

D-instantons and twistors

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Compactifications of string theory

Type II superstring theory in 10d

low energy limit



$$\alpha' E \sim \alpha' R_{(10)} \ll 1$$

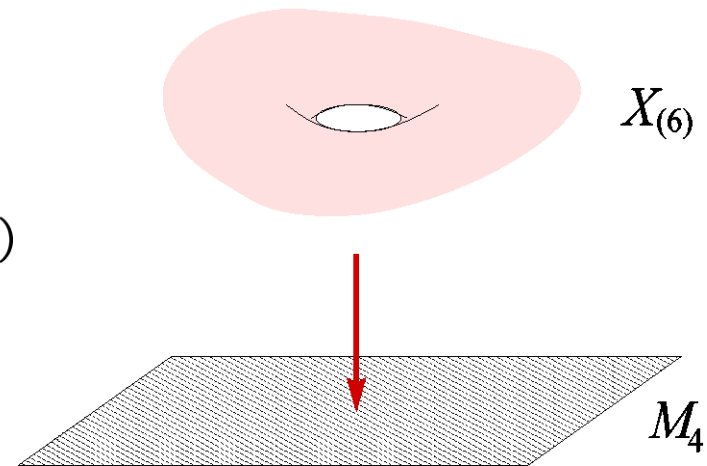
effective theory of fields $G_{\mu\nu}, B_{\mu\nu}, \Phi, \dots$:
 $\mathcal{N}=2$ supergravity in 10d

compactification
on a Calabi-Yau



$$X_{(10)} = M_4 \times X_{(6)}$$

effective theory in M_4 :
 $\mathcal{N}=2$ supergravity in 4d coupled to matter



The aim: to find the complete non-perturbative effective action in 4d
for type II string theory compactified on arbitrary CY

approximation:

- 2-derivatives
- massless sector

Vector multiplets and hypermultiplets

$\mathcal{N}=2$ supergravity

vector multiplets
(gauge fields & scalars)

supergravity multiplet
(metric)

hypermultiplets
(only scalars)

$$\mathcal{L}_{\text{VM}} = \mathcal{G}_{a\bar{b}}(z) \partial_\mu z^a \partial^\mu \bar{z}^{\bar{b}} + \mathcal{N}_{\Lambda\Sigma}(z) F_{\mu\nu}^\Lambda F^{\Sigma,\mu\nu}$$

$$\mathcal{L}_{\text{HM}} = g_{\alpha\beta}(q) \partial_\mu q^\alpha \partial^\mu \bar{q}^\beta$$

determined by

holomorphic prepotential $F(z)$

non-linear σ -model

moduli space – *quaternion-Kähler*
(known only at *tree level*)

moduli space – *special Kähler*
(known)

The main complication:

HM contain the dilaton giving string coupling $g_s \sim e^\phi$ \rightarrow $g_{\alpha\beta}$ receives perturbative and non-perturbative (instanton) g_s corrections

Instantons – (Euclidean) branes wrapping non-trivial cycles of CY

Symmetries and dualities

The problem: the rules of the string instanton calculus are *not* known

The idea: to use non-perturbative symmetries

Type IIA

symplectic invariance

- the invariance is related with the choice of the basis of 3-cycles

$$\gamma^\Lambda \# \gamma^\Sigma = \emptyset \quad \gamma_\Lambda \# \gamma_\Sigma = \emptyset \quad \gamma^\Lambda \# \gamma_\Sigma = \delta_\Sigma^\Lambda$$

$$\gamma^\Lambda \text{ — A-cycle} \quad \gamma_\Lambda \text{ — B-cycle}$$

- the physical fields form a representation of $\text{Sp}(h_{1,2}, \mathbb{Z})$

Type IIB

$\text{SL}(2, \mathbb{Z})$

- generated by T and S dualities
- truly non-perturbative

$$\tau = c^0 + ie^{-\phi} \mapsto \frac{a\tau + b}{c\tau + d}$$

- the physical fields form a representation of $\text{SL}(2, \mathbb{Z})$

mirror symmetry

$$\text{IIA}/\text{CY} = \text{IIB}/\tilde{\text{CY}}$$

(known only at *tree level*)

A-D2
B-D2

D(-1), D1
D3, D5

Symmetries and instanton corrections

Hypermultiplet sector

Vector multiplet sector

IIA / X

IIB / Y

IIA / Y

IIB / X

$\{1-l, D2, NS5\}$ $\xleftarrow{\text{mirror}}$ $\{\alpha', D(-1), D1, D3, D5, NS5\}$

\uparrow $SL(2, \mathbb{Z})$

$\xrightarrow{\text{mirror}}$

$\{\alpha', D(-1), D1, D3, D5\}$

 we are here

$\xleftarrow{\text{mirror}}$

$\{1-l, A-D2\}$ $\xleftarrow{\text{mirror}}$ $\{\alpha', D(-1), D1\}$

\uparrow $SL(2, \mathbb{Z})$

$\{\alpha', 1\text{-loop}\}$

$\xleftarrow{\text{c-map}}$

$\{\alpha'\}$

$\xleftarrow{\text{mirror}}$

$\{-\}$

$\{1-l, A-D2, B-D2\}$

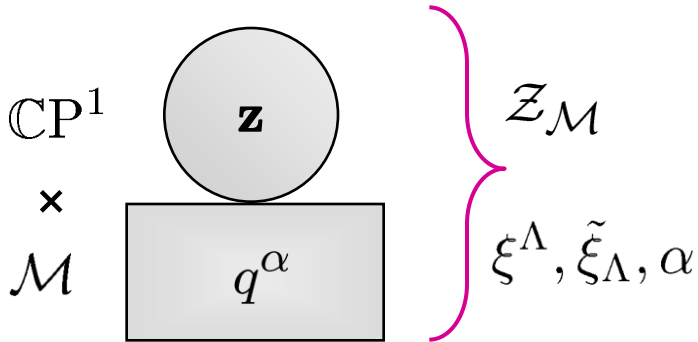
$\{\alpha', D(-1), D1, D3, D5\}$

\uparrow e/m
 duality

Twistor space formulation

How to conveniently describe (parametrize) a quaternion-Kähler manifold?

Twistor space



Properties:

- Einstein-Kähler
- has odd complex dimension
- carries *contact structure*

$$\mathcal{X}^{[i]} \equiv d\alpha^{[i]} + \xi_{[i]}^\Lambda d\tilde{\xi}_\Lambda^{[i]}$$

complex Darboux coordinates

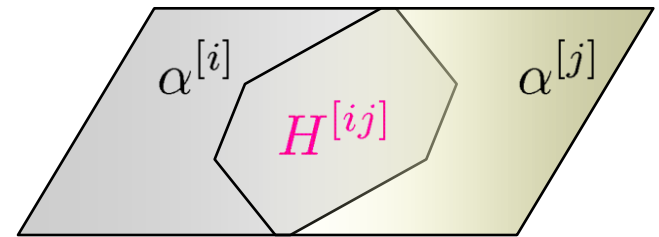
- symmetries of \mathcal{M} can be lifted to $\mathcal{Z}_{\mathcal{M}}$!!!

such that $\mathcal{X}^{[i]} \wedge (d\mathcal{X}^{[i]})^d \neq 0$

The main ingredient:

transition functions between different patches

$$H^{[ij]}(\xi, \tilde{\xi}, \alpha)$$



metric on \mathcal{M}



twistor lines

$$\xi_{[i]}^\Lambda(q^\alpha, \mathbf{z}), \tilde{\xi}_\Lambda^{[i]}(q^\alpha, \mathbf{z}), \alpha^{[i]}(q^\alpha, \mathbf{z})$$



$$\xi_{[i]}^\Lambda = \xi_{[j]}^\Lambda + \partial_{\tilde{\xi}_\Lambda^{[j]}} H^{[ij]} - \xi_{[j]}^\Lambda \partial_{\alpha^{[j]}} H^{[ij]}$$

$$\tilde{\xi}_\Lambda^{[i]} = \tilde{\xi}_\Lambda^{[j]} - \partial_{\xi_\Lambda^{[i]}} H^{[ij]}$$

$$\alpha^{[i]} = \alpha^{[j]} - H^{[ij]} + \xi_{[i]}^\Lambda \partial_{\xi_\Lambda^{[i]}} H^{[ij]}$$

Perturbative HM moduli space

HM moduli space at tree level

c-map \rightarrow

$$H^{[+0]} = F(\xi) \quad H^{[-0]} = \bar{F}(\xi)$$

Twistor lines:

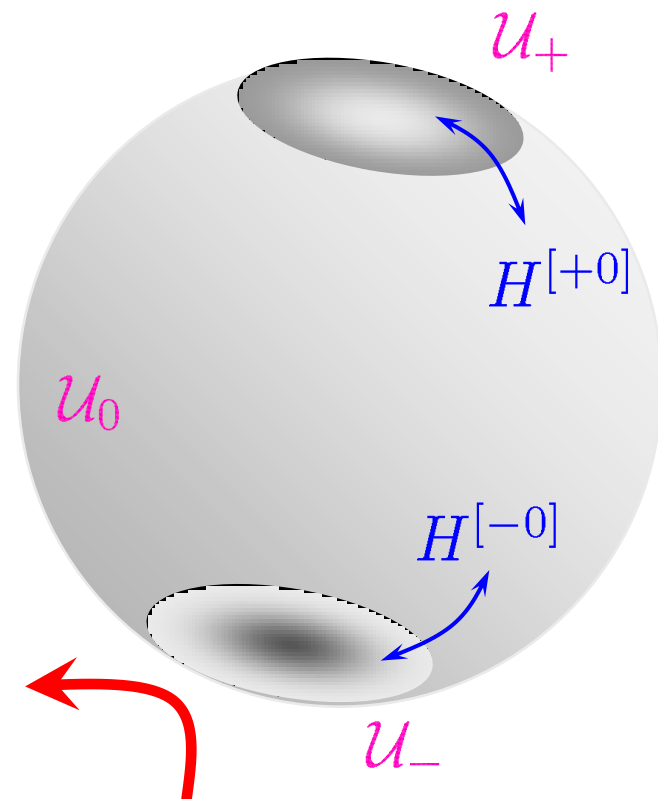
$$\xi_{[0]}^\Lambda = \zeta^\Lambda + \mathcal{R} (z^{-1} z^\Lambda - z \bar{z}^\Lambda)$$

$$\tilde{\xi}_{\Lambda}^{[0]} = \tilde{\zeta}_\Lambda + \mathcal{R} (z^{-1} F_\Lambda(z) - z \bar{F}_\Lambda(\bar{z}))$$

$$\alpha^{[0]} = \sigma + \mathcal{R} (z^{-1} W(z) - z \bar{W}(\bar{z})) + \frac{i\chi_{CY}}{24\pi} \log z$$

where $W(z) \equiv F_\Lambda(z)\zeta^\Lambda - z^\Lambda \tilde{\zeta}_\Lambda$

- respect symplectic invariance, $SL(2, \mathbb{Z})$ -duality and mirror symmetry
- reproduce the known HM metric



One-loop correction

- appears as a singular boundary condition for twistor lines
- determined by the Euler number

Non-perturbative HM moduli space (Type IIA picture)

Every D2 brane has the charge $\gamma = (q_\Lambda, p^\Lambda)$
(it wraps the cycle $q_\Lambda \gamma^\Lambda + p^\Lambda \gamma_\Lambda$)

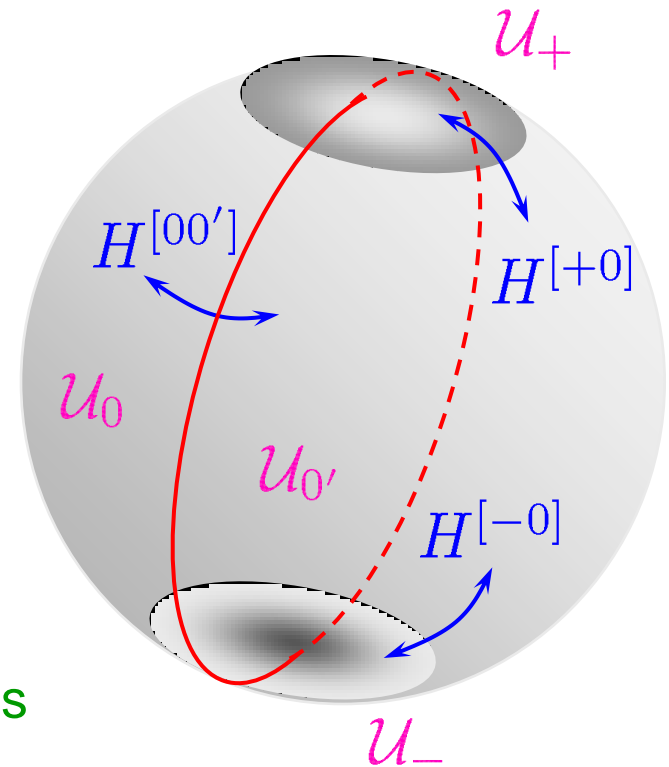


It defines a direction in the complex plane

$$\arg(q_\Lambda z^\Lambda - p^\Lambda F_\Lambda(z))$$



It gives rise to two rays introducing discontinuities



$$H^{[00']} = \frac{n_\gamma}{(2\pi)^2} \text{Li}_2 \left(e^{2\pi i (q_\Lambda \xi^\Lambda - p^\Lambda \tilde{\xi}_\Lambda)} \right)$$

$$\text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Non-perturbative mirror map



- was known only classically ($\alpha' \rightarrow 0, g_s \rightarrow 0$)
Bohm, Gunther, Hermann, Louis '99
- we can include α' , D(-1) and D1 corrections
- the method – requirement that the twistor lines form a representation of $SL(2, \mathbb{Z})$
- $SL(2, \mathbb{Z})$ transformations of the twistor lines do *not* get any corrections

Conclusions

- We developed the twistor description of QK spaces
- We found all D-instanton corrections to HM moduli space and some of them to the mirror map
- D-instantons are captured by one *dilogarithm* function



The dilogarithm is
the only function
with a sense of humor

Don Zagier

Our results are relevant for:

- BPS black holes
- wall-crossing (lines of marginal stability)
- further – NS5-instantons (non-abelian Fourier expansion)