# D-instantons and twistors

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# Compactifications of string theory

Type II superstring theory in 10d

low energy limit $\quad \alpha' E \sim \alpha' R_{(10)} \ll 1$

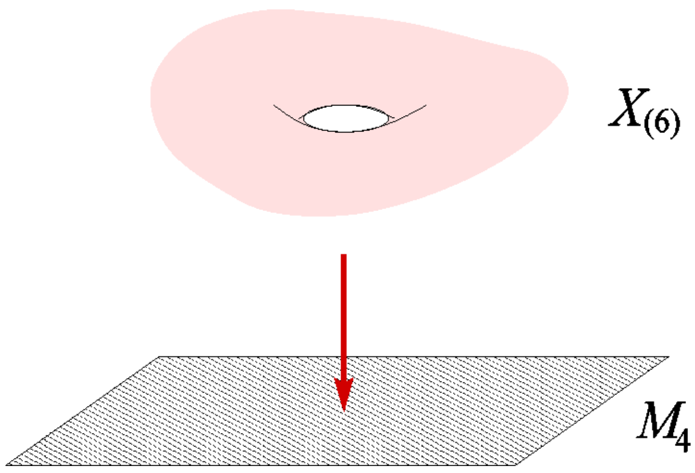
effective theory of fields $\quad G_{\mu\nu},\ B_{\mu\nu},\ \Phi, \ldots$:

$\mathcal{N}$=2 supergravity in 10d

compactification on a Calabi-Yau $\quad X_{(10)} = M_4 \times X_{(6)}$

effective theory in $M_4$:

$\mathcal{N}$=2 supergravity in 4d coupled to matter




**The aim:** **to find the complete non-perturbative effective action in 4d for type II string theory compactified on arbitrary CY**

approximation:
- 2-derivatives
- massless sector

# Vector multiplets and hypermultiplets

$\mathcal{N}$=2 supergravity

vector multiplets
(gauge fields & scalars)

supergravity multiplet
(metric)

hypermultiplets
(only scalars)

$$\mathcal{L}_{\rm VM} = \mathcal{G}_{a\bar{b}}(z)\partial_\mu z^a \partial^\mu \bar{z}^{\bar{b}} + \mathcal{N}_{\Lambda\Sigma}(z) F^{\Lambda}_{\mu\nu} F^{\Sigma,\mu\nu}$$

$$\mathcal{L}_{\rm HM} = g_{\alpha\beta}(q)\partial_\mu q^\alpha \partial^\mu \bar{q}^\beta$$

determined by
*holomorphic prepotential* $F(z)$

moduli space – *special Kähler*
(known)

non-linear **σ**-model
moduli space – *quaternion-Kähler*
(known only at *tree* level)

**The main complication:**

HM contain the dilaton
giving string coupling $g_s \sim e^\phi$ ⇒ $g_{\alpha\beta}$ receives perturbative and non-perturbative (instanton) $g_s$ corrections

Instantons – (Euclidean) branes wrapping non-trivial cycles of CY

# Symmetries and dualities

The problem: the rules of the string instanton calculus are *not* known

The idea: to use non-perturbative symmetries

## Type IIA

**symplectic invariance**

- the invariance is related with the choice of the basis of 3-cycles

$$\gamma^\Lambda \# \gamma^\Sigma = \emptyset \quad \gamma_\Lambda \# \gamma_\Sigma = \emptyset \quad \gamma^\Lambda \# \gamma_\Sigma = \delta^\Lambda_\Sigma$$

$$\gamma^\Lambda \text{ — A-cycle} \qquad \gamma_\Lambda \text{ — B-cycle}$$

- the physical fields form a representation of $\mathrm{Sp}(h_{1,2},\mathbb{Z})$

## Type IIB

**SL(2,$\mathbb{Z}$)**

- generated by T and S dualities
- truly non-perturbative

$$\tau = c^0 + \mathrm{i}e^{-\phi} \mapsto \frac{a\tau + b}{c\tau + d}$$

- the physical fields form a representation of $\mathrm{SL}(2,\mathbb{Z})$

**mirror symmetry**

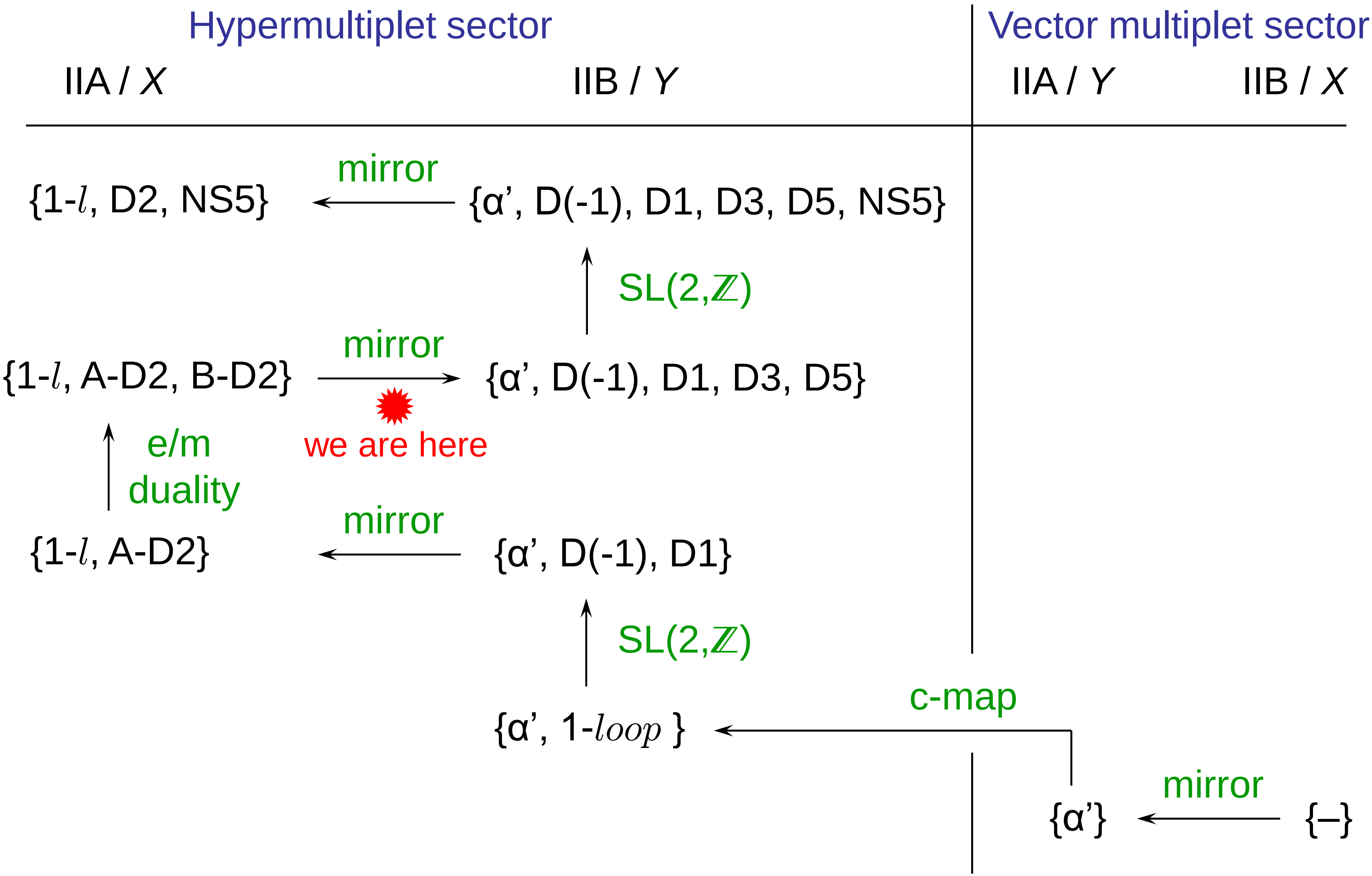
$$\mathrm{IIA}/CY \;=\; \mathrm{IIB}/\widetilde{CY}$$

(known only at *tree* level)

A-D2
B-D2

D(-1),D1
D3,D5

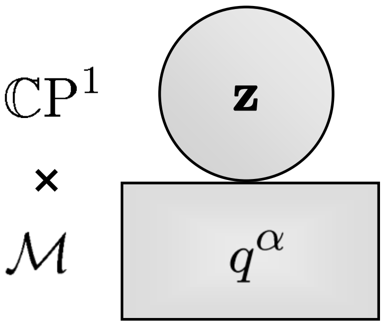
# Symmetries and instanton corrections



# Twistor space formulation

How to conveniently describe (parametrize) a quaternion-Kähler manifold?

**Twistor space**




$\mathcal{Z}_\mathcal{M}$

$\xi^\Lambda, \tilde\xi_\Lambda, \alpha$

Properties:

- Einstein-Kähler
- has odd complex dimension
- carries *contact stricture*

$$\mathcal{X}^{[i]} \equiv \mathrm{d}\alpha^{[i]} + \xi^\Lambda_{[i]}\, \mathrm{d}\tilde\xi^{[i]}_\Lambda$$

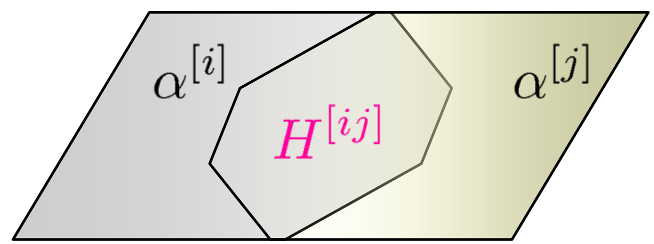
complex Darboux coordinates

such that $\mathcal{X}^{[i]} \wedge (\mathrm{d}\mathcal{X}^{[i]})^d \neq 0$

- symmetries of $\mathcal{M}$ can be lifted to $\mathcal{Z}_\mathcal{M}$ !!!

The main ingredient:

transition functions between different patches
$H^{[ij]}(\xi, \tilde\xi, \alpha)$




$$\xi^\Lambda_{[i]} = \xi^\Lambda_{[j]} + \partial_{\tilde\xi^{[j]}_\Lambda} H^{[ij]} - \xi^\Lambda_{[j]} \partial_{\alpha^{[j]}} H^{[ij]}$$
$$\tilde\xi^{[i]}_\Lambda = \tilde\xi^{[j]}_\Lambda - \partial_{\xi^\Lambda_{[i]}} H^{[ij]}$$
$$\alpha^{[i]} = \alpha^{[j]} - H^{[ij]} + \xi^\Lambda_{[i]} \partial_{\xi^\Lambda_{[i]}} H^{[ij]}$$

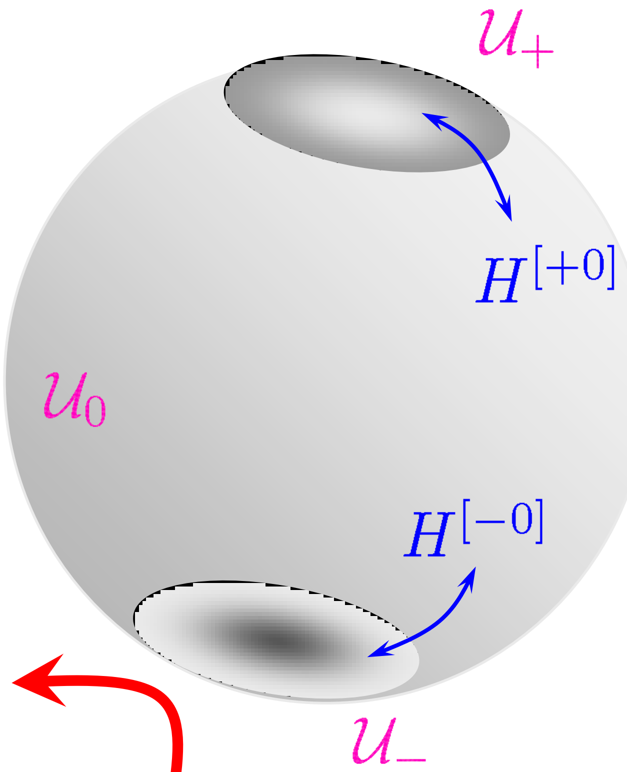
twistor lines
$\xi^\Lambda_{[i]}(q^\alpha, \mathbf{z}),\ \tilde\xi^{[i]}_\Lambda(q^\alpha, \mathbf{z}),\ \alpha^{[i]}(q^\alpha, \mathbf{z})$

metric on $\mathcal{M}$

# Perturbative HM moduli space

HM moduli space at tree level

c-map → $$H^{[+0]} = F(\xi) \qquad H^{[-0]} = \bar{F}(\xi)$$




Twistor lines:

$$\begin{aligned}
\xi^{\Lambda}_{[0]} &= \zeta^{\Lambda} + \mathcal{R}\left(\mathbf{z}^{-1} z^{\Lambda} - \mathbf{z}\,\bar{z}^{\Lambda}\right) \\
\tilde{\xi}^{[0]}_{\Lambda} &= \tilde{\zeta}_{\Lambda} + \mathcal{R}\left(\mathbf{z}^{-1} F_{\Lambda}(z) - \mathbf{z}\,\bar{F}_{\Lambda}(\bar{z})\right) \\
\alpha^{[0]} &= \sigma + \mathcal{R}\left(\mathbf{z}^{-1} W(z) - \mathbf{z}\,\bar{W}(\bar{z})\right) + \frac{\mathrm{i}\chi_{\mathrm{CY}}}{24\pi} \log \mathbf{z}
\end{aligned}$$

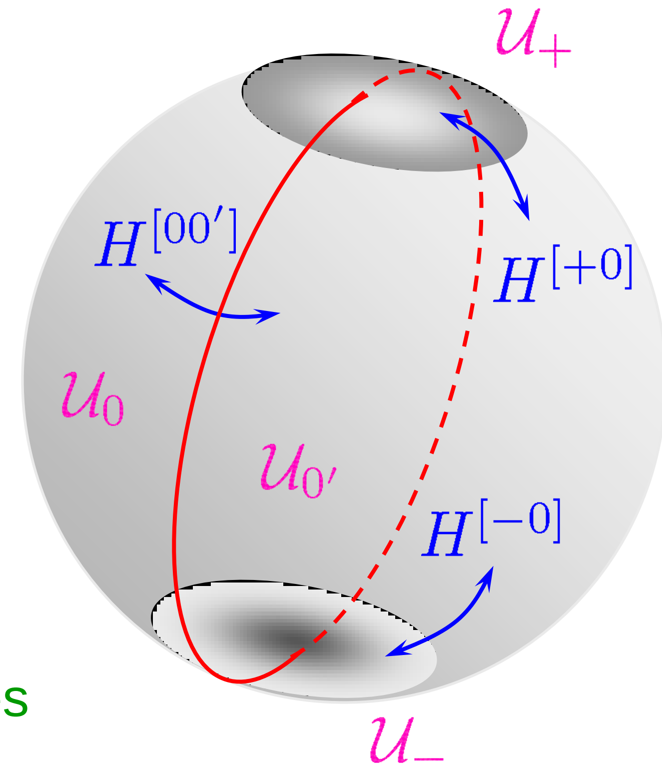
where $W(z) \equiv F_{\Lambda}(z)\zeta^{\Lambda} - z^{\Lambda}\tilde{\zeta}_{\Lambda}$

- respect symplectic invariance, SL(2,$\mathbb{Z}$)-duality and mirror symmetry
- reproduce the known HM metric

One-loop correction

- appears as a singular boundary condition for twistor lines
- determined by the Euler number

# Non-perturbative HM moduli space
## (Type IIA picture)




Every D2 brane has the charge $\gamma = (q_\Lambda, p^\Lambda)$
(it wraps the cycle $q_\Lambda \gamma^\Lambda + p^\Lambda \gamma_\Lambda$ )

⬇

It defines a direction in the complex plane

$$\arg\left(q_\Lambda z^\Lambda - p^\Lambda F_\Lambda(z)\right)$$

⬇

It gives rise to two rays introducing discontinuities

$$H^{[00']} = \frac{n_\gamma}{(2\pi)^2}\,\mathrm{Li}_2\left(e^{2\pi \mathrm{i}(q_\Lambda \xi^\Lambda - p^\Lambda \tilde{\xi}_\Lambda)}\right)$$

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

# Non-perturbative mirror map

Type IIA physical fields $\xleftrightarrow{\text{mirror map}}$ Type IIB physical fields

- was known only classically $(\alpha' \to 0, \;\; g_s \to 0)$
  Bohm,Gunther,Hermann,Louis ’99

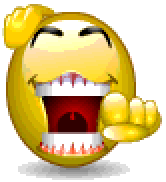
- we can include $\alpha'$, D(-1) and D1 corrections

- the method – requirement that the twistor lines
form a representation of SL(2,$\mathbb{Z}$)

- SL(2,$\mathbb{Z}$) transformations of the twistor lines
do *not* get any corrections

# Conclusions

- We developed the twistor description of QK spaces
- We found all D-instanton corrections to HM moduli space and some of them to the mirror map
- D-instantons are captured by one *dilogarithm* function



**The dilogarithm is the only function with a sense of humor**

Don Zagier

Our results are relevant for:

- BPS black holes
- wall-crossing (lines of marginal stability)
- further – NS5-instantons (non-abelian Fourier expansion)