_{n,k}$ for a suitable $n$. As affine variety $T$ can be embedded to the affine space $\mathbb{A}^m$. Let us state the following problems

(1) (the strong optimization problem) Find the minimal $m = m(T)$ such that $T$ can be regularly embedded into $\mathbb{A}^m$.
(2) (the weak optimization problem) Find the minimal $n = n(T)$ such that $T$ can be regularly embedded into $GL_{n,k}$.

In the case of quasisplit $k$-tori $m(T) = n(T) + 1$, and $n(T) = \dim T$, in the case of norm tori $m(T) = n(T) = \dim T + 1$. In the rest cases $m(T) \leqslant n(T)$ and we have found the series of examples when the last inequality is strong. Finally, we obtain the complete solution of the weak optimization problem for three dimensional algebraic tori. The result is following

**Theorem** *Let $T$ be a three dimensional $k$-torus such that $X(T)$ is an indecomposable $G$-module then the following values of $n(T)$ are possible: 3, 4, 6, 8, 12.*

We have extended this result and obtained the corresponding affine embeddings of the tori. For example, consider the maximal finite subgroups in $GL(3,\mathbb{Z})$, which are three nonequivalent integral representations of $S_4 \times S_2$. Then for the tori with these decomposition groups the affine representation are following:

$T_{34} = R_{F_3/k}\left(R^1_{F_6/F_3}(G_m)\right)$, here $F_i/k$ are non normal extensions of degree i, i=3 or 6 and $k \subset F_3 \subset F_6$.

$$T_{35} = R_{F_6^{(1)}/k}\left(R^1_{F_{12}/F_6^{(1)}}(G_m)\right) \bigcap$$
$$\bigcap ker\left(R_{L/k}(G_m) \overset{N_{L/F_{16}^{(1)}}}{\longrightarrow} R_{F_{16}^{(1)}/k}(G_m)\right) \bigcap$$
$$\bigcap ker\left(R_{L/k}(G_m) \overset{N_{L/F_{16}^{(2)}}}{\longrightarrow} R_{F_{16}^{(2)}/k}(G_m)\right) \bigcap$$
$$\bigcap ker\left(R_{L/k}(G_m) \overset{N_{L/F_6^{(2)}}}{\longrightarrow} R_{F_6^{(2)}/k}(G_m)\right),$$

here $F_{12}/k$ is a non normal extension of degree 12, $F_i^{(j)}/k$ are non normal extensions of degree $i$, $i$=6 or 16, $j$=1 or 2 and $k \subset F_6^{(1)} \subset F_{12}$.

$T_{36} = R_{F_4/k}\left(R^1_{F_8/F_4}(G_m)\right) \bigcap R_{F_2/k}\left(R^1_{F_8/F_2}(G_m)\right)$,

here $F_i/k$ are non normal extensions of degree $i$, $i$=4 or 8 and $k \subset F_4 \subset F_8$.

Such affine representations of three dimensional tori allow us to provide the following calculations in an explicit form using Computer Algebra

description of groups $T(F)$ of $F$-points for a torus $T$
finding the parameterization of rational tori $T$.

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## Chisini's Conjecture for generic projections of the surfaces and applications

V. S. Kulikov (Moscow, Russia)

Let $B \subset \mathbb{P}^2$ be an irreducible plane algebraic curve over $\mathbb{C}$ with ordinary cusps and nodes, as the only singularities. Denote by $2d$ the degree of $B$, and let $g$ be the genus of its desingularization, $c$ the number of its cusps, and $n$ the number of its nodes. A curve $B$ is called *the discriminant curve of a generic covering of the projective plane* if there exists a finite morphism $f : S \to \mathbb{P}^2$, $\deg f \geqslant 3$, satisfying the following conditions:

($i$) $S$ is a non-singular irreducible projective surface;

($ii$) $f$ is unramified over $\mathbb{P}^2 \setminus B$;

($iii$) $f^*(B) = 2R + C$, where $R$ is a non-singular irreducible reduced curve and a curve $C$ is reduced;

($iv$) $f_{|R} : R \to B$ coincides with the normalization of $B$.

Such $f$ is called *a generic covering of the projective plane* $\mathbb{P}^2$.

A generic covering $f : S \to \mathbb{P}^2$ is called a *generic projection* if the surface $S$ is embedded in some projective space $\mathbb{P}^N$ and $f = \mathrm{pr}_{|S}$ is a restriction to $S$ of a linear projection $\mathrm{pr} : \mathbb{P}^N \to \mathbb{P}^2$.

Chisini's Conjecture (see [1]) claims that if $f : S \to \mathbb{P}^2$ is a generic covering of the projective plane of $\deg f \geqslant 5$ then $f$ is determined uniquely up to an isomorphism of $S$ by its discriminant curve.

It was proved in [3] that Chisini's Conjecture holds for the discriminant curve $B$ of a generic covering $f : S \to \mathbb{P}^2$ if

$$\deg f > \frac{4(3d + g - 1)}{2(3d + g - 1) - c}. \tag{§}$$

Furthermore, it was observed in [5] that, by Bogomolov – Miaoka – Yau inequality, the right side of inequality (1) takes the values less then 12, that is, Chisini's Conjecture holds for the discriminant curves of the generic coverings of degree greater than 11. Besides, also it was shown in [5] that if $S$ is a surface of non-general type, then Chisini's Conjecture holds for the discriminant curves of the generic coverings $f : S \to \mathbb{P}^2$ if $\deg f \geqslant 8$.

The following theorem gives the answer to Chisini's conjecture in the case of generic projections.

**Theorem 8.** *Let $f : S \to \mathbb{P}^2$ be a generic projection. Then the generic covering $f$ is uniquely determined up to an isomorphism of $S$ by its discriminant curve $B \subset \mathbb{P}^2$ except the case when $S \simeq \mathbb{P}^2$ is embedded in $\mathbb{P}^5$ by the polynomials of degree two (the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$) and $f$ is the restriction to $S$ of a linear projection $pr : \mathbb{P}^5 \to \mathbb{P}^2$.*

Let $(X, L)$ be a polarized projective surface, where $L$ is an ample line bundle on $X$, For $k >> 1$, the sections of $L^{\otimes k}$ define an imbedding of $X$ into some projective space $\mathbb{P}^r$. The restriction of generic projection

$\mathbb{P}^r \to \mathbb{P}^2$ to $X$ gives a generic covering $f_k : X \to \mathbb{P}^2$ branched over a cuspidal curve $\bar{B}_k$, the degree of the covering is equal $k^2 c_1(L)^2$.

The type of braid monodromy factorization of the curve $\bar{B}_k$ (see [4], [2]; the definition will be given in the talk) is called the *k-th braid nonodromy invariant* $\beta_{L,k}(X, L)$ of $(X, L)$.

In the talk we discuss the possibility to use the braid nonodromy invariants of the polarized projective surfaces in order to distinguish the connected components of their moduli spases in the case of the surfaces of general type with canonical polarizations.

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# The Lerch zeta-function with algebraic irrational parameter

A. Laurinčikas, R. Macaitienė (Vilnius, Lithuania)

The Lerch zeta-function $L(\lambda, \alpha, s)$, $s = \sigma + it$, with parameters $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R}$, $0 < \alpha \leqslant 1$, is defined, for $\sigma > 1$, by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda m}}{(m+\alpha)^s},$$

and by analytic continuation elsewhere. If $\lambda \notin \mathbb{Z}$, then the function $L(\lambda, \alpha, s)$ is entire. For $\lambda \in \mathbb{Z}$, the Lerch zeta-function reduces to the Hurwitz zeta-function $\zeta(s, \alpha)$.

We apply probabilistic methods for the investigation of value distribution of the function $L(\lambda, \alpha, s)$. The simplest case is of transcendental $\alpha$ because in this case the system

$$L(s) = \{\log(m+\alpha) : m \in \mathbb{N}_0\}, \quad \mathbb{N}_0 = \mathbb{N}\bigcup\{0\},$$

is linearly independent over the field of rational numbers $\mathbb{Q}$. If $\alpha$ is algebraic irrational number, then by a Cassel's result at least 51 persent of elements of the system L($\alpha$) are linearly independent over $\mathbb{Q}$. We use this fact to obtain limit theorems in the sense of weak convergence of probability measures on the complex plane as well as in the space of analytic functions for the function $L(\lambda, \alpha, s)$. Also, joint limit theorems for a collection of Lerch zeta-functions $L(\lambda_1, \alpha_1, s), ...., L(\lambda_r, \alpha_r, s)$ are proved in the case when $\alpha_1, ..., \alpha_r$ are distinct algebraic irrational numbers such that the set

$$\bigcup_{j=1}^{r} I(\alpha_j)$$

is linearly independent over $\mathbb{Q}$. Here $I(\alpha_j)$ denote the maximal linearly independent over $\mathbb{Q}$ subset of $L(\alpha_j)$, $j = 1, ..., r$.

Let $\mathcal{M}(\alpha) = \{m \in \mathbb{N}_0 : \log(m+\alpha) \in I(\alpha)\}$. Define

$$\Omega = \prod_{m \in \mathcal{M}(\alpha)} \gamma_m,$$

where $\gamma_m = \{s \in \mathbb{C} : |s| = 1\}$ for all $m \in \mathcal{M}(\alpha)$. With the product topology and pointwise multiplication, $\Omega$ is a compact topological Abelian group, therefore on $(\Omega, \mathcal{B}(\Omega))$ ($\mathcal{B}(\Omega)$ denotes the class of Borel sets of $\Omega$) the probability Haar measure $m_H$ can be defined, and this leads to a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space $\gamma_m$, $m \in \mathcal{M}(\alpha)$.

If $m \notin \mathcal{M}(\alpha)$, then there exists a finite number of $m_1, ..., m_n \in \mathcal{M}(\alpha)$ and numbers $k_0(m), ..., k_n(m) \in \mathbb{Z} \setminus \{0\}$ such that

$$m+\alpha = (m_1+\alpha)^{-\frac{k_1(m)}{k_0(m)}} \cdots (m_n+\alpha)^{-\frac{k_n(m)}{k_0(m)}}.$$

Then we extend the function $\omega(m)$ to the whole set $\mathbb{N}_0$ putting, for $m \notin \mathcal{M}(\alpha)$,

$$\omega(m) = \omega^{-\frac{k_1(m)}{k_0(m)}}(m_1) \cdots \omega^{-\frac{k_n(m)}{k_0(m)}}(m_n),$$

and, for $\sigma > \frac{1}{2}$, define on $(\Omega, \mathcal{B}(\Omega), m_H)$ a complex-valued random variable

$$L(\lambda, \alpha, \sigma, \omega) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda m} \omega(m)}{(m+\alpha)^\sigma}.$$

Then the limit measure in a limit theorem on the complex plane coincides with the distribution of the random variable $L(\lambda, \alpha, \sigma, \omega)$. We have also a similar situation in the space of analytic functions.

## Parametrized families of Thue equations – On the structure of their sets of solutions

Günter Lettl (Graz, Austria)

In 1993 E. Thomas conjectured that for families of Thue equations of a certain shape there are – up to those solutions arising from finitely many polynomials over the integers $\mathbb{Z}$ – only finitely many further solutions over $\mathbb{Z}$. This conjecture was verified by Thomas himself for equations of degree 3 and by C. Heuberger (2001) for equations of arbitrary degree, with some exceptional cases still remaining open.

During the last 15 years, several parametrized families of Thue-equations were completely solved. In all these results the same structure of the set of solutions appeared, although these families do not belong to that special type, for which Thomas made his conjecture: there are solutions given by finitely many polynomials as well as finitely many further ("sporadic") solutions. Recent examples show that Thomas's conjecture in its original form does not hold for arbitrary families, but one has to make some modification.

This leads to the notion of $\mathbb{Z}$-parameter solutions. Using a theorem of C.L. Siegel one can give rather narrowing conditions for their existence. Nevertheless, the known results of Yu.I. Manin (1963), H. Grauert (1965) and R.C. Mason (1981) do not suffice to prove that any familiy of Thue equations does only have finitely many $\mathbb{Z}$-parameter solutions.

# On the $(2\times 2,2)$-generation of the group $PSL_n(\mathbb{Z}+i\mathbb{Z})$

D. V. Levchuk, Ya. N. Nuzhin (Krasnoyarsk, Russia)

A group is said to be $(2\times 2,2)$-generated, if it is generated by three involutions, two of which commute.

**Theorem.** *The projective special linear group $PSL_n(\mathbb{Z}+i\mathbb{Z})$ over the ring of Gauss numbers for $n \geqslant 8$ is $(2\times 2,2)$-generated.*

The groups $PSL_2(9)$ and $PSL_3(9)$ are not $(2\times 2,2)$-generated. It follows from the description of $(2\times 2,2)$-generated groups of Lie type over finite fields [1–3]. Therefore, the groups $PSL_2(\mathbb{Z}+i\mathbb{Z})$ and $PSL_3(\mathbb{Z}+i\mathbb{Z})$ are not $(2\times 2,2)$-generated in view homomorphism $PSL_n(\mathbb{Z}+i\mathbb{Z})$ onto $PSL_n(9)$.

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# Invariants of classical linear groups

A. A. Lopatin (Omsk, Russia)

We work over an infinite field $K$ of arbitrary characteristic that is different from 2 whenever we consider $O(n)$. Let a subgroup $G$ of $GL(n)$ act on the $d$-tuple of $n\times n$ matrices $M(n)^d = M(n)\oplus\cdots\oplus M(n)$ by the diagonal conjugation, i.e.

$$g\cdot(A_1,\ldots,A_d) = (gA_1g^{-1},\ldots,gA_dg^{-1}),$$

where $g\in G$, $A_1,\ldots,A_d\in M(n)$. Denote the algebra of polynomial functions on $M(n)^d$ by $K[M(n)^d] = K[x_{ij}(k)\,|\,i,j\in\overline{1,n},\ k\in\overline{1,d}]$ and the *matrix algebra of $G$-invariants* by $K[M(n)^d]^G = \{f\in K[M(n)^d]\,|\,f(g\cdot A) = f(A)$ for all $g\in G$ and $A\in M(n)^d\}$.

A study of this algebra for classical linear groups $GL(n)$, $O(n)$, and $Sp(n)$ was originated by Sibirskii, Razmyslov, and Procesi more than 30 years ago (see [5], [4], [3]). For zero characteristic case, generators for the algebras of invariants and relations between them were found out. For example, it was shown that the $K$-algebra $K[M(n)^d]^{GL(n)}$ is generated by the traces of products of "generic" $n \times n$ matrices

$$X_k = \begin{pmatrix} x_{11}(k) & \cdots & x_{1n}(k) \\ \vdots & & \vdots \\ x_{n1}(k) & \cdots & x_{nn}(k) \end{pmatrix},$$

where $1 \leqslant k \leqslant n$. Similarly, $K[M(n)^d]^{O(n)}$ is generated by the traces of products of $X_1, \ldots, X_d, X_1^t, \ldots, X_d^t$ and $K[M(n)^d]^{Sp(n)}$ is generated by the traces of products of $X_1, \ldots, X_d, X_1^*, \ldots, X_d^*$, where $X_k^*$ stands for the symplectic transpose matrix $(1 \leqslant k \leqslant n)$.

The importance of characteristic-free approach to the matrix invariants was pointed out by Formanek in overview [2] in 1991. Shortly afterwards, the case of arbitrary characteristic was considered. Generators and relations for $GL(n)$-invariants as well as generators for $O(n)$- and $Sp(n)$-invariants were established by Donkin and Zubkov (see [1], [6], and [7]). We completed this description by finding out relations for $O(n)$- and $Sp(n)$-invariants.

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## Cohen-Macaulay modules of infinite projective dimension

José M$^a$López, Agustín Marcelo, Félix Marcelo, and César Rodríguez (Las Palmas de Gran Canaria, Spain)

A characterization of finitely generated torsion modules of not necesarily finite projective dimension over a Cohen-Macaulay ring, is given in terms of the non-Cohen-Macaulay loci and the Fitting invariants of a free resolution of such module.

Let $R$ be a Noetherian ring and let $F_1 \xrightarrow{\varphi} F_0 \to M \to 0$, $\operatorname{rank} \varphi = r$, $\operatorname{rank} F_0 = n$, be a finite free presentation of the $R$-module $M$. Let $I(\varphi)$ denote the $(n-r)$-th Fitting invariant of $\varphi$ and let $\mathfrak{n}(M)$ denote the corresponding radical ideal of the non-Cohen-Macaulay locus of $M$.

Let $R$ be a Cohen-Macaulay local ring of dimension $d$ and let $T$ be a finitely generated torsion $R$-module of not necessarily finite projective dimension, with $\dim T < d$. Let

$$F_\bullet : \; F_m \xrightarrow{\varphi_m} F_{m-1} \to \ldots \to F_1 \xrightarrow{\varphi_1} F_0 \to T \to 0$$

be a free resolution of $T$ such that,

(i) $\operatorname{depth} \operatorname{Im} \varphi_m = \operatorname{depth} R$, and

(ii) $\operatorname{depth} \operatorname{Im} \varphi_i < \operatorname{depth} R$ for $i = 1, \ldots, m-1$.

(iii) $\operatorname{rank} \varphi_i > 0$ for $i = 1, \ldots, m$.

The aim of this work is to prove that $T$ is a Cohen-Macaulay $R$-module if and only if,

$$\operatorname{rad} \operatorname{Ann}(T) = \operatorname{rad} I(\varphi_i) = \mathfrak{n}(\operatorname{Im} \varphi_{i-1}), \quad i = 2, \ldots, m.$$

## Local cohomology, ideal modules and torsionfree modules

José M$^a$López, Agustín Marcelo, Félix Marcelo, and César Rodríguez (Las Palmas de Gran Canaria, Spain)

Let $R$ be a Noetherian ring and let $M$ be a finitely generated $R$-module. As is well known, the set of points $p \in \mathit{SpecR}$ such that $M_p$ is a free $R_p$-module, is an open subset in the Zariski topology. Hence, its complement $C$ is a closed subset -called the non-free locus of $M$- whose corresponding radical ideal is denoted by $\mathfrak{a} = \mathfrak{a}(M) = \Im(C)$. In this work the groups of local cohomology with supports in the non-free locus of a module are used in order to obtain classifications of two classes of modules. First we obtain a classification of the ideal modules over a local regular ring by means of $H^1_{\mathfrak{a}}(M)$. By applying this result and from the existence of a dualizing functor we also obtain a classification of the torsionfree finitely generated and nonfree modules of projective dimension one over a regular local ring.

## Recognizability by spectrum of the finite simple group $L_2(7)$.

D. V. Lytkina (Krasnoyarsk, Russia)

This is an joint work with A.A.Kuznetsov.

The *spectrum* of a periodic group $G$ is the set $\omega(G)$ consisting of all element orders of $G$. A group $G$ is *recognizable* by spectrum in the class of all groups if every group with spectrum coinciding with the spectrum of $G$ is isomorphic to $G$.

The goal of this communication is to announce the positive answer to the question 16.57 from [1] (see also [2]) on recognizability by spectrum of the simple group $L_2(7)$ in the class of all groups.

**Theorem.** *If the spectrum of a group $G$ is equal to* $\{1, 2, 3, 4, 7\}$ *then* $G \simeq L_2(7)$.

For finite groups this result is proved in [3].

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## Groups with given spectrum

V. D. Mazurov (Novosibirsk, Russia)

For a finite group $G$, denote by $\omega(G)$ the spectrum i.e. the set of element orders of $G$. This talk is devoted to the following question: For which groups $G$ the spectrum of $G$ defines $G$ uniquely up to isomorphism? As an example, we give the following result obtained recently by G.Y.Chen and the author.

**Theorem** *Let $m \geqslant 2$ be a natural number and $L$ be any of the finite simple groups $L_4(2^m)$ or $U_4(2^m)$. If $G$ is a finite group such that $\omega(G) = \omega(L)$ then $G \simeq L$.*

This work is supported by Russian Foundation of Basic Researches (grants NN 05-01-00797 06-01-39001), Siberian Branch of RAS (complex integration project No. 1.2).

## $\tau$-closed local formations defined by Hall subgroups

A. P. Mekhovich, N. N. Vorob'ev (Vitebsk, Belarus)

All groups considered are finite and soluble. We use the terminology of [1-3]. The symbols $\mathfrak{S}$, $\mathfrak{S}_\pi$ and $\mathrm{Proj}_{\mathfrak{F}}G$ denote respectively the class of all groups, the class of all $\pi$-groups and the set of all $\mathfrak{F}$-projectors of a group $G$.

Let $\mathfrak{X}$ be a non-empty class of groups. For any group $G \in \mathfrak{X}$ we associate some subgroup system $\tau(G)$ of $G$. The map $\tau$ is called *a subgroup $\mathfrak{X}$-functor* (or in other words $\tau$ is *a subgroup functor on $\mathfrak{X}$*) [3] if for any epimorphism $\varphi : A \longrightarrow B$ where $A$, $B \in \mathfrak{X}$ the following inclusions satisfy:

$$(\tau(A))^{\varphi} \subseteq \tau(B),$$
$$(\tau(B))^{\varphi^{-1}} \subseteq \tau(A)$$

and for any group $G \in \mathfrak{X}$ we have $G \in \tau(G)$.

Recall that a formation is a class of groups closed under taking of homomorphic images and subdirect products.

A formation $\mathfrak{F}$ is called *$\tau$-closed* [3] if $\tau(G) \subseteq \mathfrak{F}$ for any group $G \in \mathfrak{F}$.

Functions $f: \mathbb{P} \longrightarrow \{\text{formation of groups}\}$ are called satellites [2]. For any satellite $f$ we consider the class

$$LF(f) = (G \mid G/F_p(G) \in f(p) \text{ for all } p \in \pi(G))$$

where $\pi(G)$ is the set of all prime divisors of the order of a group $G$.

Recall that for any class of groups $\mathfrak{F} \supseteq (1)$ the symbol $G_{\mathfrak{F}}$ denotes the product of all normal $\mathfrak{F}$-subgroups of $G$ and the symbol $G^{\mathfrak{F}}$ denotes the intersection of all such normal subgroups $N$ that $G/N \in \mathfrak{F}$. In particular $F_p(G) = G_{\mathfrak{S}_{p'}\mathfrak{N}_p}$.

Let $\mathfrak{F}$ be a formation. If $\mathfrak{F} = LF(f)$ for some satellite $f$ then it is said that $\mathfrak{F}$ is a local formation defined by $f$ [1].

Let $\mathfrak{F}$ be a local formation. We define a class $L'_\pi(\mathfrak{F})$ as follows: $G \in L'_\pi(\mathfrak{F})$ if and only if every $\mathfrak{F}$-projector of a group $G$ contains a normal Hall $\pi$-subgroup of $G$.

We note if $\mathfrak{F} = \mathfrak{S}$ then the class $L'_\pi(\mathfrak{S})$ coincides with the class of all $\pi$-closed groups $\mathfrak{S}_\pi\mathfrak{S}_{\pi'}$.

Recall that a subgroup $H$ of a group $G$ is called an $\mathfrak{F}$-projector of $G$ if the following conditions satisfy:

1) $H \in \mathfrak{F}$;

2) $HU^{\mathfrak{F}} = U$ for any subgroup $U$ containing $H$.

We prove the following

**Theorem**. Let $\mathfrak{F}$ be a $\tau$-closed local formation. Then a class $L'_\pi(\mathfrak{F})$ is a $\tau$-closed local formation defined by a local satellite $f$ such that

$$f(p) = \begin{cases} (G \mid \mathrm{Proj}_{\mathfrak{F}} G \subseteq H(p)), & \text{if } p \in \pi; \\ \mathfrak{S}, & \text{if } p \in \pi' \end{cases}$$

where $\mathfrak{H} = \mathfrak{F} \bigcap \mathfrak{S}_\pi\mathfrak{S}_{\pi'}$ is a formation defined by the canonical local satellite $H$ such that

$$H(p) = \mathfrak{N}_p H(p) \text{ and } H(p) \subseteq LF(H) \text{ for all } p \in \mathbb{P}.$$

If $\tau$ is a trivial subgroup functor [3] i.e. $\tau(G) = G$ for any group $G$, we have the main result of [4].

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## Hurwitz numbers of seamed surfaces

S. M. Natanzon (Moscow, Russia)

A seamed surface is a two-dimensional complex from modern models of mathematical physics [1]. We extend the definition of Hurwitz numbers on seamed surfaces and we prove that these Hurwitz numbers form a system of correlators for a Klein topological field theory. Klein topological field theories describe open-closed topological string theories with oriented and non-oriented world-sheets. They correspond one-to-one to structure algebras [2]. We describe the structure algebra corresponding to $n$-degree Hurwitz numbers of seamed surfaces. Non-trivial part of this algebra is an associative algebra on a vector space, that has bichromatic graphs with $n$ edges as a basis. We prove that this algebra is isomorphic to the algebra of intertwining operators for the representation of symmetric group $S_n$ in the set of all partition of $n$ elements to batches. The talk is based on joint with A.Alexeevski papers [3,4].

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## Universal norms and Greenberg's conjecture

T. Nguyen Quang Do (Besançon, France)

Greenberg's conjecture, which could be considered as a reasonable generalization of Vandiver's conjecture, asserts that the "lambda"invariant attached to the cyclotomic $Z_p$-extension of a totally real number field $F$ should be null. In spite of numerical evidence (mainly for $p = 3$ and $F$ quadratic), no general theoretical result is known. In this talk, we study Iwasawa (co)descent for the so called groups of "unit classes"(=units modulo circular units) in terms of universal norms, and we construct a certain finite cohomological invariant whose vanishing implies Greenberg's conjecture when $p$ is totally split in $F$.

## Certain aspects of non-abelian cohomology theory

A. L. Onishchik (Yaroslavl, Russia)

Homological methods were used widely in different branches of mathematics during the XXth century. Γ special branch, Homological Algebra has been appeared, and D.K. Faddeyev is known to be one of the pioneers in this discipline. The main objects of Homological Algebra are chain and cochain complexes of abelian groups. However, there exist certain non-abelian analogues of such complexes which have important applications in algebra, differential and algebraic geometry, topology, and also in the theory of complex manifolds and supermanifolds. Our goal is to discuss certain attempts of constructing a systematic theory of non-abelian cochain complexes which give cohomology sets in degrees $\leqslant 2$ as well as certain applications of this theory.

Here we limit ourselves to the cohomology of degrees 0 ï 1. In this case, the following general concept of non-abelian cochain complex seems to be appropriate. It includes the triple $K = \{K^0, K^1, K^2\}$, where $K^0$, $K^1$ are groups and $K^2$ a point with a distinguished point $e$, two actions $\sigma_p$ of the group $K^0$ on $K^p,\ p = 1, 2$, by automorphisms of the group and the set with a distinguished point respectively, and two coboundary mappings $\delta_p : K^p \to K^{p+1},\ p = 0, 1$. Here $\delta_0$ should be a crossed homomorphism with respect $\sigma_1$, while $\delta_1$ should satisfy $\delta_1(e) = e$ and $\delta_1 \circ \rho(a) = \sigma_2(a) \circ \delta_1,\ a \in K^0$, where $\rho$ is the "affine"action of $K^0$

on $K^1$ associated with $\delta_0$. Such a complex $K$ generates the cocycle sets $Z^p(K) = \operatorname{Ker}\delta_p,\ p = 0,1$, and the cohomology sets $H^0(K) = Z^0(K)$ ï $H^1(K) = Z^1(K)/\rho$, where $H^0(K)$ is a group, while $H^1(K)$ is a set with a distinguished point. No natural group operation in $H^1(K)$ is defined, but instead of it we have an operation of "twisting"the complex by any cocycle from $Z^1(K)$. For the non-abelian complexes certain versions of exact cohomology sequences can be constructed.

The following general constructions of non-abelian complexes should be mentioned: the Čech complex of a sheaf of non-abelian groups, the complex of sections of a resolution of a sheaf of non-abelian groups, the complex associated with a differential graded Lie superalgebra. We intend to discuss some special cases of these constructions (and their applications), in particular the following ones:

1. The de Rham complex of a smooth manifold $M$ with values in a Lie group (the monodromy of differential equations, the relation between flat bundles and homomorphisms of the fundamental group of $M$, classification and deformations of homomorphisms of discrete groups) [1, 5, 6].

2. The Dolbeault complex of a complex manifold $M$ with values in a Lie group (classification and deformations of holomorphic bundles over $M$) [4, 5].

3. A complex related to a holomorphic vector bundle (classification of complex supermanifolds with a given retract) [7].

4. The "quadratic complex", associated with an orthogonal module of a Lie algebra (classification of invariant bilinear forms on Lie algebras) [2].

5. The complex of a group $G$ with values in a non-abelian $G$-module (the Galois cohomology, lifting of actions of a group) [5, 8, 9].

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## Reduction of quantum Schubert cells

A. N. Panov (Samara, Russia)

Let $N = \mathrm{UT}(n, K)$ be the lower unitriangular group of order $n$ over the field of $K$ of characteristic zero. The old problem of classification of coadjoint orbits of $N$ is not solved up today. The solution of this problem is important for the representation theory, the geometry of Poisson varieties and the theory of ideals in universal enveloping algebra $U(\mathfrak{n})$ of the Lie algebra $\mathfrak{n}$ of $N$. We reduce the Quantum Schubert cells to obtain ideals in $K[\mathfrak{n}^*]$ invariant with respect to the coadjoint action.

Let $B$ be the lower Borel subgroup in $\mathrm{GL}(n, K)$, $q$ be a variable and $\mathbb{C}_q[B]$ be the algebra of regular functions on the corresponding quantum subgroup. The localizations $\mathbb{C}'_q[B]$ and $U'_q(\mathfrak{n})$ with respect to $q - q^{-1}$ are isomorphic. The quantum Schubert cell corresponding to $w \in W = S_n$ is a pair $(QJ_w, QS_w)$ where $QJ_w$ is an ideal in $\mathbb{C}_q[B]$, generated by certain quantum minors, and $QS_w$ is a denominator subset(see [1],[2]). The set of pairs $(QJ_w, QS_w)_{w\in W}$ form a stratification of the set of primitive ideals in $\mathbb{C}_q[B]$. It is known that any primitive ideal in $\mathbb{C}_q[B]$ that contains $QJ_w$ and has empty intersection with $QS_w$ has dimension $l(w) + s(w)$, where $l(w)$ (resp. $s(w)$) is a length of a reduced decomposition of $w$ into a product of simple (resp. arbitrary) reflections.

Localization $QJ'_w$ of $QJ_w$ with respect to $q - q^{-1}$ is an ideal in $\mathbb{C}'_q[B] = U'_q(\mathfrak{b})$. We correspond the Quantum Schubert cell to the pair $(J_w, S_w)$ where

$$J_w = \mathrm{gr}\,(QJ'_w \cap U_q(\mathfrak{n}) \bmod (q-1))$$

is an ideal in $S(\mathfrak{n}) = K[\mathfrak{n}^*]$ and $S_w = \mathrm{gr}(QS_w \bmod (q-1))$ is a denominator subset in $S(\mathfrak{n})$.

Let $\sigma$ be an involution in $S_n$. Denote $\Omega_\sigma = \mathrm{Ann} J_\sigma \cap \{f \in \mathfrak{n}^* | \ a(f) \neq 0 \ \forall f \in S_\sigma\}$. The set $\Omega_\sigma$ is a union of coadjoint orbits.

Decompose $\sigma$ into a product of reflections $\sigma = r_1 r_2 \cdots r_s$ with respect to positive roots $\{\xi_m = \varepsilon_{j_m} - \varepsilon_{i_m}\}$. Let $\{y_{ij}\}_{1 \leqslant j < i \leqslant n-1}$ be a standard basis in $\mathfrak{n}$. Consider the subset $X_\sigma \subset \mathfrak{n}^*$ that consists of all $f \in \mathfrak{n}^*$ such that $f(y_{i_m, j_m}) \neq 0$, $1 \leqslant m \leqslant s$, and that annihilate on all other vectors of the standard basis.

**Theorem 1.** *$\Omega_\sigma$ is a union of coadjoint orbits $\Omega(f)$, $f \in X_\sigma$.*

**Theorem 2.** $\dim \Omega(f) = l(\sigma) - s(\sigma)$*.*

For any $1 \leqslant t \leqslant n$ consider the involution $\sigma_{t-1}$, that is a product of all $r_m$,$1 \leqslant m \leqslant s$ such that $j_m < t$. Put $\sigma_0 = \mathrm{id}$. Consider the set of pairs $P_\sigma = \{(i,t) : \ 1 \leqslant t < i \leqslant n, \ \sigma_{t-1}(t) < \sigma_{t-1}(i)\}$. Denote $\mathfrak{p}_\sigma = \mathrm{span}\{y_{ij} | \ (i,j) \in P_\sigma$.

**Theorem 3**. *$\mathfrak{p}_\sigma$ is a subalgebra in $\mathfrak{n}$ and a polarization of any $f \in X_\sigma$.*

In the talk we will present generators of the defining ideal of any coadjoint orbit $\Omega(f)$ for any $f \in X_\sigma$.

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## On Drinfeld quasi-modular forms

Federico Pellarin (Caen, France)

The forms of the title are variants in positive characteristic of classical quasi-modular forms for the group $SL_2(\mathbb{Z})$ as defined by Kaneko and Zagier (whose main properties will be reviewed). We will show that the rings generated by Drinfeld quasi-modular forms can be endowed with Hyperdifferential (or Hasse-Schmidt differential) structures. Analogies occur between Drinfeld quasi-modular forms and classical quasi-modular forms; these will be described in the lecture. The proofs however, are not similar at all. This work follows from a collaboration with V. Bosser of Basle University.

## One-relator algebras and noncommutative geometry

D. I. Piontkovski (Moscow, Russia)

Suppose that $A$ is a graded associative algebra defined by a single quadratic relation. We show that $A$ is graded coherent, that is, the category coh-mod $A$ of graded finitely presented modules over it is abelian. It follows that there is a well-defined coherent noncommutative spectrum coh-proj $A$ (in the sense of Polishchuk), that is, a quotient category of coh-mod $A$ by the category of torsion modules.

Suppose that the relation of $A$ is generic enough. Then we show that the category coh-proj $A$ is abelian Ext-finite hereditary with Serre duality, like the category of coherent sheaves on $\mathbb{P}^1$; therefore, it can be considered as a noncommutative spectrum of the projective line. On the other hand, the category coh-proj $A$ is derived equivalent to noncommutative $\mathbb{P}^{n-1}$ in the sense of Kontsevich and Rosenberg, where $n$ is the number of generators of the algebra $A$.

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## Logically separable algebras in varieties

B. I. Plotkin (Jerusalem, Israel)

We consider algebras in a variety of algebras $\Theta$. A well known invariant of every algebra $H \in \Theta$ is its elementary theory $Th(H)$. Two algebras $H_1$ and $H_2$ are elementary equivalent if $Th(H_1) = Th(H_2)$. We introduce a more strong notion of logically geometrical equivalence of two algebras ($LG$-equivalence). This $LG$-equivalence implies elementary equivalence, but not vice versa.

In the talk we consider problems related to the notion of $LG$-equivalence of algebras. In particular, let us mention the following one:

Let $\Theta$ be an arbitrary variety of algebras, $W = W(X)$ a free algebra in this variety with the finite set $X$. We say that this algebra $W$ is $LG$-separable in $\Theta$, if any other algebra $H$, $LG$-equivalent to $W$, is isomorphic to $W$. It is proved that this property holds for free semigroups and free inverse semigroups. A study of other interesting cases is in progress.

## On the solvable radical of a finite group and around

E. B. Plotkin (Jerusalem, Israel)

In the talk we discuss the progress obtained within the last years in the problem of the characterization of the solvable radical of a finite group. The main attention is paid on commutator-like descriptions of the solvable radical. We also give an insight on Burnside-type problems from the positions of solvability property.

The Baer-Suzuki theorem characterizes the nilpotent radical of a finite (or linear) group by the property that an element $g$ is in the nilpotent radical of $G$ if and only if any two conjugates of $g$ generate a nilpotent group.

We discuss a general setting whose part is the sharp analog of the Baer-Suzuki theorem for the solvable radical of a finite (linear) group.

In the talk we outline the proof of this theorem.

The Baer-Suzuki theorem for the solvable case is independently announced by Flavell, Guest, Guralnick.

Joint work with N.Gordeev, F.Grunewald, B.Kunyavskii

## On the topology of real decomposable 7th degree curves

G. M. Polotovskiy (Nizhny Novgorod, Russia)

We shall give a survey of the results in the problem of classification of decomposable algebraic curves of degree 7 in the real projective plane $\mathbf{R}P^2$ with respect to isotopies preserving cofactors, under natural conditions of maximality and general position: i) each cofactor is an $M$-curve; ii) cofactors are in general position; iii) every two cofactors have the maximal number of common real points and these points belong to the same real branch of each cofactor. This problem belongs to the topic of the Hilbert 16th problem.

For degree 6 similar problem was solved in [1] and the results have many applications. For degree 8 the solution of the problem in such formulation is unreal: the number of types is very big. In the case of degree 7 the solution approaches to the end – mainly due to efforts of E. Shustin, S. Orevkov, A. Korchagin, and the author. For example, in the series of works of Shustin, Orevkov, and Korchagin the classification of affine $M$-sextics was obtained (final paper is here [2]); classification of unions of a quartic and a cubic with 12 common points on ovals was obtained by Orevkov and the author [3].

We shall consider in detail following two cases.

(i) Classification of unions of a quintic and a pair of lines. When each of two disks of the complement of $\mathbf{R}P^2$ to the pair of lines has only one arc of odd branch of the quintic with ends in different lines of the pair, the classification was obtained by Korchagin and the author [6] and consists of 20 isotopy types. In the rest cases classification was completed recently in papers of Korchagin and the author [7] and Orevkov [8].

(ii) Classification of arrangements of a quartic and a cubic with 12 common points on odd branch of the cubic and on one oval of the quartic. S. Orevkov [4] has constructed 237 mutually nonisotopic types of such curves. There is the base to think that this list gives the complete classification. By Orevkov's method [5] based on link theory we have success in proving of this statement: having considered more than 3500 models of such arrangements (total number of models for consideration is more than 7000), we proved that there are realizable from these models only the models which belong to the Orevkov 237-list. So, the proof is in progress but is not finished now because of a big volume of necessary computations.

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## Finite linear groups, lattices, and products of elliptic curves

V. L. Popov (Moscow, Russia)

Studying complex tori (in particular, abelian varieties) with finite group actions (and, more generally, with certain endomorphisms) is a subject matter of several recent papers, see, e.g., [DL], [LR], [Vo]. In particular, it is a starting point for examples of compact Kähler manifolds that do not have the homotopy type of projective complex manifolds [Vo]. Among such tori, abelian varieties are of a special interest. For instance, they arise as the jacobians of smooth projective curves with group actions. These actions induce decompositions of jacobians up to isogeny. For hyperelliptic curves, such decompositions go back to classical interest in hyperelliptic integrals expressible in terms of elliptic integrals, because the problem boils down to decomposing jacobians, up to isogeny, as products of elliptic curves.

Complex tori with group actions arise in the following way. Let $V$ be a complex linear space of nonzero dimension $n < \infty$ and let $G$ be a finite subgroup of $\mathrm{GL}(V)$. If there is a $G$-invariant lattice $\Lambda$ in $V$ of rank $2n$ (hereinafter a *lattice* is a discrete additive subgroup of a complex or real linear space), then $V/\Lambda$ is a complex torus with $G$-action. This naturally leads to the following questions:

(1) when is there a $G$-invariant lattice $\Lambda$ of rank $2n$?

(2) if $\Lambda$ exists, when is $V/\Lambda$ an abelian variety?

(3) if $\Lambda$ exists and $V/\Lambda$ is an abelian variety, what can one say about decomposition of $V/\Lambda$, up to isogeny?

Note that the Riemann condition reduces (2) to the linear algebra problem: when is there a polarization of $V/\Lambda$, i.e., a positive definite Hermitian bilinear form $V \times V \to \mathbf{C}$, whose imaginary part takes integral values on $\Lambda \times \Lambda$?

We address and answer these questions for irreducible $G$.

Namely, we give a criterion of the existence of a nonzero $G$-invariant lattice $\Lambda$ in $V$ in terms of the character and the Schur $\mathbf{Q}$-index of the $G$-module $V$. For $\Lambda$ of rank $2n$, we describe the structure of complex torus $V/\Lambda$. In particular, we prove that in the majority of cases (but not in all) $V/\Lambda$ is an abelian variety. Moreover, we show that if the latter holds, then in many cases $V/\Lambda$ is isogenous to a self-product of an elliptic curve or even isomorphic to a product of mutually isogenous elliptic curves with complex multiplication, while in the other cases, $V/\Lambda$ is isogenous to a self-product of an abelian surface. We prove that $G$ and $\Lambda$ such that the complex torus $V/\Lambda$ is not an abelian variety do exist, but one can always replace $\Lambda$ by another $G$-invariant lattice $\Delta$ such that $V/\Delta$ is a product of mutually isogenous elliptic curves with complex multiplication. A separate discussion concerns the interesting example of groups $G$ generated by (complex) reflections, in which case a complete classification of $G$-invariant lattices is available, [Po].

These results are obtained jointly with Y. G. Zarhin and published in [PZ].

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## Degeneration of del Pezzo surfaces and local structure of del Pezzo fibrations

Y. G. Prokhorov (Moscow, Russia)

We will discuss issues on biregular local classification of del Pezzo fibrations over curves. More presicely we study the local structure of fibrations $f\colon X \to Z \ni o$, where $X$ is a threefold with worst terminal singularities, $Z \ni o$ is a curve germ, and the anti-canonical divisor $-K_X$ is $f$-ample. These fibrations naturally appear in the birational geometry of varieties of negative Kodaira dimension. On the other hand, the generic fibre of such an $f$ is a smooth del Pezzo surface, so $X/Z$ can be considered as a total space of degeneration of del Pezzo surfaces.

We will present some recent progress in the classification problem of such fibrations. One of our main goal is a partial classification of singular fibres in the "semistable case" (joint work with P. Hacking). In this case the classification is given in terms of Markov-type equations.

## Modules whose Maximal Submodules are Supplements over Dedekind Domains

D. Pusat-Yilmaz (Izmir, Turkey)

Let $R$ be a ring and $M$ be a unital left $R$-module. $M$ is called an $ms-module$ if every maximal submodule of $M$ is a supplement in $M$, and $M$ is called an $md-module$ if every maximal submodule of $M$ is a direct summand of $M$.

Throughout $R$ is always a Dedekind domain.

The following is proved by Zöschinger:

**Lemma 1.** *Let $M$ be an $R$-module and $V \leqslant M$. Then $V$ is coclosed if and only if $V$ is closed.*

Lemma 1 implies that a maximal submodule is a supplement if and only if it is a direct summand. Hence over a Dedekind domain, ms- and md-modules coincide. We characterize these modules as in the following Theorem:

**Theorem.** *Let $M$ be an $R$-module. Then $M$ is an md-module (ms-module) if and only if*

(*i*) $T(M) = M_1 \oplus M_2$, *where $M_1$ is semisimple and $M_2$ is divisible,*

(*ii*) $M/T(M)$ *is divisible.*

In the following theorem we investigate weather being an md-module imply every simple submodule is a direct summand.

**Theorem.** *For an $R$-module $M$, the following are equivalent:*

(*i*) *$M$ is an md-module and* $Rad(T(M)) = 0$,

(*ii*) *every simple submodule of $M$ is a direct summand and $M/T(M)$ has no maximal submodules.*

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# Some minimal representation-infinite algebras

C. M. Ringel (Bielefeld, Germany)

We consider finite-dimensional $k$-algebras $A$ (associative, with 1), where $k$ is a field. Recall that any finite-dimensional $A$-module can be written as a direct sum of indecomposable $A$-modules and such a decomposition is unique up to isomorphism. In case there are only finitely many isomorphism classes of indecomposable $A$-modules, then $A$ is said to be representation-finite, otherwise representation-infinite. In case $A$ is representation-infinite, but any proper factor algebra is representation-finite, then to be minimal representation-infinite. The minimal representation-infinite algebras have not yet been classified, but the known classes seem to be of great interest, most of them are related to the extended Dynkin diagrams of Lie theory (and the indecomposable modules correspond to suitable positive roots).

The lecture will present two classes of minimal representation-infinite algebras which exhibit some new features: the barbells and the wind wheels. Both are obtained from a hereditary algebra $H$ of type $\widetilde{A}_n$ for some $n$ by a process which we call barification (and actually turn out to be subalgebras of $H$). The barbell algebras are tame algebras of non-polynomial growth. In contrast, a wind-wheel algebra $W$ is 1-domestic, and its module category is obtained from the module category of $H$ by rearranging the modules in the non-homogeneous tubes and inserting countably many additional indecompoosable modules. The Auslander-Reiten components obtained in this way are coverings of a plane with a hole, and there are examples with an arbitary finite number of such components.

It should be remarked that all the algebras presented are (reduced) semigroup algebras of finite semigroups with zero, and they are (together with the hereditary algebras of type $\widetilde{A}_n$) the only minimal representation-infinite algebras which are special biserial.

## On Galois cohomology, Steenrod operations and Cycles

Markus Rost (Bielefeld, Germany)

The Bloch-Kato conjecture (bijectivity of the norm residue homomorphism) describes the (mod p) Galois cohomology ring of a field. The proof of this conjecture relies on work of Suslin and Voevodsky and the understanding of the so called norm varieties. Special examples of norm varieties are Severi-Brauer varieties and Pfister quadrics. Important properties of norm varieties are that they are generic splitting varieties of the corresponding symbol and that they split off a certain Chow motive uniquely determined by the symbol. Both properties are established using the "basic correspondence of norm variety". We will discuss some details concerning the basic correspondence and its role in the proof of the Bloch-Kato conjecture.

# Computation of complex representations of certain finite groups

A. V. Rukolaine (St. Petersburg, Russia)

Computer calculation of some complex irreducible representations of a certain finite group G may use information about all complex irreducible characters of the group G.

Now all complex irreducible characters of a finite group can be received with the help of Computer System GAP [1], beginning in Aachen, Germany. Another technology for computer calculation of irreducible characters of some finite groups was proposed in [2, 3].

It is known that minimal central idempotents in complex group algebra of the group G may be calculated with the help of irreducible characters of the group. Such minimal central idempotents are used for decompositon of G-modules as a direct sum of homogeneous G-submodules that themselves are direct sums of similar irreducible G-submodules. Such homogeneous G-submodules are known as homogeneous components of G-modules.

Sometimes such homogeneous components are the irreducible G-modules. In this case it is possible to construct a complex irreducible representation of the group G.

If the homogeneous components are not the irreducible G-modules then some information about the decomposition of the homogeneous components may be received from centralizer algebras.

In this work we propose to use the G-modules induced from irreducible characters of maximal cyclic subgroups of the group G. Preliminary results show that this technique is possible and helpful for the calculation of some complex irreducible representations of certain finite groups.

Presently work for such calculations with the help of GAP has been started.

Of course, it is possible to extend this technology to G-modules induced from irreducible characters of maximal abelian subgroups of a group.

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## Schur Multiplier and Pair of Groups

Ali Reza Salemkar and Vahid Alamian (Sari, Iran)

Let $G$ be a finite $p$-group of order $p^n$, then there exists a non-negative integer $t(G)$, such that $|\mathrm{M}(G)| = p^{1/2n(n-1)-t(G)}$. In 1999 G. Ellis gave the structure of $p$-groups, when $t(G) = 0$, 1, 2 or 3. In fact, Ya.G. Berkovich was already obtained the structure of $p$-groups in the case $t(G) = 0$ or 1, and X. Zhou gave the positive answer for $t(G) = 2$. But Ellis proved all the cases including $t(G) = 3$ with different technique. Using this technique, in chapter three we have characterized the structure of $p$-groups $G$ for which $t(G) = 4$. Also, if $(G, M)$ is a pair of groups with $G/M$ and $M/Z(M, G)$ are of orders $p^m$ and $p^n$, respectively. Then there exists a non-negative integer $s(G, M)$ such that $|[M, G]| = p^{1/2n(n+2m-1)-s(G,M)}$. Also for the pair $(G, M)$, if $M$ and $G/M$ are of orders $p^n$ and $p^m$,respectively. Then there exists a non-negative integer $t(G, M)$ such that $|M(G, M)| = p^{1/2n(n+2m-1)-t(G,M)}$.

**Theorem.** *Let $(M, G)$ be a pair of groups such that $|G/M| = p^m$, $|M/Z(M, G)| = p^n$ and $|[M, G]| = p^{\frac{1}{2}n(n+2m-1)-1}$. If $|[M/Z(M, G), G/Z(M, G)]| \leqslant p$, then one of the following holds:*

*(i) $M/Z(M, G)$ is an elementary abelian $p$-group;*

*(ii) the pair $(G/Z(M, G), M/Z(M, G))$ is an extra-special pair of finite $p$-groups;*

*(iii) $Z_2(M, G)/Z(M, G)$ is a group of order $p^2$, and there exist normal subgroups $K$ and $N$ of $Z_2(M, G)$ in such a way that $Z_2(M, G) = NK$, $N \cap K = Z(M, G)$, $M/N$ is elementary abelian and the pair $(G/K, M/K)$ is extra special.*

**Theorem.** *Let $(G, M)$ be a pair of groups with $| \frac{G}{Z(M,G)} | = p^\alpha$ and $| \frac{G}{M} | = p^\beta$, where $2^{n-1} < p^\alpha < 2^n$ and $0 < p^\beta < 2^m$ and $m, n > 5$. Then*

$$| [M, G] | \leqslant 2^{\frac{1}{2}(n-1)(n+2m-2)}.$$

**Theorem.** *Let $(G, M)$ be a pair of finite $p-$groups and $N$ be a complement of $M$ in $G$. Also, assume $M$ and $N$ are of orders $p^n$ and $p^m$, respectively. If*

$$\mid \mathcal{M}(G, M) \mid = p^{\frac{1}{2}n(n+2m-1)-2}$$

*then*
$(i)$ *if $N$ is elementary abelian, then*

$$G \simeq E_1 \times C_p \quad , \quad D_8 \quad , \quad C_{p^2} \times C_p$$

$(ii)$ *if $G$ and $N$ elementary abelian $p-$groups such that $(G, N)$ is non-capable, then*

$$G \simeq C_p \times C_{p^2}.$$

**Theorem.** *Let $(G, M)$ be a pair of finite p-groups and $N$ be a non-trivial complement of $M$ in $G$. Assume*

$$\mid M \mid = p^n, \mid N \mid = p^m, \mid \mathcal{M}(G, M) \mid = p^{\frac{1}{2}n(n+2m-1)-t}$$

*and $\mid \mathcal{M}(N) \mid = p^{\frac{1}{2}m(m-1)-s}$, where $s$ and $t$ are non-negative integers with $s \leqslant t$. Then*

*(i) $t = 0$ if and only if $G$ is an elementary abelian p-group;*

*(ii) $t = 1$ if and only if $(G, M) \cong (E_1, C_p \times C_p), (C_p \times C_{p^2}, C_p)$ or $(C_p \times E_1, C_p)$;*

*(iii) $t = 2$ if and only if $(G, M) \cong (C_p \times C_{p^2}, C_{p^2}), (C_p \times C_p \times C_{p^2}, C_p \times C_p)$ or $(C_p \times C_p \times C_p \times C_{p^2}, C_p \times C_p)$, when $G$ is abelian;*

*(iv) $t = 2$ if and only if $(G, M) \cong (D_8, C_4 D_8), (E_2 \times C_p, E_2)$,$(C_p \times C_p \times E_1, C_p \times C_p)$,*
*$(C_2 \times C_2 \times D_8, C_2 \times C_2)$ or $(C_p \times C_p \times E_1, C_p \times C_p)$, when $G$ is non-abelian and its centre is elementary abelian.*

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## On $\pi$-normal Fitting classes

N. V. Savelyeva (Vitebsk, Belarus)

All groups considered are finite and solvable. We use standard notations and definitions from [1].

A normally hereditary class of groups $\mathfrak{F}$ is called a Fitting class if it is closed under taking of normal $\mathfrak{F}$-subgroups products.

A Fitting class $\mathfrak{F}$ is called a maximal (by inclusion) subclass of a Fitting class $\mathfrak{X}$ (denoted by $\mathfrak{F} < \cdot\mathfrak{X}$), if $\mathfrak{F} \subset \mathfrak{X}$ and $\mathfrak{F} \subseteq \mathfrak{M} \subseteq \mathfrak{X}$, where $\mathfrak{M}$ is a Fitting class, then $\mathfrak{M} \in \{\mathfrak{F}, \mathfrak{X}\}$.

In the theory of Fitting classes it is known the result of Cossey [2] that if a Fitting class $\mathfrak{F}$ is maximal in a Fitting class $\mathfrak{S}$ of all groups, then $\mathfrak{F}$ is normal.

Recall that a Fitting class $\mathfrak{F} \neq (1)$ is called normal in a Fitting class $\mathfrak{X}$, if for every $\mathfrak{X}$-group $G$ a subgroup $G_{\mathfrak{F}}$ is an $\mathfrak{F}$-maximal subgroup of $G$.

If a Fitting class $\mathfrak{F} < \cdot\mathfrak{S}_\pi$, then we call $\mathfrak{F}$ $\pi$-maximal. If a Fitting class $\mathfrak{F} \neq (1)$ is normal in the class $\mathfrak{S}_\pi$, then $\mathfrak{F}$ is called $\pi$-normal. The interrelation of properties of $\pi$-maximality and $\pi$-normality is characterized by the following

**Theorem.** *Every $\pi$-maximal Fitting class is $\pi$-normal.*

Note, if $\pi$ is a set of all primes, the theorem implies the result of Cossey [2].

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## $J$-invariant and the higher Tits indices of algebraic groups

Nikita Semenov (Muenchen, Germany)

Let $G$ be a semisimple algebraic group over a field $k$ of inner type and $X$ be a projective $G$-homogeneous variety such that $G$ splits over the field $k(X)$ of $X$ (e.g. $X$ is the variety of complete flags). We introduce an invariant of $G$ called the $J$-invariant which characterizes the motivic behavior of $X$.

This generalizes the respective notion invented by A. Vishik in the context of quadratic forms.

As the main applications we obtain in a uniform way motivic decompositions of all generically cellular projective homogeneous varieties (e.g. Severi-Brauer varieties (N. Karpenko), Pfister quadrics (M. Rost), maximal orthogonal Grassmannians, $G_2$- (J.-P. Bonnet), $F_4$-, $E_6$-, and $E_8$-varieties (S. Nikolenko, V. Petrov, N. Semenov, K. Zainoulline)), classify all exceptional generically cellular varieties and provide a list of the higher Tits indices for groups of type $E_7$.

We also discuss relations with torsion indices, canonical dimensions and cohomological invariants of the group $G$.

## Adjoint orbits of the unitriangular group

V. V. Sevostiyanova (Samara, Russia)

We consider the Lie group $G = \mathrm{SL}_n(K)$ and its Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(K)$, where char $K = 0$. Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be any Cartan decomposition of $\mathfrak{g}$.

An irreducible component of intersection of nilpotent adjoint orbit with $\mathfrak{n}$ is called an *orbital variety.* Generators of the ideal of definition of given orbital variety for $n \leqslant 6$ are constructed in [1].

Every orbital variety splits into disjoint union of nilpotent orbits of $\mathfrak{n}$ via the adjoint action of Lie group of strictly upper-triangular matrices $N = \mathrm{UT}_n(K)$.

Our goal to find generators of defining ideal for any adjoint orbits of $N$.

**Theorem.** *Let $n \leqslant 6$. One can choose the generators of defining ideal of every nilpotent orbit of adjoint action of $N$ of the form $P_i - c_i$, where $P_i$ is some coefficient of minor of the characteristic matrix, $c_i \in K$.*

We present the full description of adjoint $N$–orbits and orbital varieties in $\mathfrak{n}$ for $n \leqslant 6$. We note that they can be described in terms of coefficients of minors of characteristic matrix. We also consider some special series of orbits (regular and subregular) for arbitrary $n$.

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# Rectangular low level case of modular branching problem for $\mathrm{GL}_n(K)^\star$

V. V. Shchigolev (Moscow, Russia)

Let $K$ be an algebraically closed field of characteristic $p > 0$. Consider a rational irreducible $\mathrm{GL}_n(K)$-module $L_n(\lambda)$ with highest weight $\lambda$. An important problem is to find all nonzero $\mathrm{GL}_{n-1}(K)$-high weigh vectors of $L_n(\lambda)$, that is the vectors stabilized by the subgroup $\mathrm{U}_{n-1}(K)$ of $\mathrm{GL}_{n-1}(K)$ consisting of the upper triangular matrices with 1 on the main diagonal. Here $\mathrm{GL}_{n-1}(K)$ is naturally identified with the subgroup of $\mathrm{GL}_n(K)$ consisting of the matrices with 0 everywhere in the last column and the last row except the position of their intersection where they have 1. Actually, the problem is to find which weights such vectors can have.

In my report, I plan to speak on my recent result — an explicit combinatorial criterion for the existence of a nonzero $\mathrm{GL}_{n-1}(K)$-high weight vector of weight $(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - d, \lambda_{i+1}, \ldots, \lambda_{n-1}, \lambda_n + d)$, where $1 \leqslant d < p$ in the module $L_n(\lambda_1, \ldots, \lambda_n)$. For this purpose, I introduce new lowering operators.

To formulate this result explicitly, let us introduce the strict partial order $\dot{<}$ on $\mathbf{Z}^2$ and the subsets $\mathfrak{Y}_d^\lambda(i,n)$ and $\mathfrak{C}^\lambda(i,n)$ of $\mathbf{Z}^2$ as follows: $(a,c) \mathrel{\dot{<}} (b,d)$ holds if and only if $a < b$ and $c < d$; for any $\lambda \in \mathbf{Z}^n$, denote

$$\mathfrak{Y}_d^\lambda(i,n) :=$$

$$\{(t,h) \in \{i+1, \ldots, n\} \times \{1, \ldots, d\} \mid t - i + \lambda_i - \lambda_t - h \equiv 0 \pmod p\},$$

$$\mathfrak{C}^\lambda(i,n) := \{s \in \{i+1, \ldots, n-1\} \mid s - i + \lambda_i - \lambda_s \equiv 0 \pmod p\}.$$

Moreover, a map $\varphi : A \to B$, where $A, B \subset \mathbf{Z}^2$, is called *strictly decreasing* if $\varphi(\alpha) \mathrel{\dot{<}} \alpha$ for any $\alpha \in A$. *Column $t$* of the plain $\mathbf{Z}^2$ is the subset $\{(t,k) \mid k \in \mathbf{Z}\}$. Denote also $\alpha(s,t) := (0, \ldots, 0, 1, 0, \ldots, 0, -1, \ldots, 0)$, where 1 is at position $s$, $-1$ is at position $t$ and $1 \leqslant s < t \leqslant n$.

**Main Theorem.** *Let $\lambda \in \mathbf{Z}^n$ be a dominant weight, $1 \leqslant i < n$ and $1 \leqslant d < p$. Then the module $L_n(\lambda)$ contains a nonzero $\mathrm{GL}_{n-1}(K)$-high* *weight vector of weight $\lambda - d\,\alpha(i,n)$ if and only if for each subset $\Delta$ of $\mathfrak{Y}_d^\lambda(i,n)$ whose points are incomparable with respect to $\dot{<}$, there exists a strictly decreasing injection from $\Delta$ to $\mathfrak{C}^\lambda(i,n) \times \{0\}$.*

The existence of the above mentioned injection can be checked using only subsets of $\mathbf{Z}$. Let $\pi_1 : \mathbf{Z}^2 \to \mathbf{Z}$ denote the projection to the first component. The set $\Delta$, as well as $\mathfrak{Y}_d^\lambda(i,n)$, contains at most one point in each column, since $d < p$. Therefore, there exists a strictly decreasing injection from $\Delta$ to $\mathfrak{C}^\lambda(i,n) \times \{0\}$ if and only if there exists a weakly decreasing injection from $\pi_1(\Delta) - 1$ to $\mathfrak{C}^\lambda(i,n)$. If $d = 1$ then all points of $\mathfrak{Y}_1^\lambda(i,n)$ are automatically incomparable with respect to $\dot{<}$ and one gets the criterion of [1].

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## On the formation with given characteristic of residual

V. V. Shpakov, N. T. Vorob'ev (Vitebsk, Belarus)

In the theory of the Fitting classes of finite groups Lockett operators [1] „*“, „*“ are well known. We shall recall that if $\mathfrak{F}$ is a nonempty Fitting class then $\mathfrak{F}^*$ is the least of the Fitting classes so if $\mathfrak{F}^*$-radical of the direct product of any groups $G$ and $H$ is a direct product $\mathfrak{F}^*$-radical of these groups, and the class $\mathfrak{F}_* = \bigcap\{\mathfrak{X} : \mathfrak{X}$ Fitting classes and $\mathfrak{X}^* = \mathfrak{F}^*\}$. The dual structure in the theory of the formation was determined by Doerk and Hawkes [2]. It was installed in the work [3] that each local Fitting class $\mathfrak{F}$ is defined by Hartley functions $f^*$ and $f_*$ such a way that $f^*(p) = (f(p))^*$ and $f_*(p) = (f_*(p))$ for each simple $p$.

In the theory of the formation the dual result is achieved.

Let $f$ be satellite, local satellite $f^o$ and $f_o$ such as $f^o(p) = (f(p))^o$ and class $f_o(p) = \cap\{\varphi(p) : \varphi(p)$ formations and $(\varphi(p))^o = f^o(p)\}$ for all simple $p$. Class $f^o(p)$ is the least of the formations, such as $f(p) \subseteq f^o(p)$ and $(G \times H)^{f^o(p)} = G^{f^o(p)} \times H^{f^o(p)}$. If formation is defined local satelitte then mark as $\mathfrak{F} = LF(f)$.

**Teorema.** If $\mathfrak{F} = LF(f)$, then $\mathfrak{F} = LF(f^o) = LF(f_o)$.

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# Non-commutative plane curves

Arne B. Sletsjøe (Oslo, Norway)

In this talk we study noncommutative plane curves, i.e. non-commutative $k$-algebras, over a field $k$ for which the 1- dimensional simple modules form a plane curve. We study extensions of simple modules and we try to enlighten the completion problem, i.e. understanding the connection between simple modules of different dimensions.

# Generalized subrings of some classical rings

A. L. Smirnov (St. Petersburg, Russia)

The old idea to apply algebro-geometric methods to number theory and to diophantine problems was considerably developed recently by M. J. Shai Haran and by N. Durov.

The approach of N. Durov is based on a generalization of rings and on corresponding generalization of schemes. These generalized algebra and algebraic geometry are in the very beginning of their evolution. For example, the classification of classical finite fields was obtained at least several decades before heyday of algebraic geometry. A description of generalized finite fields is unsolved and interesting problem.

Another feature of new world is its very unusual properties. For example, a dimension of a proper closed part of a generalized irreducible scheme can be greater than the dimension of an open subscheme, the tensor product of finite generalized rings can be an infinite ring and so on.

To apply new algebraic geometry it is necessary to elaborate new intuition and to develop new computing facilities. To do this it is desirable to explore a number of examples. Results of such investigation for generalized subrings of classical rings $\mathbb{F}_q$, $\mathbb{Z}/p^2$, $\mathbb{Z}_p$, $\mathbb{Z}$, $\mathbb{R}$ will be presented in the talk.

## On idempotents in generalized rings

Mirela Ştefănescu and Camelia Ciobanu (Constanţa, România)

The generalized rings considered in the paper are nearrings with some properties, infra-nearrings and ringoids.We use semigroup properties of the multiplication for nearrings and infra-nearrings, finding Peirce decompositions and structure theorems. The idempotents in the nearrings generated by the endomorphisms or the infra-endomorphisms of a group are also studied.

For ringoids, we find some unexpected properties of idempotents, when some chain conditions are fulfilled.

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# On the behaviour of unipotent elements from subsystem subgroups in modular representations of the classical algebraic groups with large highest weights

I. D. Suprunenko (Minsk, Belarus)

The goal of this talk is to discuss some regularities concerned with the Jordan block structure of the images of unipotent elements from proper subsystem subgroups in irreducible representations of the classical algebraic groups over fields of positive characteristic. The main subject is the behaviour of such elements in irreducible representations of the special linear and symplectic groups with highest weights large enough with respect to the ground field characteristic and a fixed element.

Throughout the text a subsystem subgroup of a semisimple algebraic group is a subgroup generated by it root subgroups associated with all roots from a subsystem of the group root system. Let $K$ be an algebraically closed field of characteristic $p > 0$ and $G$ be a simply connected simple algebraic group of type $A_r$ or $C_r$ over $K$. Denote by $|x|$ the order of an element $x \in G$. Assume that $p > 2$ for $G = C_r(K)$. Recall that $G \cong SL_{r+1}(K)$ if $G = A_r(K)$ and $G \cong Sp_{2r}(K)$ for $G = C_r(K)$. For an element $u \in G$ with $|u| = p^{m+1}$ ($m$ may be zero) define the collection $S(u)$ as follows. Let $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_t$ denote the sizes of all Jordan blocks of $u^{p^m}$ in the standard realization of $G$. Here $k_1 + k_2 + \ldots + k_t = r + 1$ for $G = A_r(K)$ and $2r$ for $G = C_r(K)$. Then put

$$\begin{aligned} S(u) =& \{k_1 - 1, k_1 - 3, \ldots, 1 - k_1, k_2 - 1, k_2 - 3, \\ & \ldots, 1 - k_2, \ldots, k_t - 1, k_t - 3, \ldots, 1 - k_t\}. \end{aligned}$$

For $1 \leqslant i \leqslant r$ let $\omega_i$ be the $i$th fundamental weight of $G$ (the labeling is standard) and $b_i(u)$ be the sum of $i$ largest members of $S(u)$. For an irreducible representation $\varphi$ of $G$ with highest weight $\sum_{i=1}^r a_i\omega_i$ set $b(\varphi, u) = \sum_{i=1}^r a_i b_i(u)$. Recall that such representation is $p$-restricted if all the coefficients $a_i < p$. We consider the behaviour of $u$ in $p$-restricted representations of $G$ with $b(\varphi, u)$ large enough.

**Theorem 1.** *Assume that a unipotent element $u \in G$ lies in a subsystem subgroup of type $A_l$ with $l < r-1$ for $G = A_r(K)$ and in such subgroup of type $C_l$ with $l < r$ for $G = C_r(K)$. Let $\varphi$ be a $p$-restricted representation* *of $G$ with $b(\varphi,u) \geqslant p + b_1(u)$. Then the element $\varphi(u)$ has at least $r-l$ Jordan blocks of size $|u|$ for $G = A_r(K)$ and at least $2(r-l)$ such blocks for $G = C_r(K)$.*

It follows from [2, Theorem 1.10] that $\varphi(u)$ has no blocks of size $|u|$ if $b(\varphi,u) < p-1$. We have some reasons to expect that in the majority of cases the assertion of Theorem 1 holds under a weaker assumption that $b(\varphi,u) \geqslant p$. However, the following proposition shows that one cannot simply take this weaker assumption in Theorem 1.

**Proposition 2.** *Let $G = A_r(K)$, $\varphi$ be an irreducible representation of $G$ with highest weight $a_1\omega_1 + a_r\omega_r$, $a_1a_r \neq 0$, and $a_1 + a_r = p$. Assume that $p^s < r$ and $u$ is a regular unipotent element in a subsystem subgroup of $G$ of type $A_{p^s}$. Then $\varphi(u)$ has just two Jordan blocks of size $|u|$.*

One can deduce that in the assumptions of Proposition 2 $b(\varphi,u) = p$ and $b_1(u) = 1$.

Earlier lower estimates for the number of Jordan blocks of the maximal possible size in the images of unipotent elements of fixed order in irreducible representations of the classical algebraic groups with large highest weights were obtained [1]. But those results concern only representations where all unipotent elements have the minimal polynomials of degree equal to their orders. In the assumptions of Theorem 1 it may occur that $\sum_{i=1}^{r} a_i < p-1$, but $b(\varphi,u) \geqslant p+b_1(u)$. Then it follows from [2, Theorem 1.10] that all unipotent elements whose power is a root element have the minimal polynomials of degrees smaller than their orders and hence have no blocks of such size.

Some other results concerning the Jordan block structure of the images of unipotent elements from subsystem subgroups of small rank in irreducible representations of the classical groups can be discussed as well.

This research is a part of a more general program concerned with investigating properties of unipotent elements in modular representations of semisimple algebraic groups and elaborating machinery for recognizing representations by such properties. For this purpose it is worth to distinguish some "rare"classes of unipotent elements whose presence can be effectively used for recognizing representations and linear groups. To find such classes, it is necessary to investigate in details properties of these elements (strictly speaking, of their images) in representations.

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## Cyclicity of algebras after a scalar extension

S. V. Tikhonov (Minsk, Belarus)

The goal of the talk is to present our recent result (joint with U. Rehmann and V.I. Yanchevskii) concerning properties of central simple algebras.

In particular, we are going to discuss in detail the proof of the following result.

**Theorem.** Let $A_1 \dots A_m$ be central simple algebras over a field $k$. Assume $\xi_p \in k$ for any prime $p$ dividing $\prod_{i=1}^{m} ind(A_i)$ where $\xi_p$ is a primitive $p$-th root of unity. Then there exists a regular field extension $E/k$ such that all the algebras $A_i \otimes_k E$ are cyclic, $ind(C \otimes_k E) = ind(C)$ and $exp(C \otimes_k E) = exp(C)$ for any central simple $k$-algebra $C$.

## Amply Supplemented Lattices

S. Eylem Toksoy (Izmir, Turkey)

$L$ will mean a complete modular lattice with greatest element 1. An element $c$ of a lattice $L$ is called *compact* if for every subset $X$ of $L$ with $c \leqslant \bigvee X$ there exists a finite subset $F$ of $X$ such that $c \leqslant \bigvee F$. A lattice $L$ is said to be *compact* if 1 is compact. An element $a$ of a lattice $L$ has *ample supplements* in $L$ if for every element $b(\neq 1)$ of $L$ with $a \vee b = 1$, $b/0$ contains a supplement of $a$ in $L$, that is a minimal element $c \in L$ with respect to $a \vee c = 1$. A lattice $L$ is said to be *amply supplemented* if every element of $L$ has ample supplements.

The following theorem generalizes the result for modules: a finitely generated module $M$ is amply supplemented if and only if every maximal submodule has ample supplements in $M$ (see 20.24 in [3]).

**Theorem.** *A compact lattice $L$ is amply supplemented if and only if every maximal element has ample supplements in $L$.*

If $a \vee b = 1$ and $a \wedge b \ll L$, then $b$ is said to be *weak supplement* of $a$. $L$ is called *weakly supplemented* if every element of $L$ has a weak supplement in $L$. Given elements $a \leqslant b$ of $L$, the inequality $a \leqslant b$ is called *cosmall* in $L$ if $b \ll 1/a$. An element $c$ of $L$ is called *coclosed* in $L$ if there is no proper element $a$ of $c/0$ for which $a \leqslant c$ is cosmall in $L$, i.e. the inequality $a \leqslant c$ is cosmall in $L$ implies $a = c$. An element $a \leqslant b$ is said to be *coclosure* of $b$ in $L$ if the inequality $a \leqslant b$ is cosmall in $L$ and $a$ is coclosed in $L$.

**Theorem.** (c.f. [3], 20.25) *$L$ is amply supplemented if and only if it is weakly supplemented and every element of $L$ has a coclosure in $L$.*

**Joint work with:** Refail Alizade, İzmir Institute of Technology

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## Totally Weak Supplemented Modules

Serpil Top (Izmir, Turkey)

$M$ will mean an $R$-module where $R$ is an arbitrary ring with identity. A module $M$ is weakly supplemented if every submodule $U$ of $M$ has a weak supplement in $M$, i.e. $U + V = M$ and $U \cap V \ll M$ for some submodule $V$ of $M$. $M$ is totally weak supplemented if every submodule of $M$ is weakly supplemented. A module $M$ is called *hollow* if every proper submodule of $M$ is small(superfluous) in $M$. A module is called *linearly compact* if for every family of cosets $\{x_i + M_i\}_{\triangle}$, $x_i \in M$, and submodules $M_i \subset M$ (with $M/M_i$ finitely cogenerated) such that the intersection of any finitely many of these cosets is not empty, the intersection is also not empty. (see [1])

The following example shows that if $N$ and $M/N$ are weakly supplemented, $M$ need not be weakly supplemented in general.

**Example 1.** *Let $F$ be a field and the commutative ring $S$ be the direct product $\prod_{n\in\mathbb{N}} F_n$, where $F_n = F (n \geqslant 1)$ and the element of $S$ are the sequences $\{a_n\}$ where $a_n \in F(n \in \mathbb{N})$. Let $R$ be the subring of $S$ consisting of all sequences $\{a_n\}$ such that there exists $a \in F$, $k \in \mathbb{N}$ with $a_n = a$ for all $n \geqslant k$. The $\operatorname{Soc} R$ of the $R$-module $R$ and $R/\operatorname{Soc} R$ are weakly supplemented but $R$-module $R$ is not weakly supplemented.*

In special case we have the following result.

**Предложение 1.** *$M$ is weakly supplemented if and only if $M/\oplus_{i=1}^{n} L_i$ is weakly supplemented for a finite direct sum of hollow submodules $L_i$ of $M$.*

**Theorem.** (c.f. [2], [3]) *$M$ is totally weak supplemented if and only if $M/U$ is totally weak supplemented for a linearly compact submodule $U$ of $M$.*

**Joint work with:** Rafail Alizade, İzmir Institute of Technology

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## Generated Fischer formations

A. A. Tsarev, N. N. Vorob'ev (Vitebsk, Belarus)

All groups considered are finite and soluble. We use the terminology of [1]. The symbol $\mathfrak{N}$ denotes the class of all nilpotent groups. Recall that for any class of groups $\mathfrak{F} \supseteq (1)$ an intersection of all such normal subgroups $N$ of a group $G$ that $G/N \in \mathfrak{F}$ is called an $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of

$G$. A class of groups is called a formation if it is closed under taking homomorphic images and subdirect products.

Let $\mathfrak{X}$ be a class of groups. Then $\mathrm{S_F}\mathfrak{X} = (G \mid G \leqslant H \in \mathfrak{X}$ and $G^{\mathfrak{N}} \lhd\lhd H)$.

The class of groups $\mathfrak{X} \neq \varnothing$ is called a Fischer class if $\mathfrak{X} = \mathrm{S_F}\mathfrak{X}$ and $\mathfrak{X} = \mathrm{N_0}\mathfrak{X}$. We call a *Fischer formation* (by N.T. Vorob'ev's proposition) the class of groups which is simultaneously a formation and a Fischer class. We write $\mathrm{S_F Fit}(\mathfrak{Y})$ to denote the smallest $\mathrm{S_F}$-closed class containing a non-empty set of groups $\mathfrak{Y}$.

**Theorem.** *Let $\mathfrak{X}_1, ..., \mathfrak{X}_n, \mathfrak{Y}_1, ..., \mathfrak{Y}_m$ be Fischer formations. Then*

$$\mathrm{S_F Fit}((\bigcap_{i=1}^{n} \mathfrak{X}_i) \bigcup (\bigcap_{j=1}^{m} \mathfrak{Y}_j)) = \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} \mathrm{S_F Fit}(\mathfrak{X}_i \bigcup \mathfrak{Y}_j).$$

The symbol $\mathrm{SFit}(\mathfrak{Y})$ denotes the smallest hereditary class containing a non-empty set of groups $\mathfrak{Y}$.

**Corollary [2].** *Let $\mathfrak{X}_1, ..., \mathfrak{X}_n, \mathfrak{Y}_1, ..., \mathfrak{Y}_m$ be hereditary Fitting classes. Then*

$$\mathrm{SFit}((\bigcap_{i=1}^{n} \mathfrak{X}_i) \bigcup (\bigcap_{j=1}^{m} \mathfrak{Y}_j)) = \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} \mathrm{SFit}(\mathfrak{X}_i \bigcup \mathfrak{Y}_j).$$

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## On the recognition of a finite simple group by its spectrum

A. V. Vasilev (Novosibirsk, Russia)

Let $G$ be a finite group, $\pi(G)$ be the set of prime divisors of its order and $\omega(G)$ be the spectrum of $G$, that is the set of element orders of $G$. The prime graph $GK(G)$ of a group $G$ is defined as follows. The vertex set of $GK(G)$ is $\pi(G)$ and two primes $r, s \in \pi(G)$ considered as vertices of the graph are adjacent by the edge if and only if $rs \in \omega(G)$. K. W. Gruenberg and O. Kegel introduced this graph (it is also called the Gruenberg — Kegel graph) in the middle of 1970th and gave a characterization of finite groups with a disconnected prime graph (we denote the number of connected components of $GK(G)$ by $s(G)$). This deep result and a classification of finite simple groups with $s(G) > 1$ obtained by

J. S. Williams and A. S. Kondrat'ev (see [1–2]) implied a series of important corollaries.

The proof of the Gruenberg–Kegel Theorem relies substantially upon the fact that $G$ contains an element of odd prime order which is disconnected with 2 in $GK(G)$. It turned out that disconnectedness could be successfully replaced in most cases by a weaker condition for the prime 2 to be nonadjacent to at least one odd prime.

Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $GK(G)$. In other words, $t(G)$ is a maximal number of vertices in independent sets of $GK(G)$. In graph theory this number is usually called an independence number of the graph. By analogy we denote by $t(r, G)$ the maximal number of vertices in independent sets of $GK(G)$ containing the prime $r$. We call this number an $r$-independence number. Recently, in [3] it was given a characterization of finite groups with $t(G) \geqslant 3$ and $t(2, G) \geqslant 2$, and in [4] it was proved that all finite nonabelian simple groups except the alternating permutation groups satisfy the condition $t(2, G) \geqslant 2$. Here we give a refinement of the main theorem of [3].

**Theorem 1.** *Let $G$ be a finite group with $t(G) \geqslant 3$ and $t(2, G) \geqslant 2$. Then*

(1) *There exists a finite simple nonabelian group $S$ such that $S \leqslant \overline{G} = G/K \leqslant \mathrm{Aut}(S)$ for maximal soluble normal subgroup $K$ of $G$.*

(2) *For every independent subset $\rho$ of $\pi(G)$ with $|\rho| \geqslant 3$ at most one prime in $\rho$ divides the product $|K| \cdot |\overline{G}/S|$. In particular, $t(S) \geqslant t(G) - 1$.*

(3) *One of the following holds:*

(a) *Every prime $r \in \pi(G)$ non-adjacent in $GK(G)$ to 2 does not divide the product $|K| \cdot |\overline{G}/S|$. In particular, $t(2, S) \geqslant t(2, G)$.*

(b) *There exists the unique prime $r \in \pi(K)$ non-adjacent in $GK(G)$ to 2, in which case $t(G) = 3$, $t(2, G) = 2$, and $S \simeq Alt_7$ or $A_1(q)$ for some odd $q$.*

The above characterization with the description of prime graph of every finite nonabelian simple group (see [4]) can be applied to a so-called recognition problem. Namely, the question is as follows: for a given finite nonabelian simple group $L$ and a finite group $G$ with $\omega(G) = \omega(L)$, what can we say about the structure of $G$? If it turns out that the equality $\omega(G) = \omega(L)$ implies $G \simeq L$, then $L$ is said to be recognizable. Clearly, the equality $\omega(G) = \omega(L)$ implies the coincidence of the prime graphs of $G$ and $L$. Thus, if $L$ satisfies the condition of Theorem 1, then so does $G$. The statement (1) of the conclusion of Theorem 1 implies that $G$ has the unique nonabelian composition factor $S$. On the other hand, the statements (2) and (3) help to prove that this factor $S$ is isomorphic to $L$. The described method allowed to obtain a series of results on the recognition of finite nonabelian simple group. In particular, the recognizability of infinite series of finite simple linear groups with connected prime graph was established (see [5]). Here, we mention the following general result on exceptional groups of Lie type.

**Theorem 2.** *Let $L$ be a finite simple exceptional group of Lie type and $G$ be a finite group with $\omega(G) = \omega(L)$. Then $L \leqslant G/K \leqslant \mathrm{Aut}(L)$ for maximal soluble normal subgroup $K$ of $G$.*

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## On the existence of regular orbits of solvable subgroups

E. P. Vdovin (Novosibirsk, Russia)

The following theorems are the main result of our work.

**Theorem 1.** *Let $G$ be a solvable subgroup of $GL_n(q)$ and $(|G|, q) = 1$. Then there exist vectors $u, v \in GF(q)^n$ such that $C_G(u) \cap C_G(v) = \{e\}$.*

**Theorem 2.** *Let $G$ be a solvable subgroup of $GL_n(q)$ and $(|G|, q) = 1$. Consider the action of $G$ on $V \times V$ given by $(u, v)g = (ug, vg)$. Then there exists a regular orbit of $G$ on the set $V \times V$. In particular $|G| < |V|^2$.*

Known examples show that Theorems 1 and 2 are not valid without the restriction $(|G|, q) = 1$. The following result is immediate as a corollary of Theorem 1.

**Theorem 3.** *Let $\pi$ be a set of primes, $G$ a finite $\pi$-solvable group, and $H$ a Hall $\pi$-subgroup of $G$. Then there exist $x, y \in G$ such that $H \cap H^x \cap H^y = O_\pi(G)$.*

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## Modular elements of the lattice of semigroup varieties

B. M. Vernikov (Ekaterinburg, Russia)

An element $x$ of a lattice $\langle L; \vee, \wedge\rangle$ is called *modular* (*upper-modular*) if $(x\vee y)\wedge z = (x\wedge z)\vee y$ (respectively $(z\vee y)\wedge x = (z\wedge x)\vee y$) for all $y, z \in L$ with $y \leqslant z$ (respectively $y \leqslant x$). *Lower-modular* elements are defined dually to upper-modular ones. A semigroup variety is called *modular* (*upper-modular, lower-modular*) if it is a modular (upper-modular, lower-modular) element of the lattice of all semigroup varieties. A number of results about varieties of these three types are obtained in [1–6].

An identity of the form $u = 0$ is called 0-*reduced*. We call an identity $u = v$ *substitutive* if $u$ and $v$ depend on the same letters and $v$ may be obtained from $u$ by renaiming of letters. We denote by $\mathcal{T}$, $\mathcal{SL}$ and $\mathcal{SEM}$ the trivial variety, the variety of all semilattices and the variety of all semigroups respectively. The following theorem essentially sharpen [2, Proposition 1.6].

**Theorem 1.** *If a semigroup variety $\mathcal{V}$ is modular then either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V} = \mathcal{X} \vee \mathcal{N}$ where $\mathcal{X} \in \{\mathcal{T}, \mathcal{SL}\}$ and $\mathcal{N}$ is a nil-variety given by* 0-*reduced and substitutive identities only.*

It is proved independently in [1, Corollary 3] and [2, Proposition 1.1] that if a variety is given by 0-reduced identities only then it is modular.

**Theorem 2.** *For a commutative semigroup variety $\mathcal{V}$ the following are equivalent*: (i) *$\mathcal{V}$ is modular*; (ii) *$\mathcal{V}$ is modular and upper-modular*; (iii) *$\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{N}$ where $\mathcal{N}$ satisfies the identities $x^2y = 0$ and $xy = yx$.*

A commutative upper-modular and commutative lower-modular varieties were completely determined in [5] and [6] respectively.

**Theorem 3.** *If a modular semigroup variety $\mathcal{V}$ satisfies a permutable identity of length n then it satisfies also all permutable identities of length $n+1$. If, besides that, $n \geqslant 4$ and $\mathcal{V}$ is a nil-variety then it satisfies also all identities of the form $u = 0$, where u is a word of length n depending on $n-1$ letters.*

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# Hochschild cohomology for a family of self-injective algebras of tree class $D_n$

Y. V. Volkov, A. I. Generalov (St. Petersburg, Russia)

Let $R$ be a representation-finite self-injective basic algebra over an algebraically closed field. It is known that the stable $AR$-quiver of the such algebra is described with using an associate tree and this tree is one of the Dynkin diagram $A_n, D_n, E_6, E_7$, or $E_8$ [1]. If this tree is $A_n$, then the algebra $R$ is stably equivalent to some serial self-injective algebra or to so called "Möbius algebra" [2]. In [3] the Hochschild cohomology algebra $\mathrm{HH}^*(R)$ was calculated for serial self-injective algebras, and for Möbius algebra the subalgebra $\mathrm{HH}^{*r}(R)$ of the algebra $\mathrm{HH}^*(R)$ was calculated in [4] (here, $r$ is a parameter related with the algebra $R$). In these papers the fact that the syzygy of an appropriate order for the $R$-bimodule $R$ can be described as twisted bimodule, was essentially used. More direct approach to the calculation of Hochschild cohomology for Möbius algebra $R$ was initiated in [5, 6]. Namely, the minimal projective resolution for algebra $R$ as a $\Lambda$-module, where $\Lambda$ is the enveloping algebra of the algebra $R$, was constructed, and then this resolution was used for calculation of the additive structure for the algebra $\mathrm{HH}^*(R)$, i.e. dimensions of the groups $\mathrm{HH}^n(R)$ were calculated.

In the talk we give the description of the additive structure of the algebra $\mathrm{HH}^*(R)$ for a family of representation-finite self-injective algebras with the associate tree $D_n$ ($n \geqslant 4$). As in [5], first we construct the minimal projective bimodule resolution for these algebras.

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## Modular forms with multiplicative coefficients and group representations

G. V. Voskresenskaya (Samara, Russia)

Let $G$ be a finite group. We can associate with each $g \in G$ a modular form by means of a representation $T$. Such representation has some special properties but it can be constructed for every finite group. In particular, $24 \mid dim\ T$. Modular forms in this correspondence are products of Dedekind $\eta-$ functions of various arguments.

We consider general properties of this correspondence and study in details the open problem of finding such finite groups that the modular forms associated with all elements of these groups by means of a certain faithful representation belong to a special class of modular forms. These groups are called $M\eta P$ groups.

Modular forms from this class are completely determined by the following conditions: they are cusp forms of integer weights with characters, all their zeroes are in the cusps and each zero has the multiplicity one. A priori we don't give other assumptions but in fact these functions are eigenforms of Hecke algebra and can be expressed as products of Dedekind $\eta-$functions of various arguments. Their Fourier coefficients are multiplicative and they are called *multiplicative* $\eta-$*products.*

G.Mason proved that $M_{24}$ is an $M\eta P-$group. But there are many $M\eta P-$groups which are not subgroups in $M_{24}$. So we have a nontrivial problem of classification. We present at the conference our results on this problem.

Also we investigate the arithmetic properties of the Fourier coefficients of multiplicative $\eta-$products. We consider Shimura sums related to the modular forms and prove several families of identities involving them. The type of identity obtained depends on the splitting of primes in certain imaginary quadratic number fields. Shimura sums are in a sense analogous to Gaussian and Jacobian sums.

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## The classical reciprocity law as an analog of an Abelian integral theorem

S. V. Vostokov (St. Petersburg, Russia)

Already Kronecker understood that there was a deep relationship between algebraic numbers and algebraic functions. From this point of view, the Hilbert reciprocity law (the product of local norm residue symbols is equal to 1), as emphasized by Hilbert himself and later by

Shafarevich, is an analog of the Cauchy theorem stating that the sum of residues of a differential form on a Riemann surface is zero. In this talk, we consider the classical reciprocity law for algebraic number fields from the same point of view. It is well known that the main problem in the classical reciprocity law is finding an explicit formula for the product of power residues. Class field theory connects this product with the product of local norm residue symbols (Hilbert symbols). Thus, an analogy between the classical reciprocity law and the theorem stating that the integral of a differential form on a Riemann surface is equal to the sum of residues of this form arises. Using explicit formulas for the Hilbert symbol, we make this analogy explicit and obtain an explicit global reciprocity law.

## On (2,3) generation of matrix groups over the ring of integers

M. A. Vsemirnov (St. Petersburg, Russia)

The group is called $(2,3)$-generated if it can be generated by an involution and an element of order three. The list of (2,3)-generated classical matrix groups over finite fields is known (see [1] and references therein). A similar problem for matrix groups over integers still remains open. For instance, it is known that the groups $\mathrm{SL}_n(\mathbb{Z})$, $n \geqslant 13$, and $\mathrm{GL}_n(\mathbb{Z})$, $n \geqslant 19$, are (2,3)-generated [4], while $\mathrm{SL}_2(\mathbb{Z})$, $\mathrm{SL}_3(\mathbb{Z})$, $\mathrm{SL}_4(\mathbb{Z})$, $\mathrm{GL}_2(\mathbb{Z})$, $\mathrm{GL}_3(\mathbb{Z})$, $\mathrm{GL}_4(\mathbb{Z})$ are not [3],[5].

The main result presented here is the following.

**Theorem 9.** *For $n = 5, 6, 7$, the groups $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{Z})$ are $(2,3)$-generated.*

The results are constructive and explicit generators are provided.

In the case of $\mathrm{GL}_5(\mathbb{Z})$ Luzgarev and Pevzner [2] reduced the problem to the study of ten explicit pairs of matrices. However, they were not able to determine whether these pairs actually generate $\mathrm{GL}_5(\mathbb{Z})$ or not. We strengthen their results and give a complete description of $(2,3)$-generating pairs.

**Theorem 10.** *The groups* $\mathrm{GL}_5(\mathbb{Z})$ *and* $\mathrm{SL}_5(\mathbb{Z})$ *are* $(2,3)$*-generated. Moreover, any* $(2,3)$*-generating pair for* $\mathrm{GL}_5(\mathbb{Z})$ *(resp.,* $\mathrm{SL}_5(\mathbb{Z})$*) is* $\mathrm{GL}_5(\mathbb{Z})$*-conjugated to one the pairs* $(x,y)$ *(resp.,* $(-x,y)$*), where*

$$x = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 & 0 & a_1 \\ -1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 1 & a_3 \\ 0 & 0 & -1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

*and the set* $(a_1, a_2, a_3, a_4)$ *is one of the following sets:*

$$(1,-1,-2,-2), \quad (0,-1,-2,-2), \quad (-1,1,-2,-2),$$
$$(0,1,-2,-2), \quad (1,-1,1,-3), \quad (0,-1,0,-1).$$

One may expect a complete classification of (2,3)-generating pairs also in the case of $\mathrm{SL}_6(\mathbb{Z})$. The corresponding was reduced [6] to the study of 32 explicit pairs of matrices. Now it is known that these pairs generate $\mathrm{SL}_6(\mathbb{Z})$ in four cases, they generate a proper subgroup in 16 cases, while 12 cases remain open.

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# Prescribed properties and imbedding problems for central simple algebras under central field extensions

V. I. Yanchevskii (Minsk, Belarus)

Let $A_1, \dots, A_s$ be central simple finite-dimensional algebras over a field $F$. We will be interested in the following question: does there exist a field extension $E/F$ such that $A_1 \otimes_F E, \dots, A_s \otimes_F E$ have some prescribed property $\mathcal{P}$?

Example 1. $\mathcal{P} = \{A_i \otimes_F E$ is cyclic and $ind(A_i) = ind(A_i) \otimes_F E$ for each $i = 1, \dots, s\}$, where $ind(A)$ denotes the index of the algebra $A$.

Example 2. $\mathcal{P} = \{ind(A_i) = ind(A_i) \otimes_F E$ and $exp(A_i \otimes_F E)$ is a prescribed divisor of $exp(A_i)$ for each $i = 1, \dots, s\}$, where $exp(A)$ denotes the exponent of the algebra $A$.

For $s = 1$ and $\mathcal{P}$ as in examples 1,2 the solutions of the corresponding problems are well known [1], [2] (see also [3] and [4]). The talk will be devoted to the case $s > 1$.

Among others we are going to discuss a few recent (joint with U. Rehmann and S.V. Tikhonov) results on this topic such as theorems below.

**Theorem 1.** Let $A_1 \dots A_m$ be central simple algebras over a field $k$. Assume $\xi_p \in k$ for any prime $p$ dividing $\prod_{i=1}^m ind(A_i)$ where $\xi_p$ is a primitive $p$-th root of unity. Then there exists a regular field extension $E/k$ such that all the algebras $A_i \otimes_k E$ are cyclic, $ind(C \otimes_k E) = ind(C)$ and $exp(C \otimes_k E) = exp(C)$ for any central simple $k$-algebra $C$.

**Theorem 2.** Let $A_1, \dots, A_n$ be central simple algebras over a field $k$, $exp(A_i) = ind(A_i) = n_i$. Assume

$$< [A_i] > \cap < [A_1], \dots, [A_{i-1}], [A_{i+1}], \dots, [A_n] >= \{0\}.$$

Let also $m_i$, $1 \leqslant i \leqslant n$, be such that $m_i | n_i$ and for any prime $p$ dividing $n_i$ one has $p | m_i$. Then there exists a field extension $E/k$ such that $exp(A_i \otimes_k E) = m_i$ and $ind(A_i \otimes_k E) = ind(A_i)$, $1 \leqslant i \leqslant n$.

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## Motivic decompositions of projective homogeneous varieties

Kirill Zainoulline (St. Petersburg, Russia)

Let $G$ be a linear algebraic group over a field $F$ and $X$ be a projective homogeneous $G$-variety such that $G$ splits over the function field of $X$. In the present talk we introduce an invariant of $G$ called $J$-invariant which characterizes the motivic behaviour of $X$. This generalizes the respective notion invented by A. Vishik in the context of quadratic forms. As a main application we obtain a uniform proof of all known motivic decompositions of generically split projective homogeneous varieties (Severi-Brauer varieties, Pfister quadrics, maximal orthogonal Grassmannians, $G_2$- and $F_4$-varieties) as well as provide new examples (exceptional varieties of types $E_6$, $E_7$ and $E_8$). We also discuss relations with torsion indices, canonical dimensions and cohomological invariants of the group $G$.

## A characterization of finite groups by the prime graph

A. V. Zavarnitsine (Novosibirsk, Russia)

The *prime graph* $\Gamma(G)$ of a finite group $G$, also often called the Gruenberg–Kegel graph, is a graph whose vertex set is the set $\pi(G)$ of prime divisors of the order $|G|$, two vertices $p, q \in \pi(G)$ being joined by an edge if and only if $G$ contains an element of order $pq$. For example, a prime graph of the simple group $G_2(7)$ is shown in fig. 1. The group $G$ is called *recognizable by the prime graph* if, for every finite group $H$, the equality of the vertex-labeled graphs $\Gamma(H) = \Gamma(G)$ implies the isomorphism $H \cong G$. The recognition problem of finite groups by the prime graph is similar to the problem, popularly studied recently, of recognizing groups by element orders. Obviously, the recognizability by graph is stronger than the recognizability by the set $\omega(G)$ of element orders of the group $G$, since the knowledge of $\omega(G)$ allows one to determine $\Gamma(G)$, but in general not vice versa.

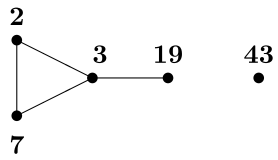


Рис. 1. $\Gamma(G_2(7))$

We obtain [1] examples of groups recognizable by the prime graph.

**Theorem.** *Suppose that the group $G \cong G_2(q)$, with $q \equiv 1 \pmod 3$ odd, acts on a vector space $V$ over a field of characteristic 3. Then $3(q^2 - q + 1) \in \omega(VL)$.*

**Corollary.** *The group $G_2(7)$ is recognizable by the prime graph.*

In particular, $G_2(7)$ is a new example of a group recognizable by the set of element orders.

**Theorem.** *Suppose that the group $G \cong {}^2G_2(q)$, with $q = 3^{2m+1} > 3$, acts on a vector space $V$ over a field of characteristic 2. Then $2(q - \sqrt{3}q + 1) \in \omega(VL)$.*

**Corollary.** *All the groups ${}^2G_2(q)$ are recognizable by the prime graph.*

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# On Iversen's formula for morphisms of algebraic surfaces

I. B. Zhukov (St. Petersburg, Russia)

We give a report on the work aimed at a 2-dimensional analog of Riemann-Hurwitz formula. The classical Riemann-Hurwitz formula compares the Euler characteristics of smooth projective algebraic curves when a finite morphism between the curves is given. Namely, if $Y$ and $X$ are such curves over an algebraically closed field $k$ (of any characteristic), and $f : Y \to X$ is a finite morphism of degree $n$, then

$$\chi_Y - n\chi_X = \sum_Q v_Q(D_Q),$$

where $\chi_X = 2g(X) - 2$, $\chi_Y = 2g(Y) - 2$, the sum is taken over all closed points of $Y$, $vQ$ is the valuation at $Q$, and $D_Q$ is the different ideal in the extension of discrete valuation rings $\mathcal{O}_{Y,Q}/\mathcal{O}_{X,f(Q)}$. In the case of surfaces one can seek for similar formulas for different discrete invariants of a surface $S$. In particular, one can consider 1) the Euler number $\chi_S$ defined as the degree of the second Chern class of $S$; 2) the Euler characteristic of the structure sheaf $\chi(S, \mathcal{O}_S)$. B. Iversen in his work "Numerical invariants and multiple planes"(Amer. J. Math. 92 (1970), 968–996) established Riemann-Hurwitz-like formulas for $\chi_S$ and $\chi(S, \mathcal{O}_S)$ of complete smooth surfaces over an algebraically closed field of characteristic zero. We make an attempt to extend his results to fields of prime characteristic, at least for the Euler numbers. The central point is the definition of an invariant describing wild ramification in codimension 2.

What is done so far? We derived the following formula comparing the Euler numbers of surfaces in any characteristic. Let $f : T \to S$ be a finite morphism of degree $n$ of complete smooth surfaces over an algebraically closed field. Let $B_f = \sum_i b_i B_i$ be the branch divisor (here $B_i$ are prime divisors on $S$). Assume that $B_f$ is a normal crossing divisor. Then

$$\chi_T - n\chi_S = \sum_i b_i \chi_{Bi} + \sum_Q \lambda_{\mathfrak{q}}(\widehat{\mathcal{O}_{T,Q}}/\widehat{\mathcal{O}_{S,f(Q)}}).$$

Here $Q$ runs over closed point of $T$, $\lambda_{\mathfrak{q}}(A'/A)$ is a certain invariant defined explicitly for an extension of complete 2-dimensional regular local rings $A'/A$ and a (sufficiently general) unramified prime ideal $\mathfrak{q}$ of $A$ of height 1, and the ideals $\mathfrak{q}$ are chosen in a certain coherent way. This invariant is defined in terms of the different of $A'/\mathfrak{q}$ over $A/(\mathfrak{q} \cap A)$, the invariants of singularity of arcs corresponding to $A'/\mathfrak{q}$ and $A/(\mathfrak{q} \cap A)$, and the invariants of intersection of the latter arc with the branch divisor. The summand $\lambda_\mathfrak{q}(\widehat{\mathcal{O}_{T,Q}}/\widehat{\mathcal{O}_{S,f(Q)}})$ is non-vanishing only for a finite number points $Q$, all of them lying on the ramification divisor of $f$. This result is only a step towards the solution of the original problem, because we could not avoid the dependence on $\mathfrak{q}$ in the definition of the term $\lambda$ that describes the ramification in codimension 2. However, examples suggest that $\lambda_\mathfrak{q}$ does not actually depend on $\mathfrak{q}$, and therefore, the formula is expected to be in its final form. What is important, this term depends on infinitesimal (rather than merely local) behavior of $f$, i. e., on the properties of extensions of completed local rings, and this reduces the further analysis to some questions related only to complete regular local rings.

## Groups acting transitively and flag-transitively on projective spaces

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Let $K$ be a (commutative or skew) field and $PG(n, K)$ be a projective space over $K$. The projective linear group $PGL_{n+1}(K)$ naturally acts on $PG(n, K)$. Under such an action, the classical group $PSO_{n+1}(\mathbb{R})$ over the real field $\mathbb{R}$ acts transitively on the projective space $PG(n, \mathbb{R})$. Moreover it acts flag-transitively, i.e. it moves a maximal flag of the space to any other one. Another example of transitive subgroup of the projective linear group is the subgroup $PSL_2(\mathbb{Z})$ of the group $PGL_2(\mathbb{Q})$. The following two problems are part of the problem 11.70 from the Kourovka Notebook. This problem was posed by P. Neumann and Ch. Praeger for infinite fields. (1) Find all subgroups of $PGL_2(K)$, which act transitively on the projective line $PG(1, K)$; (2) under what conditions does the subgroup $PGL_{n+1}(R)$ over a subring $R$ of the field $K$ act flag-transitively on $PG(n, K)$?

The next theorem gives answer for the second part of the problem for commutative fields.

**Theorem 1.** *Let $R$ be a subring of a (commutative) field $K$. Then the subgroup $PGL_{n+1}(R)$ acts flag-transitively on the projective space $PG(n, K)$ if and only if $R$ is a Bezout ring and its quotient field coincides with $K$.*

The second result gives answer for the first part of the problem for locally finite fields.

**Theorem 2.** *Let $K$ be a locally finite field and $G$ be a subgroup of $PGL_2(K)$. If $G$ acts transitively on the projective line $P(1, K)$ then the only following cases are possible:*

*(i) $G = PGL_2(K)$;*

*(ii) $G = PSL_2(K)$;*

*(iii) $G$ is a maximal subgroup of $PGL_2(K)$ that conjugated over quadratic extension of $K$ to the monomial subgroup;*

*(iv) $G$ has index 2 in the subgroup from the previous case.*

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## Фонд Эйлера

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