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St.Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences

International Conference in Spectral Theory

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Dedicated to the memory of M. Sh. Birman (1928–2009)

Supported by RFBR

Program

Abstracts

St.Petersburg, 2009

Organizers:

Alexandre Fedotov, Nikolai Filonov, Ari Laptev, Alexander Pushnitski

Organizing committee:

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Yanina Shibaeva, Nadia Zalesskaya

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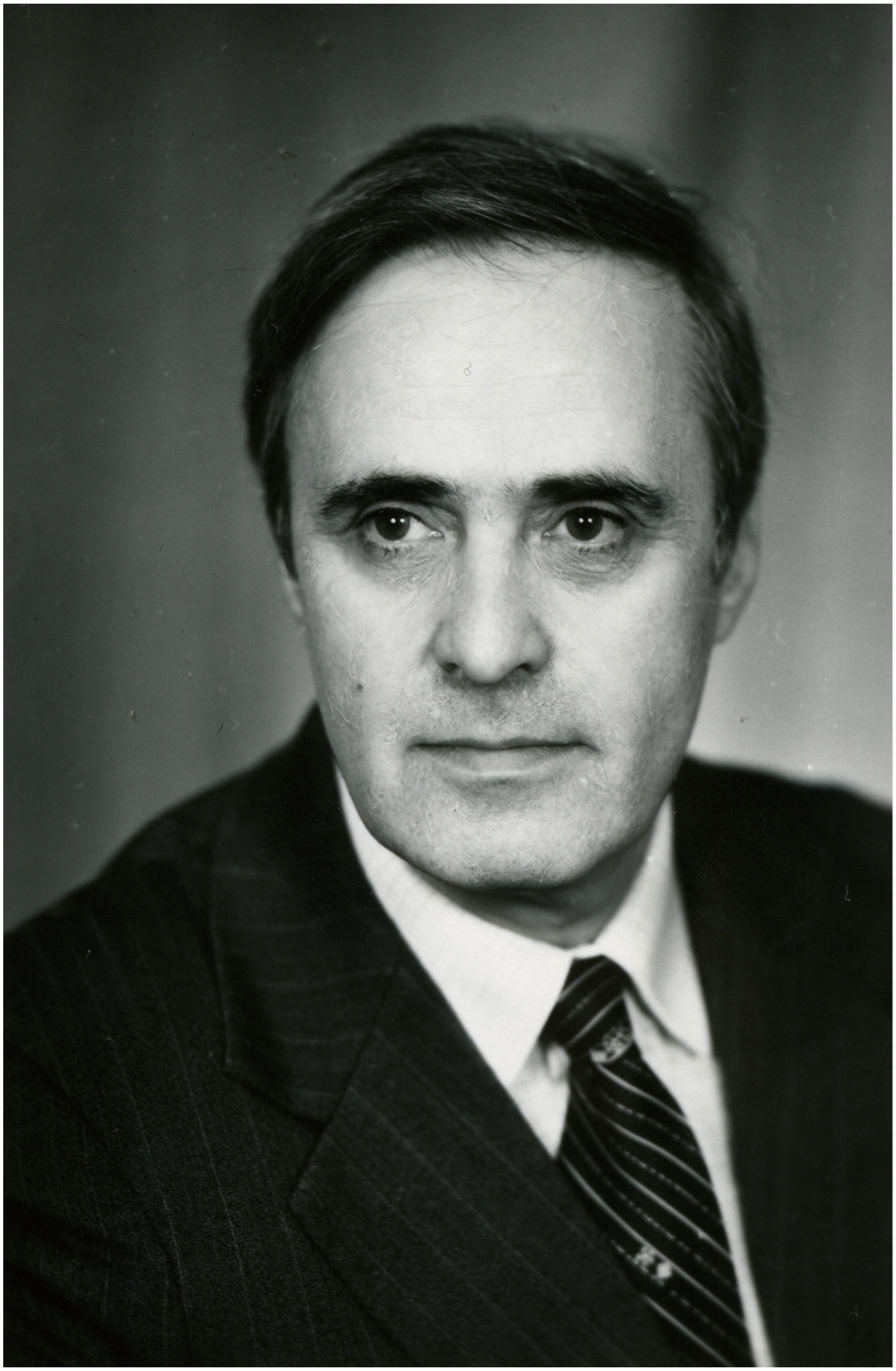

M. Sh. Birman (17 January 1928 – 2 July 2009)

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and others.

Scientific program

MONDAY 3 August:

9:40: OPENING

10:00–11:00: Michael Solomyak *On my joint work with M. Sh. Birman in 1965–1970*

COFFEE BREAK

11:30–12:30: Vladimir Buslaev *New approach to the quantum three-body scattering problem. I. One-dimensional particles*

LUNCH

14:30–15:30: Dmitri Chelkak *Weyl-Titchmarsh functions of vector-valued Sturm-Liouville operators on the unit interval*

15:40–16:40: Galina Perelman *Two soliton collision for nonlinear Schrodinger equations in dimension 1*

COFFEE BREAK

17:00–18:00: Alexander Its *On the Riemann-Hilbert approach in the theory of Toeplitz and Hankel determinants*

TUESDAY 4 August:

10:00–11:00: Dmitri Yafaev *Spectral properties of the scattering matrix*

COFFEE BREAK

11:30–12:30: Alexander Kiselev *Solutions of Surface Quasi-geostrophic equation*

LUNCH

14:30–15:30: Frederic Klopp *Resonances for large ergodic systems*

15:40–16:40: Sergei Naboko *Sharp decay estimate for the generalized eigenvectors asymptotics for unbounded Hermitean Jacobi Matrices*

COFFEE BREAK

17:00–18:00: Vladimir Peller *Hölder–Zygmund operator functions*

WEDNESDAY 5 August:

10:00–11:00: Leonid Friedlander *Heat trace asymptotics in polyhedra*

COFFEE BREAK

11:30–12:30: Georgi Raikov *Low Energy Asymptotics of the SSF for Pauli Operators with Non-Constant Magnetic Fields*

LUNCH

YOUNG SCIENTISTS SESSION:

14:30–14:50: Ilya Kachkovskiy *Absolute continuity of the spectrum of the Schrödinger operator in a layer and in a smooth multidimensional cylinder*

14:55–15:15: Sergei Matveenko *The uniqueness theorem for vector-valued Sturm-Liouville operators*

15:20–15:40: Fedor Sandomirskiy *Monodromization and the Maryland equation*

COFFEE BREAK

16:10–16:30: Sergei Simonov *Weyl-Titchmarsh type formula for discrete Schrödinger operator with Wigner-von Neumann potential*

16:35–16:55: Elizaveta Vasilevskaya *Homogenization with corrector of a periodic parabolic Cauchy problem*

17:00–17:20: Nikolai Veniaminov *Homogenization of High Order Periodic Differential Operators*

THURSDAY 6 August:

10:00–11:00: Grigori Rozenblum *Finite rank Toeplitz operators in Bergman spaces and some applications*

COFFEE BREAK

11:30–12:30: Yuri Safarov *On the relation between an operator and its self-commutator*

LUNCH

14:30–15:30: Lev Kapitanski *The Pontrjagin-Hopf invariants for Sobolev maps*

15:40–16:40: Alexander Sobolev *Szegö limit theorem for operators with discontinuous symbols: Widom's hypothesis*

CONFERENCE DINNER

FRIDAY 7 August:

10:00–11:00: Vladimir Sloushch *Double-sided estimates for the trace of the difference of two semigroups.*

COFFEE BREAK

11:30–12:30: Michael Solomyak *Counting bound states for Schrödinger operators on the lattice*

LUNCH

14:30–15:30: Tatyana Suslina *Homogenization of nonstationary periodic equations*

15:40–16:40: Vladimir Sukhanov *Inverse and Direct scattering on the half line*

COFFEE BREAK

17:00–18:00: Dmitri Yafaev *Exponential decay of eigenfunctions of first order systems*

Abstracts

New approach to the quantum three-body scattering problem. I. One-dimensional particles.

Vladimir Buslaev
St.Petersburg State University

The talk is based on a joint work with S.B. Levin.

We present a new approach to the quantum three-body scattering problem. The main idea is to propose a priory explicit formulas for the asymptotic behavior of the eigenfunctions of the continuous spectrum (of scattered plane waves type) describing them up to the simple diverging waves with a smooth amplitude. If we are able to find such asymptotic behavior even heuristically, we obtain a way for regular numerical computations of the eigenfunctions, and also a method to construct an appropriate integral equation of the same nature as the Lippmann-Schwinger equation for the scattering of the plane wave by a quickly decreasing potentials. The equation can be used to justify the asymptotic behavior rigorously.

For one-dimensional particles with quickly decreasing at infinity pair potentials we can use for the description of the mentioned asymptotic behavior the analogy between the stated problem and the classical problem of the diffraction of the plane waves by the set of semi-transparent infinite screens. This analogy was already used in [1,2]. In case of long range potentials we are able to treat the diffraction problem analogously with the replacement of the classical plane waves by plane waves that are appropriately deformed by the long range tails of the Coulomb potentials. It is important to mention that the diffraction itself and the corresponding scattering problems cannot be completely reduced to the scattering of the plane waves by the screens: we have to add to these processes some genuine diffraction components that have more complicated analytical structure but still explicit description. This more complicated structure is also dictated by the analogy with the classical diffraction theory.

The formulas we are going to present have been already used for the numerical coputations and turned out quite effective.

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The characterization of spectral data for the vector-valued Sturm-Liouville problem

D.Chelkak
St.Petersburg State University

We consider the vector-valued Sturm-Liouville operator $Ly = -y'' + Vy$ on $[0,1]$ with Dirichlet boundary conditions, where $V(x)$ is a *self-adjoint* $N \times N$ matrix-valued potential. We suppose that the mean value $V^0 = \int_0^1 V(x)dx$ of the potential is fixed (the unitary transform leads to the diagonal V^0) and all eigenvalues of V^0 are simple (the "nondegenerate" case). The spectral data consist of

(i) *eigenvalues* λ_m and their multiplicities $k_m : 1 \le k_m \le N$;

and residues $-B_m$ of the (matrix-valued) Weyl-Titchmarsh function. Each B_m is a nonnegative self-adjoint $N \times N$ matrix of rank k_m and we treat it as $P_m g_m^{-1} P_m$, where

(ii) P_m is an *orthogonal projector* in $\mathbb{C}^N$, $\mathrm{rank}(P_m) = k_m$;

(iii) g_m is a *positive quadratic form* ("normalizing matrix") defined in $\mathrm{Ran}(P_m)$.

It is well-known that the Weyl-Titchmarsh function defines the potential uniquely. We give the *complete characterization* (in other words, *necessary and sufficient conditions*) of spectral data that correspond to the square summable potentials with given V^0. Note that in "nondegenerate" case all sufficiently large eigenvalues are simple (and corresponding g_m are positive "normalizing constants"). Then, if $k_m = 1$ for all $m > m_0$ and $k_1 + .. + k_{m_0} = Nn_0$, we define the double-indexing (n, j), $n > n_0$, $j = 1, .., N$, instead of the simple-indexing $m > m_0$ by $m - m_0 = N(n - n_0 - 1) + j$.

Theorem 1 (Chelkak-Korotyaev, 2008). $(\lambda_m, P_m, g_m)_{m=1}^{\infty}$ *correspond to some potential* $V = V^* \in L^2_{N\times N}([0,1])$*:* $V^0 = \mathrm{diag}(v_1^0, .., v_N^0)$, $v_1^0 < .. < v_N^0$ *iff*
(a) the spectrum is asymptotically simple, i.e. $k_m = 1$ *for all* $m > m_0$*;*
(b) for each $j = 1, .., N$ *the following "asymptotics in* ℓ^2*-sense" are fulfilled:*

$$\left\{\lambda_{n,j} - \pi^2 n^2 - v_j^0\right\}_{n=n_0+1}^{\infty} \in \ell^2; \qquad \left\{\pi n \cdot (2\pi^2 n^2 g_{n,j} - 1)\right\}_{n=n_0+1}^{\infty} \in \ell^2;$$

$$\{\|P_{n,j} - P_j^0\|\}_{n=n_0+1}^{\infty} \in \ell^2 \qquad \textit{and} \qquad \{\pi n \cdot \|\textstyle\sum_{j=1}^N P_{n,j} - I_N\|\}_{n=n_0+1}^{\infty} \in \ell^2,$$

where P_j^0 *are the standard coordinate projectors and* I_N *is the identity* $N \times N$ *matrix;*
(c) the following "unique interpolation property" holds true for $(\lambda_m, P_m)_{m=1}^{\infty}$*:*

if $P_m \xi(\lambda_m) = 0$ *for all* $m \ge 1$ *and some entire vector-valued function* $\xi : \mathbb{C} \to \mathbb{C}^N$ *such that* $\xi(\lambda) = o(|\lambda|^{-\frac{1}{2}} e^{|\mathrm{Im}\sqrt{\lambda}|})$ *as* $|\lambda| \to \infty$*, then* $\xi \equiv 0$*.*

Remark. (i) Asymptotics of $\lambda_{n,j}$ and $g_{n,j}$ are the same as in the scalar case and their leading terms are Fourier coefficients of diagonal entries of $V(x)$. Similarly, the leading terms in the asymptotics of individual projectors $P_{n,j}$ and their sums $\sum_{j=1}^N P_{n,j}$ are given by the Fourier coefficients of nondiagonal entries of $V(x)$.

(ii) This work is a part of the *joint with E.Korotyaev project* devoted to the spectral theory of 1D Schrödinger-type operators with matrix-valued potentials. The author was supported by the Foundation of the President of the Russian Federation (grants no. MK-4306.2008.1 and NSh-2409.2008.1).

Heat Trace Asymptotics in Polyhedra

Leonid Friedlander
University of Arizona

It is well known that

$$h(t) = \mathrm{tr} e^{-t\Delta} \sim \sum_{j=0}^{\infty} c_j t^{(-n+j)/2},\ t \to \infty \tag{1}$$

where Δ is the Laplacian on a compact Rimannian manifold M, with or without boundary. The boundary is assumed to be smooth, and the above asymptotics holds for a big class of boundary conditions. We will be dealing with the Dirichlet boundary condition. All coefficients c_j are locally computable quantities: they are sums of integrals over M of polynomials of the components of the curvature tensor and integrals over the boundary of M of polynomials of components of the second fundamental form. The situation is more complicated when the boundary is not smooth. In the case when M is an n-dimensional polyhedron, the asymptotic expansion (1) holds, but computing coefficients turns out to be not that easy. A naive idea of approximating a polyhedron by smooth domains and passing to the limit does not work. The main issue is computing the contribution of a vertex. For polygons, the problem was solved by B.V. Fedosov in the early sixties: the contribution of a vertex equals $(\pi^2 - \theta^2)/(24\pi\theta)$ where θ is the corresponding interior angle. In the case $n > 2$, the answer was not known. It is not clear whether the contribution of a vertex can be explicitly written down as a simple function of different angles attached to that vertex.

Let P be a polyhedron in $\mathbf{R}^n$, let A be a vertex of P, and let C_A be the cone in $\mathbf{R}^n$ that has A as its vertex and such that $C_A \cap U = P \cap U$ for a neighborhood U of the point A that is small enough. Let ω_A be the intersection of C_A with the sphere of radius 1 centered at A. By $\theta(t)$ we denote the heat trace in ω_A, and let

$$p(t) = \exp\left\{-\frac{(n-1)(n-3)}{4}\right\}\theta(t).$$

Consider the space of Brownian paths $b(t)$, $0 \le t \le 2$ conditioned on $b(0) = b(2) = 0$, and let $\mu_{2,0}$ be the Wiener measure on this space. For a path $b(t)$, we define a function

$$\xi[r; b] = \frac{1}{2}\int_0^2 \frac{dt}{(r + b(t))^2} dt.$$

It equals $+\infty$ when $r \le m(b) = -\min\{b(t)\}$; then it is strictly monotone, and it decreases from $+\infty$ to 0. The inverse function $r[\xi; b]$ is defined for all $\xi > 0$, and it decreases from ∞ to $m(b)$. Let

$$r(\xi) = \int r[\xi; b] d\mu_{0,2}(b).$$

We show that, up to an explicitly computable expression (and we compute that expression,) the contribution of the vertex A to the heat trace expansion (1) equals to the free term in the expansion of

$$\int_\tau^\infty r(\xi) p'(\xi) d\xi$$

as $\tau \to 0$. Notice that $r(\xi)$ is a universal function. In this way, we reduce the problem of computing the contribution of a vertex to the heat trace for an $(n-1)$-dimensional domain.

On the Riemann-Hilbert approach in the theory of Toeplitz and Hankel determinants

Alexander Its
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Let $\phi(z)$ be a function defined on the unit circle,

$$C = \{z : |z| = 1\}.$$

The *Toeplitz determinant,* $D_n^T[\phi]$, is defined as

$$D_n^T[\phi] := \det T_n[\phi],$$

where

$$T_n[\phi] := \{\phi_{j-k}\}, \quad k = 0, ..., n-1,$$

and

$$\phi_k = \int_C \phi(z) z^{-k-1} \frac{dz}{2\pi i}.$$

Similarly, given a function $\phi(z)$ defined on the real line $\mathbf{R}$ the *Hankel determinant*, $D_n^H[\phi]$, is defined as

$$D_n^H[\phi] := \det H_n[\phi],$$

where

$$H_n[\phi] := \{\phi_{j+k}\}, \quad k = 0, ..., n-1,$$

and

$$\phi_k = \int_{-\infty}^{\infty} z^k \phi(z) dz.$$

The principal analytic question is evaluation of the large n asymptotics of D_n^T and D_n^H.

Starting with Onsager's celebrated solution of the two-dimensional Ising model in the 1940's, Toeplitz and Hankel determinants play an increasingly central role in modern mathematical physics. Simultaneously, the theory of Toeplitz and Hankel determinants is a very beautiful area of analysis representing an unusual combinations of profound general operator concepts with the highly nontrivial concrete formulae. The area has been thriving since the classical works of Szegö, Fisher and Hartwig and Widom, and it very much continious to do so.

In the 90s, it has been realized [3,1,2] that the theory of Toeplitz and Hankel determinants can be also embedded in the Riemann-Hilbert formalism of integrable systems . The new Riemann-Hilbert techniques have gone far beyond the classical Wiener-Hopf schemes, and they have led to the solutions of several important long-standing asymptotic problems of the theory. We shall review some of the most recent results which includes the proof of the Basor-Tracy conjecture concerning the asymptotics of Toeplitz determinants with the most general Fisher-Hartwig type symbols, the Fisher-Hartwig type asymptotics for Hankel determinants and for Toeplitz + Hankel determinants, and the asymptotics of the determinants of Toeplitz matrices with the matrix-valued algebraic symbols. The Riemann-Hilbert approach will be outlined as well.

The presentation is based on the author's joint works with P. Deift, V. Korepin, I. Krasovsky, F. Mezzadri, and M. Mo.

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The Pontrjagin-Hopf invariants for Sobolev maps

Lev Kapitanski
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I have been interested in the Faddeev S^2-nonlinear sigma-model, [3], for quite some time. Thinking about it has led me to the problem of homotopy classification of maps from a 3-dimensional manifold, M, into the sphere S^2. In fact, the classification was obtained in the 1930s starting with the work of H. Hopf [4] and ending with deep results of L. Pontrjagin [5]. In the case of a simply connected 3-manifold M, the homotopy classes of the maps $M \to S^2$ are distinguished by an integer, the Hopf invariant, and there is a neat analytical formula (due to J.H.C. Whithead, 1947) that allows one to in principle compute this integer for any sufficiently smooth map $\varphi : M \to S^2$. If M is not simply connected (take, e.g., T^3, the 3-torus), no analytical method has been known to tell whether two smooth maps belong to the same homotopy class or not (and I have discussed this with a number of world renowned geometers). Recently, Dave Auckly and I came up with a solution, [1,2]. To describe it, it is convenient to use quaternions, $q = q^0 + q^1\,\mathbf{i} + q^2\,\mathbf{j} + q^3\,\mathbf{k}$, with the usual properties. We make the following identifications. The sphere S^3 is identified with the unit (norm 1) quaternions. S^3 is the Lie group Sp(1). Its Lie algebra, $\mathfrak{sp}(1)$, is identified with the space of purely imaginary quaternions, $\mathbb{R}^3$, with the Lie bracket $[p, q] = pq - qp$. We identify the usual sphere S^2 with the unit sphere in the space of purely imaginary quaternions. Finally, we identify S^1 with the unit quaternions of the form $q^0 + q^1\,\mathbf{i}$. Thus $S^2 \subset S^3$, $S^1 \subset S^3$, and $S^2 \cap S^1 = \mathbf{i} \cup -\mathbf{i}$.

Our description of the Pontrjagin-Hopf invariants. Let M be a closed, connected, oriented 3-manifold. To any continuous map φ from M to S^2 one associates the pull-back $\varphi^*\mu_{S^2} \in H^2(M;\mathbb{Z})$ of the orientation class $\mu_{S^2} \in H^2(S^2;\mathbb{Z})$. The class $\varphi^*\mu_{S^2}$ is the *primary* invariant. For two maps, φ and ψ, to be in the same homotopy class, it is necessary that $\psi^*\mu_{S^2} = \varphi^*\mu_{S^2}$. In [2] we use the Čech picture to define the pull-backs for maps with finite Faddeev energy (such maps may be discontinuous) and prove the following result (under appropriate regularity assumptions).

Theorem 1 *$\psi^*\mu_{S^2} = \varphi^*\mu_{S^2}$ iff there exists a map $u : M \to S^3$ such that*

$$\psi(x) = u(x) \cdot \varphi(x) \cdot u(x)^{-1} \tag{1}$$

The intertwining map u is not unique. If $\tilde{u}$ is another such map, then $\tilde{u}(x) = u(x)\,\mathfrak{q}(\varphi(x), \lambda(x))$, where λ is a map $M \to S^1$ and $\mathfrak{q} : S^2 \times S^1 \to S^3$ is defined via $\mathfrak{q}(z, \lambda) = q\lambda q^{-1}$, where $z = q\,\mathbf{i}\,q^{-1}$.

Fix $\varphi : M \to S^2$. The pull-back $\varphi^*\mu_{S^2}$ is the primary invariant of the homotopy class of φ. The map $\eta \mapsto (\varphi^*\mu_{S^2} \cup \eta)[M]$ from $H^1(M;\mathbb{Z})$ to $\mathbb{Z}$ is a group homomorphism, hence has image $m\,\mathbb{Z}$ for some integer $m = m_\varphi$ depending on the class $\varphi^*\mu_{S^2} \in H^2(M,\mathbb{Z})$.

Theorem 2 *The map $\psi : M \to S^2$ with the same primary invariant as φ is homotopic to φ iff the Brouwer degree of the intertwining map u in (1), is an integer multiple of $2\,m_\varphi$, i.e., $deg\,(u) = 0\; mod\; 2\,m_\varphi$.*

Theorems 1 and 2 turned out to be new even for smooth maps. The homotopy classification of Sobolev maps between manifolds is an interesting area of current research. Although the notion of homotopy itself requires reconsideration, it is important to make sure that the homotopy invariants make sense for the relevant Sobolev maps. In [2], Dave Auckly and I prove that the homotopy invariants of theorems 1 and 2 are well defined for maps $\varphi : M^3 \to S^2$ with finite Faddeev energy,

$$E(\varphi) = \int_M |d\varphi|^2 + |d\varphi \wedge d\varphi|^2 \,.$$

The argument relies on some subtle analytical considerations.

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Blow up and regularity in some models of fluid mechanics

Alexander Kiselev
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I will talk about recent results on global existence and regularity, blow up, and properties of solutions to certain partial differential equations motivated by fluid mechanics. These are nonlocal nonlinear equations involving fractional dissipation.

I am going to describe a new technique, nonlocal maximum principle, that is particularly useful for proving global existence of smooth solutions in the critical case, when nonlinear and dissipation terms balance. The technique involve conservation of a modulus of continuity of the solution, which carries nonlocal information. It can also be used to obtain some quantitative estimates on the solutions.

I will also discuss constructions and characteristics of blow up, where available.

The results I am going to describe apply to several different models.

1. The simplest model I will mention is the Burgers equation in one dimension,

$$\theta_t = \theta\theta_x - (-\Delta)^\alpha \theta.$$

The properties of this equation are fairly well understood, even though some advances are quite recent. The value $\alpha = 1/2$ is critical. There are global smooth solutions for $\alpha \geq 1/2$, and finite time blow up (shocks) is possible if $\alpha < 1/2$.

2. The Cordoba-Cordoba-Fontelos model. This is next step up in difficulty, still one-dimensional, but the nonlinearity is now nonlocal. Let $H\theta$ denote the Hilbert transform of θ. Then the equation reads

$$\theta_t = H\theta\,\theta_x - (-\Delta)^\alpha \theta.$$

This models more complex and realistic equations of fluid mechanics, such as surface quasi-geostrophic or Euler equation in vorticity form, where the advection velocity in the nonlinearity is also given by a singular integral operator of the advected quantity.

3. Surface quasi-geostrophic (SQG) equation in two dimensions,

$$\theta_t = u \cdot \nabla\theta - \kappa(-\Delta)^\alpha \theta,$$

$u = \nabla^\perp(-\Delta)^{-1/2}\theta$. This equation arises in atmospheric studies. It can be derived, under certain assumptions, from the Boussinesq system (Navier-Stokes equations coupled with advected temperature equation via buoyancy term) describing fluid in a rotating half-space. The SQG equation should be satisfied by temperature on the surface of the half-space. The physically relevant cases are then $\kappa = 0$ (conservative case) and $\alpha = 1/2$ (models Ekman pumping boundary layer effect).

The SQG model appears to be simplest-looking equation of fluid mechanics for which the question of the global existence of smooth solutions (for $\alpha < 1/2$) is still open.

Resonances for "large" ergodic systems

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On $\ell^2(\mathbb{Z}^d)$, consider V a bounded ergodic potential and the operator

$$H = -\Delta + V$$

where $-\Delta$ is the discrete Laplace operator. By ergodic potential, we mainly think of:

- V periodic;
- $V = V_\omega$ random e.g. Anderson model;

By large "ergodic" system, we mean that we consider the ergodic potential only on a large subset, say cube, of the total space. More precisely, let $L \in \mathbb{N}$ be large and set $H_L = -\Delta + V\mathbf{1}_{|x|\le L}$. So we deal with a compact (actually finite rank perturbation) of the Laplace operator. Hence, we know that

- $\sigma_{\text{ess}}(H_L) = \sigma(-\Delta) = [-2d, 2d]$;
- outside $\sigma(-\Delta)$, H_L has only discrete eigenvalues.

The operator valued function $z \in \mathbb{C}^+ \mapsto (z - H_L)^{-1}$ admits a meromorphic continuation (valued in the operators from l^2_{comp} to l^2_{loc}) from $\mathbb{C}$ to $\mathbb{C}$ cut at finitely many half-lines staring at the critical points of the symbol of $-\Delta$. In figure 1, we represented first the spectrum of H_L and second the analytic continuation to the cut lower half-plane. The poles of this analytic

Figure 1: The meromorphic continuation

continuation are the resonances of H_L. They are associated with finite dimensional resonant subspaces. The pole width is the imaginary part of the pole. It is well known that the resonance widths play an important role in the large time behavior of e^{-itH_L}, especially the smallest width that gives the leading order contribution.

Our goal is to describe the resonances and, more specifically, relate them (their distribution, the distribution of their width) to the spectral characteristics of the full space Hamiltonian $H = -\Delta + V$.

We do this for a very simple one-dimensional model on a half-line and essentially study two cases:

- when V is periodic;
- when V is random.

Let us now describe shortly the results we obtain. Fix some energy $E_0 \in (-2, 2)$. In the case V periodic, we prove that the resonances near E_0 stay roughly at a distance of order $1/L$ of the real axis; we find a quantization condition that enables us to describe the resonances precisely.

In particular, if we rescale the imaginary parts of the resonances (locally near E_0) by multiplying it by L, we prove that they accumulate near an analytic curve the equation of which we compute (see figure 2).

The local linear density of resonances is given by the density of states of the full Hamiltonian $-\Delta + V$.

In the case when $V = V_\omega$ is random, the picture is quite different. Fix some energy $E_0 \in (-2, 2)$. Let $\rho(E_0)$ denote the Lyapunov exponent of $-\Delta + V_\omega$ at energy E_0 and $n(E_0$ be the density of states of $-\Delta+V_\omega$ at energy E_0. It is known that $\rho(E_0) > 0$. Assume that $n(E_0) > 0$. In the case V periodic, we prove that the resonances near E_0 stay roughly at a distance roughly $e^{n(E_0)L(1+o(1))}$ of the real axis.

Moreover, if one rescales the resonances such that their real parts have roughly spacing one and their imaginary parts are of order one i.e. one scales the real parts by the factor L and the imaginary parts by the factor $e^{-n(E_0)L}$, then the thus obtained point process converges weakly to a Poisson process in $\mathbb{R} \times [0, 1]$ of intensity the measure $n(E_0)\rho(E_0)dxdy$. So the picture of the rescaled resonances is roughly that shown in figure 3.

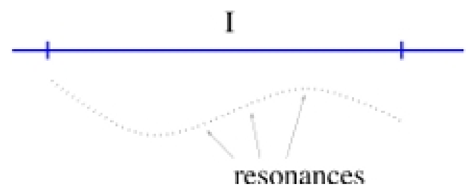

Figure 2: Rescaled resonances

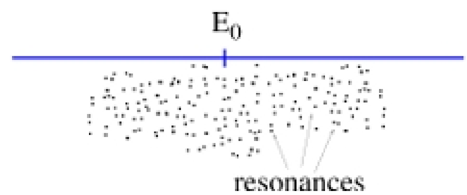

Figure 3: Rescaled resonances

Sharp decay estimate for the generalized eigenvectors asymptotics for unbounded Hermitean Jacobi Matrices

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The presentation is based on the joint work with J. Janas and G. Stolz.

Bounds on the exponential decay of generalized eigenfunctions of bounded and unbounded selfadjoint Jacobi matrices in $l^2(\mathbb{N})$ are established. Two cases are considered separately and lead to different results: (i) the case in which the spectral parameter lies in a general gap of the spectrum of the Jacobi matrix and (ii) the case of a lower semibounded Jacobi matrix with values of the spectral parameter below the spectrum. It is demonstrated by examples that both results are sharp.

We apply these results to obtain a "many barriers-type" criterion for the existence of square-summable generalized eigenfunctions of an unbounded Jacobi matrix at almost every value of the spectral parameter in suitable open sets. In particular, this leads to examples of unbounded Jacobi matrices with a spectral mobility edge, i.e. a transition from purely absolutely continuous spectrum to dense pure point spectrum.

The main results are the following theorems.

Theorem. *Let $\mathcal{J}$ be a selfadjoint Jacobi matrix (in limit point case) with off-diagonal entries $a_n \to +\infty$ as $n \to \infty$. Assume that (r, s) is a gap in the spectrum of $\mathcal{J}$. Then for arbitrary $\varepsilon \in (0, \frac{1}{2})$ there exists $N = N(\varepsilon)$ such that*

$$|(\mathcal{J}-\lambda)^{-1}e_1, e_n)| \le \frac{s-r}{\varepsilon(\lambda-r)(s-\lambda)} \exp\left\{-\left(\frac{1}{2}-\varepsilon\right)\sqrt{(\lambda-r)(s-\lambda)}\sum_{k=N}^{n-1}\frac{1}{a_k}\right\}$$

for all $\lambda \in (r, s)$ and for all $n > N$. Here e_n stands for the canonical basis in $l_2(\mathbb{N})$.

Theorem. *Let $\mathcal{J} = \mathcal{J}^*$ and assume that $\lim\limits_{n\to\infty} a_n = +\infty$. Suppose that $\mathcal{J}$ is bounded from below and denote $d := \inf \sigma(\mathcal{J})$. Fix $\varepsilon \in (0; 1)$ and complex $\lambda, Re\lambda < d$. Then there exists $N = N(\varepsilon, \lambda)$ such that*

$$|((\mathcal{J}-\lambda)^{-1}e_1, e_n)| \le [(d - Re\lambda)\varepsilon]^{-1} \exp\left\{-(1-\varepsilon)\sqrt{d-Re\lambda}\sum_{k=N}^{n-1}\frac{1}{\sqrt{a_k}}\right\},$$

for $n > N$.

Hölder–Zygmund operator functions

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This is a joint work with A. B. Aleksandrov.

It is well known that a Lipschitz function is not necessarily operator Lipschitz which means that for a Lipschitz function f on the real line it is not necessarily true that

$$\|f(A) - f(B)\| \le \text{const}\|A - B\|$$

for self-adjoint operators A and B. It is also well known that a continuously differentiable function is not necessarily operator differentiable. However, we have proved that if f is a function in the Hölder class Λ_α with $0 < \alpha < 1$, then it is operator Hölder, i.e.,

$$\|f(A) - f(B)\| \le \text{const}\|A - B\|^\alpha$$

for self-adjoint operators A and B. The same is true for functions in the Zygmund class Λ_1 . They must be operator Zygmund, i.e.,

$$\|f(A + K) - 2f(A) + f(A - K)\| \le \text{const}\|K\|$$

for self-adjoint operators A and K. The same is true for all spaces Λ_α , $0 < \alpha < \infty$ of the Hölder–Zygmund scale. Similar results also hold for functions of unitary operators and for functions of contractions.

We also obtain similar estimates in the case of Schatten–von Neumann norms.

Two soliton collision for nonlinear Schrodinger equations in dimension 1

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We consider the nonlinear Schrödinger equation

$$i\psi_t = -\psi_{xx} + F(|\psi|^2)\psi, \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{1}$$

where F is a smooth function that satisfies $F(\xi) = -2\xi + O(\xi^2)$, as $\xi \to 0$.

This equation possesses solutions of special form - solitary waves (or, shortly, solitons):

$$e^{i\Phi(x,t)}\varphi(x - b(t), E),$$

$$\Phi(x,t) = \omega t + \gamma + \frac{1}{2}vx, \; b(t) = vt + c, \; E = \omega + \frac{v^2}{4} > 0,$$

where ω, γ, c, $v \in \mathbb{R}$ are constants and φ is the ground state that is a smooth positive even exponentially decreasing solution of the equation

$$-\varphi_{xx} + E\varphi + F(\varphi^2)\varphi = 0, \quad \varphi \in H^1(\mathbb{R}).$$

We are interested in the solutions of (1) that behave as $t \to -\infty$ like a sum of two nonlinearly stable solitons

$$e^{i\Phi_0}\varphi(x - b_0(t), E_0) + e^{i\Phi_1}\varphi(x - b_1(t), E_1),$$

$\Phi_j = \omega_j t + \gamma_j + \frac{1}{2}v_j x, \quad b_j(t) = v_j t, \; v_1 - v_0 \neq 0$, our goal being to understand the collision between the solitons and to determine what happens after. We show that in the case where $E_1 \equiv \varepsilon^2 \ll 1$ (depending on $v_1 - v_0$ and E_0) the collision leads to the splitting of the small soliton into two outgoing parts, that at least up to the times $t \sim \varepsilon^{-2}|\ln \varepsilon|$ propagate independently according to the cubic NLS:

$$i\psi_t = -\psi_{xx} - 2|\psi|^2\psi. \tag{2}$$

The splitting of the small soliton is essentially controlled by the flow linearized around the "large" one: in the interaction region a small amplitude soliton behaves as a slowly modulated plane wave $\varepsilon e^{-iv_1^2 t/4 + iv_1 x/2}$ and is splitted by the large soliton into a reflected and a transmitted parts accordingly to the linear scattering theory. For the first time this phenomenon was observed by J.Holmer, J.Marzuola, M.Zworski [1], [2] in the context of soliton-potential interaction for the cubic NLS with an external delta potential:

$$i\psi_t = -\psi_{xx} + \delta(x)\psi - 2|\psi|^2\psi.$$

To control the solution in the post interaction region $\varepsilon^{-1-\delta} \leq t \leq \delta\varepsilon^{-2}|\ln \varepsilon|$ one invokes the orbital stability argument combined with the integrability of (2), again in the spirit of [1], [2].

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Low Energy Asymptotics of the SSF for Pauli Operators with Nonconstant Magnetic Fields

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Suppose that the magnetic field $\mathbf{B} : \mathbb{R}^3 \to \mathbb{R}^3$ has a constant direction, say,

$$\mathbf{B} = (0, 0, b).$$

By the Maxwell equation, $\operatorname{div} \mathbf{B} = 0$, we should then have $\frac{\partial b}{\partial x_3} = 0$. In what follows we assume that $b = b_0 + \tilde{b}$ where $b_0 > 0$ is a constant, while the function $\tilde{b} : \mathbb{R}^2 \to \mathbb{R}$ is such that the Poisson equation

$$\Delta \tilde{\varphi} = \tilde{b}$$

admits a solution $\tilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}$, continuous and bounded together with its derivatives of order up to two. For $x \in \mathbb{R}^2$ set $\varphi_0(x) := b_0|x|^2/4$, and $\varphi := \varphi_0 + \tilde{\varphi}$. Then $\Delta\varphi_0 = b_0$ and $\Delta\varphi = b$. Put $\mathbf{A} := (A_1, A_2, A_3)$ with

$$A_1 := -\frac{\partial \varphi}{\partial x_2}, \quad A_2 := \frac{\partial \varphi}{\partial x_1}, \quad A_3 = 0.$$

Then $\operatorname{curl} \mathbf{A} = \mathbf{B} = (0, 0, b)$. Let

$$H_0 := \begin{pmatrix} (-i\nabla - \mathbf{A})^2 - b & 0 \\ 0 & (-i\nabla - \mathbf{A})^2 + b \end{pmatrix} := \begin{pmatrix} H_0^- & 0 \\ 0 & H_0^+ \end{pmatrix} = H_0^- \oplus H_0^+$$

be the unperturbed Pauli operator, self-adjoint in $L^2(\mathbb{R}^3; \mathbb{C}^2)$. Note that we have

$$H_0^{\pm} = H_{\perp}^{\pm} \otimes I_{\parallel} + I_{\perp} \otimes H_{\parallel} \tag{1}$$

where $I_{\parallel}$ and $I_{\perp}$ are the identity operators in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$ respectively,

$$H_{\parallel} := -\frac{d^2}{dx_3^2},$$

$$H_{\perp}^- = H_{\perp}^-(b) := a^* a, \quad H_{\perp}^+ = H_{\perp}^+(b) := a a^*,$$

and

$$a = a(b) := -2ie^{-\varphi}\frac{\partial}{\partial \overline{z}}\, e^{\varphi}, \; a^* = a(b)^* := -2ie^{\varphi}\frac{\partial}{\partial z}\, e^{-\varphi}, \quad z := x_1 + ix_2, \quad \overline{z} := x_1 - ix_2.$$

Let $p = p(b)$ be the orthogonal projection onto

$$\operatorname{Ker} H_{\perp}^- = \operatorname{Ker} a = \left\{ u \in L^2(\mathbb{R}^2) | u = ge^{-\varphi}, \; \frac{\partial g}{\partial \overline{z}} = 0 \right\}.$$

Obviously, $\operatorname{rank} p = \infty$. Since

- $\sigma(H_\|) = [0, \infty)$, and $\sigma(H_\|)$ is purely absolutely continuous,
- $\inf \sigma(H_\perp^-) = 0$, and $H_\perp^+ \geq 0$,

we easily find that by (1) we have $\sigma(H_0) = [0, \infty)$, and $\sigma(H_0)$ is purely absolutely continuous. Further, let $V := \{v_{jk}\}_{j,k=1,2}$ be a bounded Lebesgue-measurable Hermitian matrix-valued function. On the domain of H_0 define the operator $H := H_0 + V$. Assume

$$v_{jk} \in C(\mathbb{R}^3), \quad |v_{jk}(\mathbf{x})| \leq C_0 \langle \mathbf{x} \rangle^{-m}, \quad \mathbf{x} \in \mathbb{R}^3, \quad j, k = 1, 2, \tag{2}$$

with $m > 3$. Then we have

$$(H - i)^{-1} - (H_0 - i)^{-1} \in S_1(L^2(\mathbb{R}^3; \mathbb{C}^2))$$

where S_1 denotes the trace class. Our results concern the asymptotic behavior as $E \to 0$ of *the spectral shift function* $\xi(E; H, H_0)$ for the operator pair (H, H_0).
In what follows we assume that V satisfies (2) with $m > 3$. Moreover, in the sequel we will suppose that the perturbation of the operator H_0 is of definite sign. More precisely, we will assume

$$V(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \mathbb{R}^3, \tag{3}$$

and will consider the operators $H_0 + V$ or $H_0 - V$. Set

$$W(x) := \int_{\mathbb{R}} v_{11}(x, x_3) dx_3, \quad x \in \mathbb{R}^2,$$

$$\omega(E) := \frac{1}{2\sqrt{E}} p(b) W p(b), \quad E > 0.$$

Let $T = T^*$ be a compact operator. For $s > 0$ set $n_+(s; T) := \mathrm{Tr}\, \mathbf{1}_{(s,\infty)}(T)$.

Theorem 1 *Let V satisfy (2) with $m > 3$, and (3). Then for each $\varepsilon \in (0, 1)$ we have*

$$-n_+((1-\varepsilon); \omega(E)) + O(1) \leq \xi(-E; H_0 - V, H_0) \leq -n_+((1+\varepsilon); \omega(E)) + O(1), \quad E \downarrow 0.$$

For $E > 0$ define the matrix-valued function

$$\mathcal{W}_E = \mathcal{W}_E(x) := \begin{pmatrix} w_{11}(x) & w_{12}(x) \\ w_{21}(x) & w_{22}(x) \end{pmatrix}, \quad x \in \mathbb{R}^2,$$

where

$$w_{11}(x) := \int_{\mathbb{R}} v_{11}(x, x_3) \cos^2(\sqrt{E} x_3) dx_3, \quad w_{22}(x) := \int_{\mathbb{R}} v_{11}(x, x_3) \sin^2(\sqrt{E} x_3) dx_3,$$

$$w_{12}(x) = w_{21}(x) := \int_{\mathbb{R}} v_{11}(x, x_3) \cos(\sqrt{E} x_3) \sin(\sqrt{E} x_3) dx_3.$$

Set

$$\Omega(E) := \frac{1}{2\sqrt{E}} p(b) \mathcal{W}_E p(b), \quad E > 0.$$

Evidently, $\Omega(E) = \Omega(E)^* \geq 0$ in $L^2(\mathbb{R}^2; \mathbb{C}^2)$. Moreover, $\Omega(E) \in S_1$.

Theorem 2 *Let V satisfy (2) with $m > 3$, and (3). Then for each $\varepsilon \in (0, 1)$ we have*

$$\pm \frac{1}{\pi} \mathrm{Tr}\, \arctan\left((1 \pm \varepsilon)^{-1} \Omega(E)\right) + O(1) \leq \xi(E; H_0 \pm V, H_0) \leq \pm \frac{1}{\pi} \mathrm{Tr}\, \arctan\left((1 \mp \varepsilon)^{-1} \Omega(E)\right) + O(1), \quad E \downarrow 0.$$

Using known results on the spectral asymptotics for compact Berezin-Toeplitz operators $p(b)Up(b)$ with $U \in L^\infty(\mathbb{R}^2;\mathbb{R})$, $\lim_{|x|\to\infty} U(x) = 0$, we can describe explicitly the main asymptotic term of $\xi(E; H_0 - V, H_0)$ as $E \uparrow 0$, of $\xi(E; H_0 \pm V, H_0)$ as $E \downarrow 0$ under appropriate assumptions about the decay of V at infinity.

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Theorems on finite rank Bergman-Toeplitz operators and applications

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Toeplitz operators arise in many fields of Analysis and have been an object of active study for many years. Quite a lot of questions can be asked about these operators, and these questions depend on the field where Toeplitz operators are applied.

For a Hilbert space H of functions, a bounded function f (a symbol) and a closed subspace $L \subset H$, the Toeplitz operator T_f in L acting as

$$T_f u = P f u,$$

where P is the projection $P : H \to L$. In particular, in the case when H is the space $L_2(\Omega, \rho)$ for some domain $\Omega \subset \mathbb{C}^d$ and some measure ρ and L is the Bergman space $B^2 = B^2(\Omega, \rho)$ of analytical functions in H, such operator is called Bergman-Toeplitz.

More generally, the operator T_f can be defined for more general symbols, i.e., for f being a complex regular Borel measure in Ω or even a distribution in $\mathcal{E}'(\Omega)$.

The initial question consists in the following. Suppose that the Toeplitz operator T_f has finite rank. What can be said about the distribution f? It is natural to expect that f should be degenerate in a certain sense; especially, if f is a function, it must be zero. The latter hypothesis was formulated more than 20 years ago and turned out to be important in many questions of analysis. It was only in 2007 that the first general result in this direction was established, see [1]. For the domain Ω in $\mathbb{C}^1$ and a measure f it was proved that finite rank of T_f implies that f is a finite combination of point masses. After this, a number of generalizations of this result were established and applications to different fields of analysis were found. The talk contains a description of a part of the results in this direction, especially, obtain with a participation of the author, more details can be found in [2].

First of all, the above finite rank theorem is extended to the case of Bergman spaces in $\Omega \subset \mathbb{C}^d$ for any d and for f being a distribution in $\mathcal{E}'(\Omega)$. Here, the finiteness of the rank of T_f implies that f must be a finite combination of δ–distributions and and their derivatives. Moreover, this result holds true if one considers Toeplitz operators in a proper subspace in the Bergman space, actually, the closed linear span of a sufficiently rich set of analytic monomials. Further on, the finite rank theorem (for measures) was carried over to the Bergman space of harmonic function and also to the Bergman space of solutions of the Helmholtz equation.

As applications, we mention here the results on the structure of ideals of finite codimension in the algebras of analytical functions, on the approximation of smooth functions by products of analytical and antianalytical polynomials with restrictions on entries, on the classical question about which Toeplitz operators can have zero product, on operator equations for Berezin transform, on the spectral properties of an operator determining the splitting of the spectrum of the Landau Hamiltonian, and on the spectral properties of the scattering matrix.

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On the relation between an operator and its self-commutator

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The talk will discuss the following naive question: if A is a bounded operator in a Hilbert space whose self-commutator $[A^*, A]$ is small in an appropriate sense, is there a normal operator $\tilde{A}$ close to A? There are two known positive results on this problem.

The Brown–Douglas–Fillmore (BDF) theorem: if $[A^*, A]$ is compact and the corresponding to A element of the Calkin algebra has trivial index function then there is a compact operator K such that $A + K$ is normal.

Huaxin Lin's theorem: there exists a nondecreasing function F vanishing at the origin such that the distance from A to the set of normal operators is estimated by $F\big(\|[A^*, A]\|\big)$ for all finite rank operators A.

We consider a general unital C^*-algebra L of real rank zero and denote the sets of normal and self-adjoint elements in L by L_n and L_s respectively. Let B_ε be the ball of radius ε about the origin in L, and let $M_{[A^*,A]}$ be the convex hull of $\bigcup_{U,V} U[A^*, A]V$ where the union is taken over all unitary elements of L. Our main result is the following theorem.

Theorem 1. *There exists a nonincreasing function $h : (0, \infty) \mapsto [0, \infty)$ such that*

$$A \ \in \ B_{\|A\|} \bigcap L_n + h(\varepsilon) M_{[A^*,A]} \bigcap L_s + B_\varepsilon \,, \qquad \forall \varepsilon > 0 \,,$$

for all C^*-algebras L of real rank zero and all $A \in B_1$ satisfying the following condition: for each $\lambda \in \mathbf{C}$ the operator $A - \lambda I$ belongs to the closure of the connected component of unity in the set of invertible elements of L.

Theorem 1 implies both the BDF and Huaxin Lin's theorems and allows us to extend the latter to operators of infinite rank and other unitary invariant norms. We shall outline its proof, present some corollaries and discuss possible applications to Szegö type limit theorems, which describe asymptotic behaviour of the spectra of truncations of (almost) normal operators to finite dimensional subspaces.

The research was supported by the EPSRC grant GR/T25552/01.

Double-sided estimates for the trace of the difference of two semigroups

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This is a joint work with M. Sh. Birman.

Main results. We derive double-sided estimates for the trace of the difference of two semigroups, generated by a pair of Schrodinger operators in $L_2(\mathbb{R}^3)$ with a trace class difference of resolvents. The results obtained are formulated in quite general abstract terms.

1. Let $\mathfrak{H}$ be a Hilbert space; A_k, $k=0,1$, – self-adjoint operators in $\mathfrak{H}$. Assume that the following conditions are fulfilled

$$\mathrm{Dom}A_0 = \mathrm{Dom}A_1; \tag{1}$$

$$(-\infty,\gamma]\cap\sigma(A_k)=\varnothing,\quad k=0,1; \tag{2}$$

$$V := A_1 - A_0 \in \mathfrak{B},\quad |V|^{1/2}(A_0 - zI)^{-1}\in\mathfrak{S}_2,\quad z\in\varrho(A_0). \tag{3}$$

Here for a self-adjoint operator A the following notation has been adopted: $\mathrm{Dom}A$, $\sigma(A)$, $\varrho(A)$ are the domain, spectrum and the resolvent set, respectively; $\mathfrak{B}$ and $\mathfrak{S}_2$ are the class of all bounded operators and the class of Hilbert-Schmidt operators, respectively. Consider semigroups, generated by the operators A_k, $k=0,1$,

$$\mathcal{U}_k(t) := e^{-tA_k},\quad t>0,\quad k=0,1. \tag{4}$$

Under the conditions (1)–(3), the difference $\mathcal{U}_0(t)-\mathcal{U}_1(t)$, $t>0$ is trace class. Consider

$$\Xi(t) = \mathrm{Tr}\left(\mathcal{U}_0(t)-\mathcal{U}_1(t)\right),\quad t>0.$$

Our main result is the following assertion.

Theorem 1. *Let conditions* (1) – (3) *be fulfilled. Then the difference* $\mathcal{U}_0(t)-\mathcal{U}_1(t)$ *is trace class, and the following double-sided estimate holds:*

$$\mathrm{Tr}(\mathcal{U}_1(t/2)V\mathcal{U}_1(t/2)) \leqslant t^{-1}\Xi(t) \leqslant \mathrm{Tr}(\mathcal{U}_0(t/2)V\mathcal{U}_0(t/2)). \tag{5}$$

2. Let now $\mathfrak{H}=L_2(\mathbb{R}^3)$; define the operators

$$A_0=-\Delta,\quad \mathrm{Dom}A_0 = H^2(\mathbb{R}^3),\quad A_1 = A_0+V. \tag{6}$$

Here Δ denotes the Laplace operator in $\mathbb{R}^3$, $H^2(\mathbb{R}^3)$ is the standard Sobolev space. It is assumed that

$$V=\overline{V}\in L_\infty(\mathbb{R}^3)\cap L_1(\mathbb{R}^3). \tag{7}$$

Under the condition (7) the operators (6) satisfy the conditions (1) – (3). Hence for the semigroups (4) generated by the operators (6) the estimate (5) holds. The following equality holds:

$$\mathrm{Tr}(\mathcal{U}_0(t/2)V\mathcal{U}_0(t/2)) = (4\pi t)^{-3/2}\int_{\mathbb{R}^3} V(x)dx. \tag{8}$$

From (7) the following estimate follows easily:

$$\mathrm{Tr}(\mathcal{U}_0(t/2)V\mathcal{U}_0(t/2)) - \mathrm{Tr}(\mathcal{U}_1(t/2)V\mathcal{U}_1(t/2)) = O(t^{-1/2}),\quad t\to+0. \tag{9}$$

Comparing (5), (8) and (9) we arrive at the following assertion:
Proposition 2. *Under the condition* (7)*, the following asyptotics holds*

$$\Xi(t) = (4\pi)^{-3/2}t^{-1/2}\int_{\mathbb{R}^3} V(x)dx + O(t^{1/2}), \quad t \to +0. \tag{10}$$

Well-known asymptotics (10) shows that the estimate (5) is tight for small $t > 0$.
A brief exposition of methology. We employ purely operator-theoretical technique developed by M.S. Birman and M.Z. Solomyak in 1972. Our argument is in essence based on the M.G. Krein – I.M. Lifshits formula and on the representation for the spectral shift function, obtained by M.S. Birman and M.Z. Solomyak.

We remark that related questions were discussed recently in a work by S.A. Stepin, where based on the Feynman-Kac representation for the corresponding heat kernels inequalities close to (5) were obtained in the case of Schrodinger operators in $\mathbb{R}^3$.

Szegö limit theorem for operators with discontinuous symbols: Widom's hypothesis

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The objective is to study the quasi-classical asymptotics of the spectrum for a pseudo-differential operator with a discontinuous symbol. Let $a(\mathbf{x}, \boldsymbol{\xi}),\ \mathbf{x}, \boldsymbol{\xi} \in \mathbb{R}^d$, $d \geq 1$ be a classical smooth symbol, i.e.

$$|\nabla_{\mathbf{x}}^s \nabla_{\boldsymbol{\xi}}^p a(\mathbf{x}, \boldsymbol{\xi})| \leq C_{s,p}\langle \mathbf{x}\rangle^{\gamma - s}\langle \boldsymbol{\xi}\rangle^{\sigma - p}, \langle \ \cdot \ \rangle = (1 + | \ \cdot \ |^2)^{\frac{1}{2}},$$

with $\gamma, \sigma \in \mathbb{R}$. Denote by $Op_\alpha(a), \alpha \geq 1$, the (quasi-classical) pseudo-differential operator (PDO) with the symbol a:

$$(Op_\alpha(a)u)(\mathbf{x}) = \left(\frac{\alpha}{2\pi}\right)^d \int\int e^{i\alpha(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}} a(\mathbf{x}, \boldsymbol{\xi})u(\mathbf{y})d\boldsymbol{\xi}d\mathbf{y},$$

$u \in L^2(\mathbb{R}^d)$. Let Λ and Ω be domains in $\mathbb{R}^d$. Denote by χ_Λ and χ_Ω their indicators, and let $P_\Omega = \chi_\Omega(-i\nabla)$. We are interested in the spectrum of the operator

$$A = A(a) = \chi_\Lambda Op_\alpha(a) P_\Omega \chi_\Lambda,$$

which clearly has a symbol with jump discontinuities in both variables $\mathbf{x}$ and $\boldsymbol{\xi}$. The aim is to find asymptotics of $tr f(A)$ as $\alpha \to \infty$ for suitable functions f, such that $f(0) = 0$.

In 1982 H. Widom [1] conjectured that

$$tr f(A) = \alpha^d W_0 + \alpha^{d-1}\log\alpha\ W_1 + o(\alpha^{d-1}\log\alpha),\ \alpha \to \infty, \tag{4}$$

with the coefficients

$$W_0 = \left(\frac{1}{2\pi}\right)^d \int_\Omega \int_\Lambda f(a(\mathbf{x}, \boldsymbol{\xi}))d\mathbf{x}d\boldsymbol{\xi},$$

$$W_1 = \left(\frac{1}{2\pi}\right)^{d-1} \frac{1}{4\pi^2} \int_{\partial\Omega} \int_{\partial\Lambda} |\mathbf{n_x} \cdot \mathbf{n}_{\boldsymbol{\xi}}| U(0, a(\mathbf{x}, \boldsymbol{\xi}); f) d\mathbf{x} d\boldsymbol{\xi}.$$

Here $\mathbf{n_x}$ and $\mathbf{n}_{\boldsymbol{\xi}}$ are exterior normals to the boundaries $\partial\Lambda$ and $\partial\Omega$ at the points $\mathbf{x}$ and $\boldsymbol{\xi}$ respectively, and

$$U(p, q; f) = \int_0^1 \frac{f((1-t)p + tq) - [(1-t)f(p) + tf(q)]}{t(1-t)} dt.$$

Operators of this type have been very well studied in the one-dimensional situation. In particular, the classical Szegő formula was generalized for the symbols with jump discontinuities by M. Fisher- R. Hartwig ('68), E. Basor ('79), A. Böttcher ('82). The formula (4) for $d = 1$ was proved by H. Widom ('82).

In the case $d \geq 3$ H. Widom [2] justified (4) under the assumption that one of the domains Λ, Ω was a half-space, and f was analytic in a disk of a suitably large radius. Recently D. Gioev and I. Klich ('06) discussed the relevance of (4) to the *Entanglement Entropy*, and in this context they announced a proof of the Widom's Hypothesis for $f(t) = t^2$. For general smooth f the precise order of the second term, i.e. $\alpha^{d-1} \log \alpha$, was established by D. Gioev ('06).

The main result of the talk is the following theorem:

Main Theorem *Let $d \geq 2$. Suppose that Ω, Λ are compact domains in $\mathbb{R}^d$, and that $\partial\Lambda \in C^1$, $\partial\Omega \in C^3$. Then the Widom's Hypothesis* (4) *holds for any polynomial f such that $f(0) = 0$.*

The proof has two ingredients:

1. The study of a model problem.
2. The reduction of the initial operator to the model one.

The model operator is the operator of the form $A(a)$ with $d = 1$, $\Lambda = (0, \infty), \Omega = (0, \infty)$. The required information about this operator is obtained using the methods of [1]. The reduction to the model operator requires new ideas. The main technical tool is a partition of unity, which becomes finer as one approaches the boundary $\partial\Lambda$, which enables one to localize the problem to balls of small radii. In each ball the boundary is approximated by a hyperplane, after which the ideas from [2] are used.

References

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On my joint work with M. Sh. Birman in 1965–1970

Michael Solomyak
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I am going to talk about our work on double operator integrals (DOI) and on the theory of piecewise-polynomial approximation.

1. Our joint work started in the early 1964 but the first result were published in 1965. The notion of a DOI was introduced by Y. Daletskii and S. Krein in 1956, but their comprehensive theory did not exist before our work. Its basics were developed by us in '65-67; some important additional results were obtained a little bit later, in 1973.

2. This development required a new technical tools, for estimation of eigenvalues of integral operators acting in a weighted L^2-spaces. Such technical tools were created in the process of our work. The main idea consisted in a special way of approximation of functions from the Sobolev spaces $H^l(Q)$ (where Q is a cube in $\mathbb{R}^d$) by a piecewise-polynomial functions.

3. The way of approximation suggested by us in 1967 turned out to be quite efficient not only for solving our original problem. It worked perfectly in such, quite different fields as spectral estimates and spectral asymptotics for the elliptic operators with non-smooth data, and also estimation of ε-entropy of embeddings $H^l(Q) \subset C^l(Q)$.

Counting bound states for Schrödinger operators on the lattice

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The results of a joint work with G. Rozenblum [2] are presented.

We study the estimates of the number of negative eigenvalues of the discrete Schrödinger operator $-\Delta - \alpha V$ in the Hilbert space $\ell_2(\mathbb{Z}^d)$, $d \geq 3$. The Rozenblum – Lieb – Cwikel estimate for the number of negative eigenvalues of the Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, easily extends to the discrete case:

$$N_-(-\Delta - \alpha V) \leq C(d)\alpha^{d/2} \sum_{x\in\mathbb{Z}^d} V(x)^{d/2}.$$

However, unlike in the "continuous" case, $V \in \ell_{d/2}(\mathbb{Z}^d)$ yields

$$N_-(-\Delta - \alpha V) = o(\alpha^{d/2}).$$

This shows that the discrete theory cannot be considered as just an elementary analogue of its continuous counterpart. The sharpest question here is this: how to construct discrete potentials V, that give the exact order $N_-(-\Delta - \alpha V) = O(\alpha^{d/2})$ (with "O" capital)?

We answer this question, by showing that such behavior can be achieved for the so-called sparse potentials. The property of sparseness will be defined below, after some necessary preliminaries.

Define the Hilbert space $\mathcal{H}^1(\mathbb{Z}^d)$, $d \geq 3$, consisting of all sequences $u(x)$, $x \in \mathbb{Z}^d$, such that

$$Q_0[u] = \sum_{\substack{x,y\in\mathbb{Z}^d \\ x\sim y}} |u(x) - u(y)|^2 < \infty; \qquad \sum_{x\in\mathbb{Z}^d} \frac{|u(x)|^2}{|x|^2+1} < \infty.$$

The quadratic form Q_0 is taken as the metric form in $\mathcal{H}^1(\mathbb{Z}^d)$, so that $\|U\|^2_{\mathcal{H}^1} = Q_0[u]$. If the discrete potential V is such that the quadratic form

$$\mathbf{b}_V[u] = \sum_{x\in\mathbb{Z}^d} V(x)|u(x)|^2$$

is bounded on $\mathcal{H}^1(\mathbb{Z}^d)$, then it defines on this space a bounded operator, say, $\mathbf{B}_V$. The Birman – Schwinger principle reduces the original problem to the study of the operator $\mathbf{B}_V$.

Now, consider the Green function of the discrete Laplacian. This is a discrete convolution, and its kernel is given by the explicit formula:

$$h_y(x) = h_0(x-y), \qquad \forall y \in \mathbb{Z}^d,$$

where

$$h_0(x) = (2\pi)^{-d} \int \frac{e^{ixz}}{4\sum_{j=1}^d \sin^2(z_j/2)} dz,$$

with integration over the d-dimensional torus. The function $h_0(x)$ lies in $\mathcal{H}^1(\mathbb{Z}^d)$, is harmonic outside the point $x = 0$, and its value at this point is some number μ^2. If u has finite support, then summation by parts leads to

$$(u, h_y)_{\mathcal{H}^1} = u(y).$$

This equality extends by continuity to all $u \in \mathcal{H}^1$. In particular,

$$(h_y, h_{y_1})_{\mathcal{H}^1} = h_y(y_1) = h_0(y-y_1)$$

and $\|h_y\|_{\mathcal{H}^1} = \mu$.

It follows from the formula for h_0 that

$$h_0(x)| \le C|x|^{-(d-2)}.$$

So we see that for the points $y, y_1 \in \mathbb{Z}^d$ lying far enough from each other, the functions h_y, h_{y_1} are 'almost orthogonal' in $\mathcal{H}^1$. This is the heart of the further construction. It is convenient to normalize these functions, and to work with $\widetilde{h}_y = \mu^{-1}h_y$.

Let Y be a subset in $\mathbb{Z}^d$, $d \ge 3$, and let $\mathcal{H}^1_Y$ stand for the subspace in $\mathcal{H}^1(\mathbb{Z}^d)$ spanned by the functions $h_y,\ y \in Y$. We say that the set Y is *sparse*, if in $\mathcal{H}^1(\mathbb{Z}^d)$ there exists a compact operator $\mathbf{T}$, such that the operator $\mathbf{I} - \mathbf{T}$ has bounded inverse and the functions

$$e_y = (\mathbf{I}-\mathbf{T})^{-1}\widetilde{h}_y, \qquad y \in Y,$$

form an orthonormal system in $\mathcal{H}^1$. Sparse sets do exist, their many examples can be constructed on the basis of Theorem VI.3.3 in the book [1]. We call a potential V sparse if its support is sparse.

Below we formulate one of our main results. For a function $V \ge 0$ on $\mathbf{Z}^d$, such that $V(x) \to 0$ as $|x| \to \infty$, we write $V_j^*, j \in \mathbb{N}$, for the numbers $V(x)$ rearranged in the non-increasing order.

Theorem *Let $V \ge 0$ be a sparse potential on $\mathbb{Z}^d$, $d \ge 3$. Then the operator $\mathbf{B}_V$ is compact if and only if $V(x) \to 0$ as $|x| \to \infty$. Moreover, the following two-sided inequality is satisfied for the eigenvalues $\lambda_j(\mathbf{B}_V)$*

$$CV_j^* \le \lambda_j(\mathbf{B}_V) \le C'V_j^*, \qquad \forall j \in \mathbb{N}.$$

In particular, if $V_j^* = j^{-2/d}$, then

$$N_-(-\Delta - \alpha V) = O(\alpha^{d/2}) \qquad \text{but} \neq o(\alpha^{d/2}).$$

References

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Inverse and Direct scattering on the half line

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This work is devoted to the study of the inverse and direct scattering problem for the forth order differential operator on the half line

$$L = \frac{d^4}{dx^4} + \frac{d}{dx}u(x)\frac{d}{dx} + v(x), \quad x \in [0, +\infty)$$

with smooth quickly decaying potentials $v(x)$ and $u(x)$. We will consider inverse problem for this operator in terms of corresponding Riemann-Gilbert problem on the system of rays. As an example we investigate well known inverse problem for the second order Schroedinger operator on the half line with the help of Riemann-Gilbert problem. This talk is based on the joint work with R.Shterenberg.

Homogenization of nonstationary periodic equations

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This is a joint work with M. Sh. Birman.

In $L_2(\mathbf{R}^d; \mathbf{C}^n)$, we consider a second order differential operator $\mathcal{A}_\varepsilon = b(D)^* g(\varepsilon^{-1}x) b(D)$, $\varepsilon > 0$. Here $g(x)$ is an $(m \times m)$-matrix-valued function in $\mathbf{R}^d$ such that $g, g^{-1} \in L_\infty$, $g(x) > 0$, and $g(x)$ is periodic with respect to some lattice. Next, $b(D)$ is a first order differential operator; its symbol $b(\xi)$ is an $(m \times n)$-matrix-valued linear homogeneous function of $\xi \in \mathbf{R}^d$ such that $\operatorname{rank} b(\xi) = n$, $\xi \neq 0$. We assume that $m \geq n$. We study the following Cauchy problem for the Schrödinger type equation for a function $u_\varepsilon(x, \tau)$, $x \in \mathbf{R}^d$, $\tau \in \mathbf{R}$:

$$i\partial_\tau u_\varepsilon(x, \tau) = \mathcal{A}_\varepsilon u_\varepsilon(x, \tau), \quad u_\varepsilon(x, 0) = \phi(x).$$

We also study the Cauchy problem for the hyperbolic equation for a function $v_\varepsilon(x, \tau)$, $x \in \mathbf{R}^d$, $\tau \in \mathbf{R}$:

$$\partial_\tau^2 v_\varepsilon(x, \tau) = -\mathcal{A}_\varepsilon v_\varepsilon(x, \tau), \quad v_\varepsilon(x, 0) = \varphi(x), \ \partial_\tau v_\varepsilon(x, 0) = \psi(x).$$

The corresponding "homogenized" problems look as follows:

$$i\partial_\tau u_0(x,\tau) = \mathcal{A}^0 u_0(x,\tau), \quad u_0(x,0) = \phi(x);$$

$$\partial_\tau^2 v_0(x,\tau) = -\mathcal{A}^0 v_0(x,\tau), \quad v_0(x,0) = \varphi(x), \ \partial_\tau v_0(x,0) = \psi(x).$$

Here $\mathcal{A}^0 = b(D)^* g^0 b(D)$ is the effective operator.

Theorem 1. *If $\phi \in L_2(\mathbf{R}^d;\mathbf{C}^n)$, then u_ε tends to u_0 in $L_2(\mathbf{R}^d;\mathbf{C}^n)$ for a fixed $\tau \in \mathbf{R}$, as $\varepsilon \to 0$. If $\phi \in H^s(\mathbf{R}^d;\mathbf{C}^n)$, $0 < s \le 3$, then*

$$\|u_\varepsilon(\cdot,\tau) - u_0(\cdot,\tau)\|_{L_2} \le \varepsilon^{s/3} C_s(\tau) \|\phi\|_{H^s}.$$

Here $C_s(\tau) = O(|\tau|^{s/3})$ for large values of $|\tau|$.

Theorem 2. *If $\varphi, \psi \in L_2(\mathbf{R}^d;\mathbf{C}^n)$, then v_ε tends to v_0 in $L_2(\mathbf{R}^d;\mathbf{C}^n)$ for a fixed $\tau \in \mathbf{R}$, as $\varepsilon \to 0$. If $\varphi, \psi \in H^s(\mathbf{R}^d;\mathbf{C}^n)$, $0 < s \le 2$, then*

$$\|v_\varepsilon(\cdot,\tau) - v_0(\cdot,\tau)\|_{L_2} \le \varepsilon^{s/2} \left(C_s^{(1)}(\tau) \|\varphi\|_{H^s} + C_s^{(2)}(\tau) \|\psi\|_{H^s} \right).$$

Here $C_s^{(1)}(\tau) = O(|\tau|^{s/2})$, $C_s^{(2)}(\tau) = O(|\tau|^{1+s/2})$ for large values of $|\tau|$.

We also prove analogs of Theorems 1 and 2 for more general class of operators. The results are published in [1].

References

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Exponential decay of eigenfunctions of first order systems

Dmitri Yafaev
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The first exponential estimate on eigenfunctions ψ of the discrete spectrum for second order self-adjoint elliptic operators H is due to Shnol' (1957) who proved that an eigenfunction corresponding to an eigenvalue λ satisfies the estimate

$$\int_{\mathbb{R}^d} |\psi(x)|^2 e^{2\delta|x|} dx < \infty. \qquad (*)$$

Here δ depends only on the distance $d(\lambda) = \mathrm{dist}\{\lambda, \sigma_{ess}(H)\}$ between the corresponding eigenvalue and the essential spectrum $\sigma_{ess}(H)$ of the operator H. Later Agmon (1982) has shown that estimate (*) is true with an arbitrary $\delta < \sqrt{d(\lambda)}$, but only for eigenvalues lying below $\sigma_{ess}(H)$. A natural question to ask is whether such a stronger estimate is true for eigenvalues lying in gaps of $\sigma_{ess}(H)$. We give a negative answer to this question considering a one-dimensional Schrödinger operator whose potential is a sum of a periodic function and of a function with compact support.

Another goal of our work is to study exponential decay of eigenfunctions for first order matrix differential operators

$$H = -i \sum_{j=1}^{\mathrm{d}} A_j \frac{\partial}{\partial x_j} + V(x)$$

acting in the space $\mathcal{H} = L_2(\mathbb{R}^{\mathrm{d}}; \mathbb{C}^n)$. Here $A_j = A_j^*$, $j = 1, \dots, \mathrm{d}$, are constant matrices and $V(x)$ is a symmetric matrix-valued functiion. Set

$$\gamma = \max_{|\xi|=1} \Big| \sum_{j=1}^{\mathrm{d}} A_j \xi_j \Big|, \quad \xi = (\xi_1, \dots, \xi_{\mathrm{d}}), \quad |\cdot| = |\cdot|_{\mathbb{C}^n}.$$

For example, $\gamma = 1$ for the Dirac operator. Our main result is the estimate (*) with an arbitrary $\delta < \gamma^{-1} d(\lambda)$ for all eigenvalues (including those lying in gaps of $\sigma_{ess}(H)$).

These results are published in *Contemporary Mathematics*, v. 447, 249-256, 2007.

Spectral properties of the scattering matrix

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The relation

$$\det S(\lambda) = e^{-2\pi i \xi(\lambda)}$$

between the scattering matrix $S(\lambda)$ and the spectral shift function $\xi(\lambda)$ for a pair of self-adjoint operators H_0, H was found in the famous paper by M. Sh. Birman and M. G. Kreĭn *On the theory of wave operators and scattering operators*, Soviet Math. Dokl. **3** (1962), 740-744. Actually, this relation is quite often used for the definition of the spectral shift function. To a certain extent, the theories of the scattering matrix and of the spectral shift function developed in parallel. I'm going to concentrate on spectral properties of the scattering matrix. Its spectrum consists of eigenvalues $\mu_n(\lambda)$ lying on the unit circle and accumulating at the point 1 only.

The following properties of these eigenvalues will be discussed in the talk:

1. If a perturbation $V = H - H_0$ is positive (negative), then eigenvalues $\mu_n(\lambda)$ may accumulate to 1 only from below (from above).

2. If a perturbation increases (decreases), then eigenvalues $\mu_n(\lambda)$ rotate in the clockwise (counterclockwise) direction.

These properties were discovered by M. Sh. Birman and M. G. Kreĭn in the trace class framework. Later they were extended to a broader class of perturbations (but also of trace class type) by L. S. Koplienko and S. Yu. Rotfel'd. In the talk these assertions will be obtained using only the structure of the stationary representation of the scattering matrix. Therefore this approach works both under trace class and smooth assumptions.

A typical example is the pair $H_0 = -\Delta$, $H = -\Delta + V(x)$ in the space $L_2(\mathbb{R}^d)$ where the real function $V(x)$ satisfies the estimate

$$|V(x)| \le C(1 + |x|)^{-\rho}, \quad \rho > 1.$$

If $V(x) = v(\hat{x})|x|^{-\rho} + o(|x|^{-\rho})$ where $v \in C^\infty(\mathbb{S}^{d-1})$, then asymptotics of $\mu_n(\lambda)$ can be found. The following result was obtained in the paper by M. Sh. Birman and D. R. Yafaev *The* *asymptotic behavior of the spectrum of the scattering matrix*, J. Soviet Math. **25** (1984), 793-814. Let us write eigenvalues of $S(\lambda)$ as $\mu_n^{\pm}(\lambda) = e^{\pm 2i\varphi_n^{\pm}(\lambda)}$, where $\varphi_n^{+}(\lambda) \in (0, \pi/2]$, $\varphi_n^{-}(\lambda) \in (0, \pi/2)$, $\varphi_{n+1}^{\pm}(\lambda) \leq \varphi_n^{\pm}(\lambda)$. Then asymptotics of the scattering phases is given by the formula

$$\lim_{n\to\infty} n^{\gamma}\varphi_n^{\pm}(\lambda) = c^{\pm}$$

where $\gamma = (\rho - 1)(d-1)^{-1}$ and $c^{\pm}$ are some explicit functionals of v, ρ and λ.

YOUNG SCIENTISTS SESSION ABSTRACTS:

Absolute continuity of the spectrum of the Schrödinger operator in a layer and in a smooth multidimensional cylinder

I. Kachkovskiy
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This is a joint work with N. Filonov.

We study the periodic Schrödinger operator in a d-dimensional cylinder and in a plane-parallel layer. In the case of a layer we establish the absolute continuity of its spectrum for the potentials $V \in L_{p,\mathrm{loc}}$, $p > d/2$, $d \geqslant 3$. The conditions for the cylinder are $p > d/2$, $d = 3, 4$ and $p > d - 2$, $d \geqslant 5$. The proof is based on the classical Thomas scheme. In the case of a layer we then use a modified version of Sogge's spectral cluster L_p-estimates. In the case of a cylinder we use spectral cluster estimates for manifolds with boundary, derived by Smith and Sogge as a corollary of their Strichartz estimates for the wave equation.

The uniqueness theorem for vector-valued Sturm-Liouville operators

S. Matveenko
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We consider self-adjoined Sturm-Liouville operators on the unit interval with matrix-valued potentials and separated boundary conditions of general type. We obtain the uniqueness theorem, if the boundary conditions are fixed. Moreover, we prove that in some special cases spectral data (i.e. residues of the Weyl-Titchmarsh function) uniquely determine the boundary conditions and so the whole operator.

Monodromization and the Maryland equation

F. Sandomirskiy
St.Petersburg State University

This is a joint work with A. Fedotov.

Monodromization method is a renormalization method invented by V. Buslaev and A. Fedotov in 90s to study quasi-periodic equaions. It was successfully used to solve different problems. We apply this method to the Maryland equation, the simplest finite difference Schreodinger equation with a meromorphic potential. We prove that this equation is equivalent to a difference equation invariant with respect to the renormalizations up to two constant parameters. The transformation of the frequency, one of these parameters, is described by the Gauss map. The transformation of second parameter, an effective coupling constant, is described by a simple explicit formula.

Weyl-Titchmarsh type formula for discrete Schrödinger operator with Wigner-von Neumann potential

S. Simonov
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The discrete Schrodinger operator with Wigner-von Neumann potential is considered. The classical Weyl-Titchmarsh formula for Schrödinger operator on the half-line with summable potential relates the spectral density to the behavior of solutions of the spectral equation. The analog of this formula is obtained in the considered discrete case.

Homogenization with corrector of a periodic parabolic Cauchy problem

E. Vasilevskaya
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We consider the Cauchy problem for the parabolic equation with periodic coefficients in the small period limit. The convergence of the solutions to the solution of the corresponding homogenized problem was proved by M. Sh. Birman and T. A. Suslina. Using the same spectral approach we improve the estimate for the solutions by introducing a corrector.

Homogenization of High Order Periodic Differential Operators

N. Veniaminov
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The homogenization for the second order differential operators in the small period limit is a well studied problem. An approach based on the operator theory has been developed by M. Sh. Birman and T. A. Suslina during the last ten years. In this talk, the higher order differential operators that admit factorization are considered within the same framework. The special and physically meaningfull case is the fourth order operator $DD\,aDD$ (where a is the elasticity tensor) that describes elasticity of plates. For the class of operators desribed above the approximation for the resolvent is obtained.