Strong impurity in a quantum wire

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Outline

- Impurity in Luttinger liquid : fermionic approach and bosonization
- S-matrix approach and current algebra formulation
- Analysis of interaction corrections to S-matrix – RG equation
- B function up to 3rd loop and solution of RG equation
- Comparison with CFT solution and dependence on regularization

Hamiltonian

$$
\begin{aligned}
H &= H_0 + H_1 + H_{imp} \\
H_0 &= v_F \int_{-\infty}^{\infty} dx \left[\psi_R^\dagger (i\partial_x)\psi_R - \psi_L^\dagger (i\partial_x)\psi_L\right] \\
H_1 &= g_2 \int_{-L}^{L} dx \, (\psi_R^\dagger \psi_R)(\psi_L^\dagger \psi_L) \\
H_{imp} &= v_F \int dx \left[u_1(x)(\psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L)\right. \\
&\quad \left. + (u_2(x)\psi_L^\dagger \psi_R + h.c.)\right],
\end{aligned}
$$

Fermions

The same, in bosonization:

$$
\begin{aligned}
H &= \frac{\tilde{v}_F}{2\pi} \int dx \left[K\,(\partial_x \theta)^2 + K^{-1}(\partial_x \varphi)^2\right. \\
&\quad \left. + 2u_1(x)\partial\varphi(x) + u_2(x)\cos 2\varphi(x)\right]
\end{aligned}
$$

Boundary Sine-Gordon model

Introduction: Quantum wire with single barrier

The observable: conductance G = current-density correlation functon was studied in bosonization approach

- Decrease of the conductance for a weak barrier

$$G = 1 - \left(\frac{T}{T_0}\right)^{2(K-1)}$$

- Strong barrier = weak tunneling : further suppression and vanishing of conductance in limit $T \rightarrow 0$

$$G \sim \left(\frac{T}{T_0}\right)^{2(K^{-1}-1)}$$

Kane and Fisher, 1992
Furukawa and Nagaosa, 1993

complete solution in bosonization :
Fendley, Ludwig and Saleur, 1995

Introduction: Poor man's scaling at weak interaction

Fermionic approach :
Integrating out high momentum states, reducing the band width, one finds a renormalization group equation for the transmission amplitude as a function of the bandwidth *D*

$$\frac{dt}{d\ln(D_0/D)} = -gt(1-|t|^2), \qquad g = g_2/2\pi v_F$$

The transmission coefficient as a function of energy follows as

$$T(\varepsilon) = \frac{T_0(\varepsilon/D_0)^{2g}}{R_0 + T_0(\varepsilon/D_0)^{2g}}, \qquad \varepsilon < D_0$$

Yue, Matveev and Glazman, 1995

Introduction: Further analytical studies

Boundary sine-Gordon model - complete solution in bosonization :
Fendley, Ludwig and Saleur, 1995 one-parameter scaling in RG

$$\begin{array}{c|cccc} & U & U^2 & \cdots & S=U^{\infty} \\ \hline g & & & & YMG \\ g^2 & & & & \\ \vdots & & & & \\ K=g^{\infty} & KF & \rightarrow & & FLS \end{array}$$

(bosonization)

(fermions)

Our aim

Fermionic approach is more flexible (inelastic, out-of-equilibrium)

Single particle scattering states

Scattering of spinless fermions by potential barrier:

S-matrix:

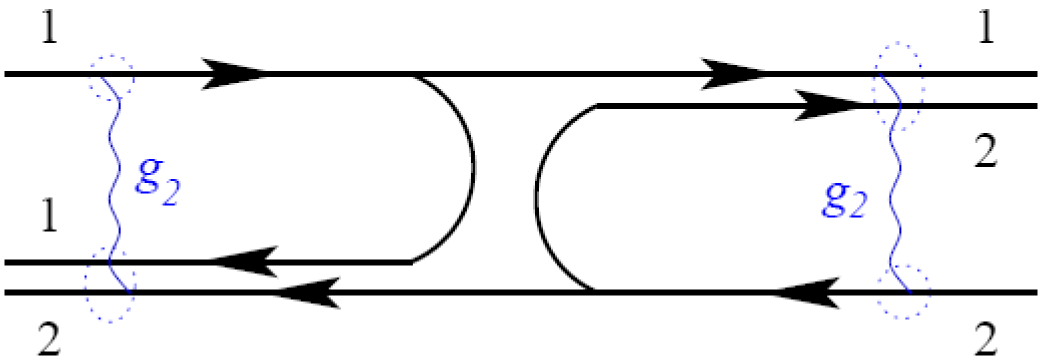
$$S = \begin{pmatrix} t & \tilde{r} \\ r & \tilde{t} \end{pmatrix} = \begin{pmatrix} \cos\theta & i\sin\theta e^{-i\phi} \\ i\sin\theta e^{i\phi} & \cos\theta \end{pmatrix}$$

Single particle scattering states for right (left) moving particles (k>0) :

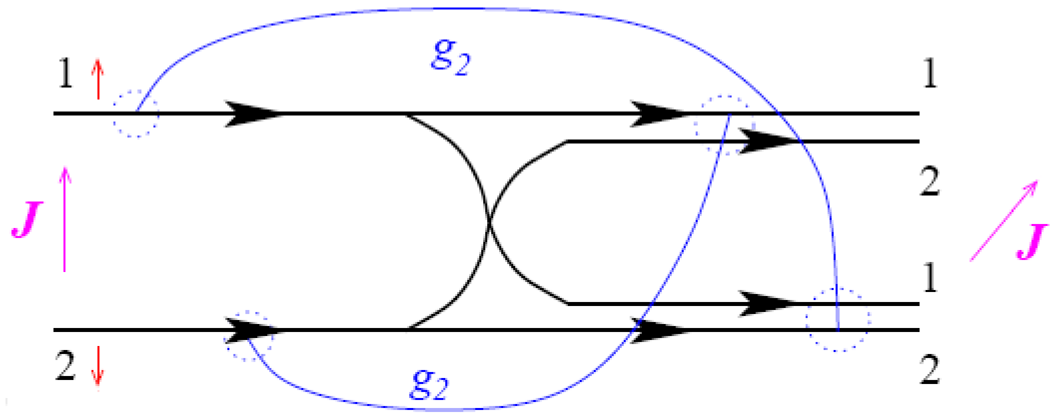
$$\begin{aligned} \psi_k(x) &= (e^{ikx} + r_k e^{-ikx}) c^\dagger_{1k} + \tilde{t}_k e^{-ikx} c^\dagger_{2k}, \quad x < 0 \\ &= t_k e^{ikx} c^\dagger_{1k} + (\tilde{r}_k e^{ikx} + e^{-ikx}) c^\dagger_{2k}, \quad x > 0 \end{aligned}$$

Neglect *k*-dependence of *t*, *r* in the following

Interaction in scattering state representation



potential barrier is viewed as a local magnetic field rotating the isospin vector of a wave packet, when it passes through the field.



"Nonlocal" interaction: (cf. M.Fabrizio, A.Gogolin, 1995)

Interaction in scattering state representation

$$\begin{pmatrix} J_0 + J_3 & J_1 - iJ_2 \\ J_1 + iJ_2 & J_0 - J_3 \end{pmatrix} = \begin{pmatrix} \psi_1^\dagger(x)\psi_1(x) & \psi_1^\dagger(x)\psi_2(-x) \\ \psi_2^\dagger(-x)\psi_1(x) & \psi_2^\dagger(-x)\psi_2(-x) \end{pmatrix}$$

Currents or chiral densities

Kac-Moody algebras :

$$\begin{aligned} [J_0(x), J_0(y)] &= \frac{i}{4\pi}\partial_x \delta(x-y) \\ [J_j(x), J_k(y)] &= \frac{i}{4\pi}\delta_{jk}\partial_x\delta(x-y) + i\varepsilon_{jkl}J_l(y)\delta(x-y) \end{aligned}$$

$$\begin{aligned} H_0 &= 2\pi v_F \int_{-\infty}^{0} dx\,(J_0^2(x) + J_3^2(x)) + 2\pi v_F \int_0^\infty dx\,(J_0^2(x) + \widetilde{J}_3^2(x)) \\ H_1 &= 2g_2 \int_a^L dx\,(J_0(-x)J_0(x) - J_3(-x)\widetilde{J}_3(x)) \end{aligned}$$

Rotated current $\widetilde{J}_3 = (R\vec{J})_3$ where $R_{\mu\nu} = \frac{1}{2}Tr[\sigma_\mu.S.\sigma_\nu.S^\dagger]$

Perturbation theory in g_2 : Feynman diagrams

Diagram rules for *n*th order contributions in (energy-position)-representation:

(1) Draw *n* vertical wavy lines representing interaction ($-2g_2$) , the *i*th line

connecting the upper point $-x_i$ with vertex $\frac{1}{2}\tau^{3}_{\alpha\beta}$

and the lower point x_i with vertex $\frac{1}{2}R_{3\mu}\tau^{\mu}_{\alpha\beta}$

(2) Connect all points with two propagator lines entering and leaving the point:

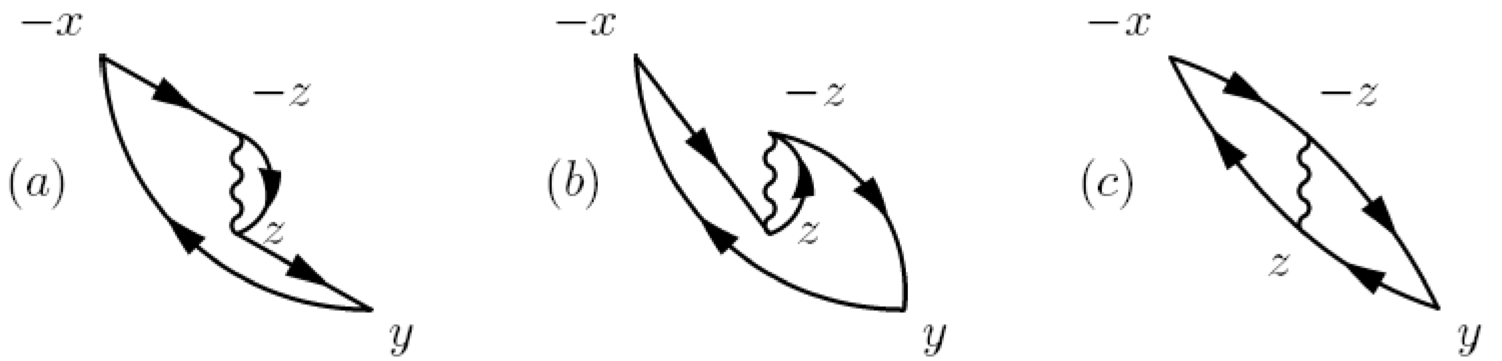
$$G_{\alpha\beta}(x,\omega_n) = -\delta_{\alpha\beta}\frac{i}{v_F} sign(\omega_n)\theta(\omega_n x)e^{-\omega_n x/v_F}$$

(3) In each fermion loop take trace of product of vertex matrices

(4) Take limit of external frequency $\Omega_m \to 0$

Computer algebra up to 3rd order in g_2 , ~10^4 diagrams

Conductance in 1 st order



Logarithmic correction: $$\int_a^L \frac{dz}{z} = \ln(\frac{L}{a}) \approx \ln(\frac{T_0}{T}), \quad L = v_F/T, \quad a = v_F/T_0$$

$$G^{(1)} = -\frac{g_2}{4\pi}\sin^2(2\theta)\ln(\frac{T_0}{T})$$

Agrees with *Yue, Matveev, and Glazman, 1995*

Corrections to conductance up to 3rd order

$$
\begin{aligned}
Y_r &= Y - c_{\{4\}} g \Lambda (1 - Y^2) \\
&+ g^2 Y (1 - Y^2)(-c_{\{6\}}(\Lambda^2 - a_4) + c_{\{4,2\}}(\Lambda - a_1)/2) \\
&+ g^3 (1 - Y^2)\Big[- c_{\{8\}}(\Lambda^3 - 3\Lambda a_4)(Y^2 - 1/3) \\
&+ c_{\{8\}}(1 - Y^2)\Lambda a_2 + c_{\{6,2\}} Y^2 \Lambda (\Lambda - a_1) \\
&- c_{\{4,4\}}(1 - Y^2)(\Lambda^2/4 - a_3 \Lambda) \\
&- c_{\{4,2,2\}}(1 + Y^2)\Lambda/4 \Big]
\end{aligned}
$$

$$\vec{a} = (2\ln 2, \pi^2/12, (\ln 2 - 1)/2, 0) \qquad c_{\{\cdot\}} = 1$$

$$
\begin{aligned}
\Lambda &= \ln(L/L_0) \ \text{(at zero temperature)} \\
&= \ln(T_0/T) \ \text{at finite temperature}
\end{aligned}
$$

RG equation in Callan-Symanzik scheme

Renormalized value via bare quantity

$$\begin{aligned} Y_r &= Y_0 + b_{11} g \Lambda \\ &+ b_{22} g^2 \Lambda^2 + b_{21} g^2 \Lambda \\ &+ b_{33} g^3 \Lambda^3 + b_{32} g^3 \Lambda^2 + b_{31} g^3 \Lambda + \ldots \end{aligned}$$

Inverting, we obtain the bare quantity via renormalized value

counterterms appear

$$\begin{aligned} Y_0 &= Y_r + \bar{b}_{11} g \Lambda \\ &+ \bar{b}_{22} g^2 \Lambda^2 + \bar{b}_{21} g^2 \Lambda \\ &+ \bar{b}_{33} g^3 \Lambda^3 + \bar{b}_{32} g^3 \Lambda^2 + \bar{b}_{31} g^3 \Lambda + \ldots \end{aligned}$$

$$0 = \frac{dY_0}{d\Lambda} = \frac{\partial Y_0}{\partial \Lambda} + \frac{\partial Y_0}{\partial Y_r} \frac{\partial Y_r}{\partial \Lambda}$$

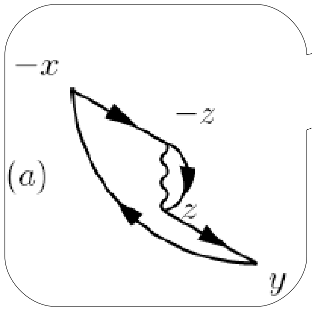
$$\beta = \frac{\partial Y_r}{\partial \Lambda} = -\frac{\partial Y_0 / \partial \Lambda}{\partial Y_0 / \partial Y_r}$$

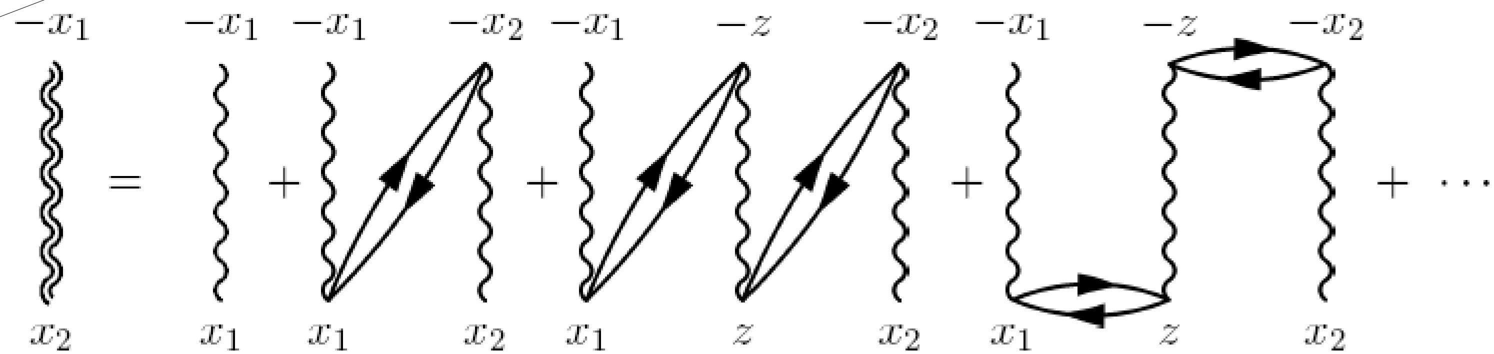
Perturbative Β-function

$$\beta(Y) = (1-Y^2)\left[-g + g^2\frac{Y}{2} - g^3\frac{Y^2+1}{4}\right] + c_3 g^3 (1-Y^2)^2 + O(g^4)$$

$$c_3 = a_2 + a_3 - a_4 = \frac{\pi^2}{12} - \frac{1-\ln 2}{2}$$

Dressing of the interaction line in 1st order correction





Ladder equation for B-function

L obeys the Wiener-Hopf integral equation

$$L(x;\omega) = e^{-\omega x}[-4\pi g - g\omega(Y + \frac{g}{2})\int_0^\infty dz e^{-\omega z} L(z;\omega)] + \frac{1}{2} g^2 \omega \int_0^\infty dz e^{-\omega|x-z|} L(z;\omega)$$

which is eventually solved, so that the RG equation now reads

$$-\frac{dY}{d\Lambda} = \frac{2g(1-Y^2)}{1+\sqrt{1-g^2}+gY} - c_3 g^3 (1-Y^2)^2$$

Kane & Fisher
Yue, Matveev, Glazman
1st loop RG

Beyond Born approximation
similar to CFT exact solution
different value c_3
3rd loop RG

Final formula for conductance

$$\begin{aligned}(T/T_0)^{2(1-K)} &= \Phi(G)/\Phi(G_0)\,,\\ \Phi(G) &= \frac{G^K}{1-G}\left(K+G(1-K)\right)^{4c_3(1-K)}\end{aligned}$$

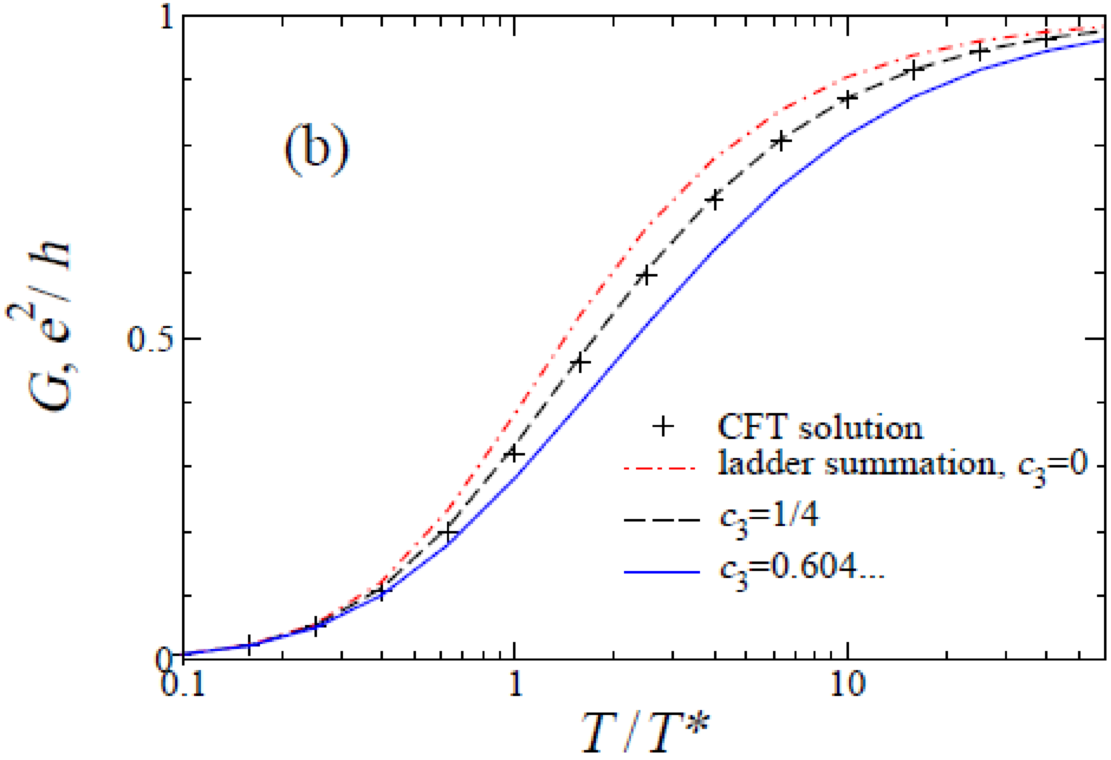


Illustration for K=1/2

CFT prediction multiplied by 2 = 1/K

Why difference with CFT ?

Transition from the Hamiltonian with backscattering potential
to the observable transmission amplitude requires **regularization of delta-function potential**

Exact solution of a boundary conformal field theory
Callan, Klebanov, Ludwig, Maldacena, Nucl. Phys. B422, 417 (1994).

CFT depends on short-scale regularization, when going beyond Born approximation even for the case of non-interacting fermions

$$G = f\left(u_{bs}(T_0/T)^{2(1-K)}, K\right)$$

$$\Phi\left(G, K\right) = u_{bs}(T_0/T)^{2(1-K)} = u_{bs}\exp(2(1-K)\Lambda), \quad \Rightarrow \qquad u_{bs} = \Phi\left(G, K=1\right)$$

$$\frac{d\Phi}{d\Lambda} = \frac{dG}{d\Lambda}\frac{d\Phi}{dG} = 2(1-K)u_{bs}\exp(2(1-K)\Lambda) = 2(1-K)\Phi$$

$$\frac{dG}{d\Lambda} = 2(1-K)\frac{\Phi}{d\Phi/dG} \equiv \beta(G, K)$$

Conclusions

- We re-visited the problem of impurity in the Luttinger liquid
- Current algebra formulation with non-local interaction
- Symbolic computation in **fermionic approach**: impurity renormalization by interaction - up to 3rd order – about 10,000 diagrams
- The ladder sequence identified and resummed, bosonization results reproduced
- A difference with CFT “complete solution” found at the level of 3rd loop RG, explained by different regularizations.
- The proposed fermionic approach is simpler in non-equilibrium (work in progress)

Fendley, Ludwig, Saleur, PRB 52, 8934 (1995)

problem, providing a sort of proof. The main source of difficulty is that the zero-temperature action must be handled with great care. Perturbation theory around the zero-temperature fixed point is ill behaved and depends on an infinite number of counterterms (see the discussion at the end of Ref. 31 in the case of the flow from tricritical Ising to Ising model), so identifying the leading term in the approach to this fixed point is not sufficient. This is equivalent to saying that what one calls the strong-barrier problem must actually be defined with great care.

The strong-barrier problem which is at the end of our renormalization-group trajectory follows formally from dimensional continuation of the integrals for the weak-barrier problem. It is not in any case a generic strong-barrier problem. For instance regularizing the integrals by putting a UV cutoff would give very different results, with a nonmonotonic conductance.[40] Also, note that the

Exact solution of a boundary conformal field theory
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$$\sin[\Delta(g, \bar{g})] = \tfrac{1}{2}(g+\bar{g})\frac{\sin(\pi\,|\,g\,|)}{|\,g\,|}. \tag{2.22}$$

This expression has a curious implication about the limit of infinite potential strength. We expect that when $|\,g\,| \to \infty$ the boundary state $|\,B\rangle$ turns into a sum over Dirichlet states with the field sitting at the minima of the potential (whose locations are in turn set by the phase of g). This means that, in the limit of infinite potential strength, Δ should approach the *phase* of the complex coupling g. According to (2.22), this happens at $|\,g\,| = \frac{1}{2}$ rather than at $|\,g\,| \to \infty$. This is just a finite renormalization effect: As we remarked earlier, our coupling is related to the usual one by a coupling constant redefinition of the form $g' = gf(g\bar{g})$ [6]. This can have the effect of mapping infinite coupling strength to $|\,g\,| = \frac{1}{2}$ if f has a singularity there.

[6] In fact, the regularization used in ref. [12] is the standard one and the coupling constant g' used there is related to ours by $g' = g(1 + \frac{1}{12}\pi^2 g\bar{g} + \dots)$.

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The coincident-point product of $e^{iX/\sqrt{2}}$ with $e^{-iX/\sqrt{2}}$ evaluates to a divergent constant which can be absorbed as a constant shift in the interaction potential. This has no effect on the physics and can be dropped. When we try to extend this sort of argument to higher order, we find another type of divergent commutator which can this time be absorbed as a finite renormalization of the coupling strength [5]. This procedure can be generalized to all orders but we will defer the details of the argument to Appendix A. There we will show explicitly how to

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[5] To be a bit more specific: With a standard short-distance cutoff ϵ, all divergences can be absorbed by choosing the "bare" coupling constant to be g/ϵ. The procedure we are outlining amounts to choosing the "bare" coupling to be a power series $g(1+c_1|g|^2+\ldots)/\epsilon$. This does not change the physics of the theory, but does change the precise meaning of the parameter g.