Instanton constituents in sigma models

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Brendel, FB, Janssen, Wipf, Wozar: PLB 676 (2009) 116, [0902.2328]

Harland: 0902.2303

the O(3) model in 2D:

$$S = \int d^2x \, \frac{1}{2}(\partial_\mu \phi^a)^2 \qquad \phi^a\phi^a = 1 \qquad (a = 1, 2, 3)$$

global O(3) symmetry $\leftrightharpoons$ $SU(2)$ Yang-Mills

later CP(N) models $\leftrightharpoons$ $SU(N \geq 3)$

condensed matter: from Heisenberg Hamiltonian $H = -J \sum_{\langle i,j \rangle} \vec{S}_i \vec{S}_j$

as a field theory similar to gauge theories:

- asymptotic freedom
- dynamical mass gap
- discretisation on a lattice: cooling ...
- instantons, constituents, fermionic zero modes

Topology

finite action:

$$\phi^a \stackrel{r\to\infty}{\longrightarrow} \text{const.}$$

$\Rightarrow \phi$ as a mapping: $\mathbb{R}^2 \cup \{\infty\} \sim S^2_x \longrightarrow S^2$ with winding number/degree

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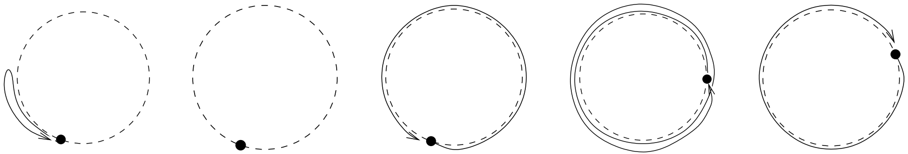
example $f:\; S^1_x \to S^1$, images:



$Q=0$ $\qquad$ $Q=0$ $\qquad$ $Q=1$ $\qquad$ $Q=2$ $\qquad$ $Q=-1$

= how often S^n is wrapped by S^n_x through ϕ, for S^2:

$$Q = \frac{1}{8\pi}\int d^2x\, \epsilon_{\mu\nu}\epsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c \in \mathbb{Z}$$

topological quantum number $\qquad$ $\leftrightharpoons$ instanton number

invariant under small deformations of ϕ

Bogomolnyi bound & Complex structure

- Bogomolnyi trick...

$$(\partial_\mu \phi^a \pm \epsilon_{\mu\nu}\epsilon_{abc}\phi^b\partial_\nu\phi^c)^2 = (\partial_\mu\phi^a)^2 \pm 2\epsilon_{\mu\nu}\epsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c + (\partial_\mu\phi^a)^2 \geq 0$$

...and bound (integrated):

$$S \geq 4\pi|Q|$$

equality iff: $\partial_\mu\phi^a = \mp\,\epsilon_{\mu\nu}\epsilon_{abc}\phi^b\partial_\nu\phi^c$ first order $\leftrightharpoons$ selfduality

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- complexification: $\leftrightharpoons$ quaternions in ADHM formalism

$$\begin{aligned} x_{1,2} &\rightarrow z = x_1 + ix_2 \\ \phi^a &\rightarrow u = \frac{\phi^1 + i\phi^2}{1-\phi^3} \end{aligned}$$

$$\begin{array}{llll} \mathrm{N}: & \phi^a = (0,0,1) & u = \infty \\ \mathrm{S}: & \phi^a = (0,0,-1) & u = 0 \end{array}$$

$\Rightarrow$ self-duality equations become Cauchy-Riemann conditions on u

Classical solutions

= any meromorphic function $u(z)$

topological charge: $Q =$ number of zeroes or poles

topological charge density: $q = \frac{1}{\pi}\frac{1}{(1+|u|^2)^2}|\partial_z u|^2$

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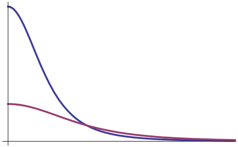
$$\left.\begin{array}{l} u(z) = \frac{\lambda}{z-z_0} \\[2em] u(z) = \frac{z-z_0}{\lambda} \end{array}\right\} \quad \begin{array}{l} q = \frac{1}{\pi}\frac{\lambda^2}{(|z-z_0|^2+\lambda^2)^2} \\[2em] \text{location } z_0 \text{ and size } \lambda \end{array}$$



1 pole and 1 zero to cover S^2_c, one of them at infinity

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1 pole and 1 zero to cover S_c^2, one of them at infinity

- $Q = 1$ instanton, pole and zero at finite z:

$$u(z) = \frac{z - z_I}{z - z_{II}}$$

constituents at $z = \{z_I, z_{II}\}$? $\rightsquigarrow$ 'instanton quarks'?!

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NO! same profile $q(x)$ as above $\Rightarrow$ one lump

conjecture: 2 complex moduli per $Q \rightsquigarrow$ locations of 2 constituents?!

Classical solutions

- $Q \geq 2$ instantons $\leftrightharpoons$ CFTW ansatz

$$u(z) = \prod_{j=1}^{Q} \frac{\lambda}{z - z_{0,j}} \qquad \text{poles at } z_{0,j}$$

or a fraction as for $Q = 1$

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finite temperature: $R^1 \times S^1_\beta$ with $\beta = 1/k_B T$

- periodic instantons $\quad\quad\quad \leftrightharpoons$ HS calorons

$$u(z) = \frac{\lambda}{\exp((z - z_0)\frac{2\pi}{\beta}) - 1} \qquad \text{poles at } z_0 + j \cdot i\beta$$

Twisted boundary conditions

top. and action density invariant under global $SO(3)$ rotations

$SO(2)$ subgroup rotating $\phi_{1,2} \Rightarrow U(1)$ on complex variable u:

$u(x) \to e^{2\pi i\omega} u(x)$

let u be periodic in Im z up to that symmetry FB '07

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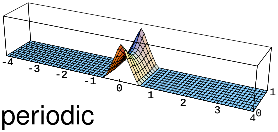
- 'generalisations of finite temperature'??
- gauge theory at finite T: gauged away by time-dep. gauge trafo: generates A_0 ... $\leftrightharpoons$ holonomy $\lim_{\vec{x}\to\infty} \mathcal{P}(\vec{x})$
- spatial $R^1 \times S^1 \simeq$ 'tube': defect, when glued together or flux through the tube ...

- $SO(3)$-inv. quantities like action density still periodic
- in which observable can we measure the twist ω? Diakonov

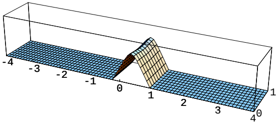
$\Rightarrow$ topological profiles ($\ln q$) for charge $Q = 1$: FB '07

$\lambda = \beta$



periodic

$\lambda = 10\beta$



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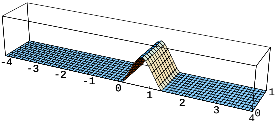

copies overlap $\sim$ dimensional reduction

Instanton constituents

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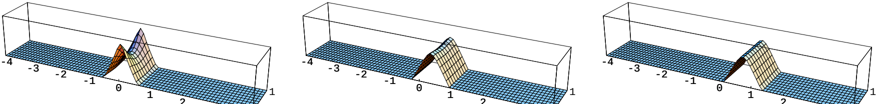
$\lambda = \beta$ | $\lambda = 10\beta$ | $\lambda = 100\beta$



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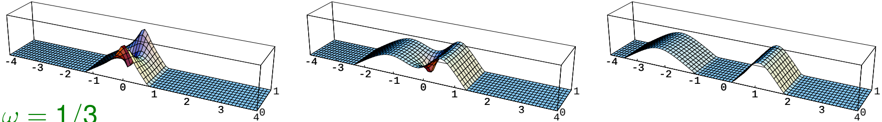
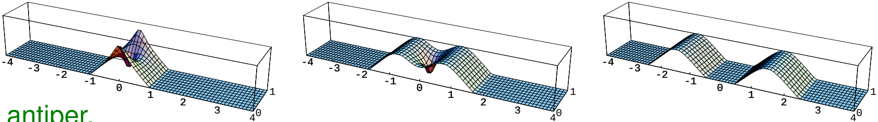


$\omega = 1/3$



antiper.

novel solution:

$$u(z) = \frac{e^{\omega(z-z_0)\frac{2\pi}{\beta}} \cdot \lambda}{\exp((z-z_0)\frac{2\pi}{\beta}) - 1}$$

at $z_0 + j \cdot i\beta$: residues $e^{2\pi i\omega \cdot j}$ = rotation $\leftrightharpoons$ KvBLL calorons

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rewrite:

$$u(z) = \frac{1}{\exp(-\omega(z-z_1)\frac{2\pi}{\beta}) - \exp((1-\omega)(z-z_2)\frac{2\pi}{\beta})}$$

with $z_1 = z_0 - \beta\,\frac{\ln\lambda}{2\pi\omega}, \qquad z_2 = z_0 + \beta\,\frac{\ln\lambda}{2\pi(1-\omega)}$

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= combination of two constituents $\leftrightharpoons$ dyons

$$u(z) = \exp\big(\omega(z-z_1)\frac{2\pi}{\beta}\big), \qquad u(z) = \exp\big(-(1-\omega)(z-z_2)\frac{2\pi}{\beta}\big)$$

transmutation of λ: instanton size $\to$ constituent distance $\leftrightharpoons$ same

Properties

- simply exponential functions

  (as such periodic in Im z up to a phase)
- fractional top. charges: $Q = \omega$ and $Q = 1 - \omega$ $\quad\leftrightharpoons$ 2ω and $1 - 2\omega$
- exp. localised in Re z, scale = β $\quad\leftrightharpoons$ decay algebraically
- static in isolation
- overlapping: one lump with $Q = 1$ and 'time-dependence'
- higher charge: more constituents

  different kinds alternate in noncompact direction (see below)
- carry the fermionic zero modes (see below)

$\leftrightharpoons$ everything very similar

CP(N) models

a constraint $(n+1)$-dim. complex vector in 2D: $\leftrightharpoons$ SU(N+1) YM

$$S = \int d^2x \left|D_\mu \vec{n}\right|^2 \qquad \left|\vec{n}\right|^2 = 1$$

$$D_\mu = \partial_\mu - iA_\mu \quad A_\mu \equiv -i\vec{n}^* \partial_\mu \vec{n} : \text{dependent gauge field}$$

note: $CP(1) \sim O(3)$ $\leftrightharpoons$ SU(2) YM

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again:

- Bogomolnyi bound and complexification
- Cauchy-Riemann condition
- instantons as meromorphic functions, use

$$\vec{n} = \frac{\vec{v}(z)}{\left|\vec{v}(z)\right|}$$

as basic variable, then: $q = \frac{1}{4\pi}\Delta \ln \left|\vec{v}(z)\right|^2$

Instanton constituents

twisted bc.s in global symmetry $U(n+1)$: FB et al. '09

$$v^j(z+i\beta) = e^{2\pi i\mu_j} v^j(z)$$

$n+1$ constituents 'instanton quarks'

with topological charges: $\mu_j - \mu_{j-1}$ $\leftrightharpoons$ dyon masses in SU(N)

$[\mu_0 \equiv 0, \mu_{n+1} \equiv 1$, add up to 1]

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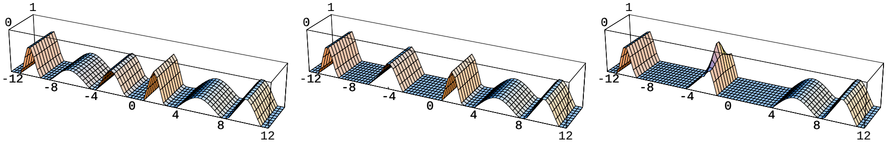
- higher charge: $Q \cdot (n+1)$ constituents

some $Q=2$ solutions in CP(2): $\leftrightharpoons$ $Q=2$ in SU(3)

$\mu_j - \mu_{j-1} \in \{0.55, 0.15, 0.3\}$



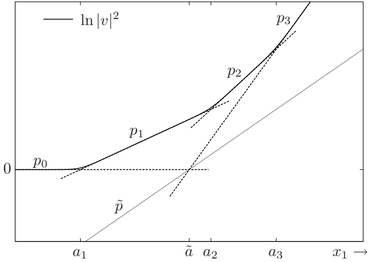
- ordered along noncompact direction
- time-dependent, when $n+1$ (or more) constituents merge

Explanation

- the $v(z)$'s in these solutions are again exponentials:

$$v_j(z) = e^{2\pi\mu_j z} \times (\ldots + \ldots e^{2\pi z} + \ldots e^{2\pi(2z)} + \ldots)$$

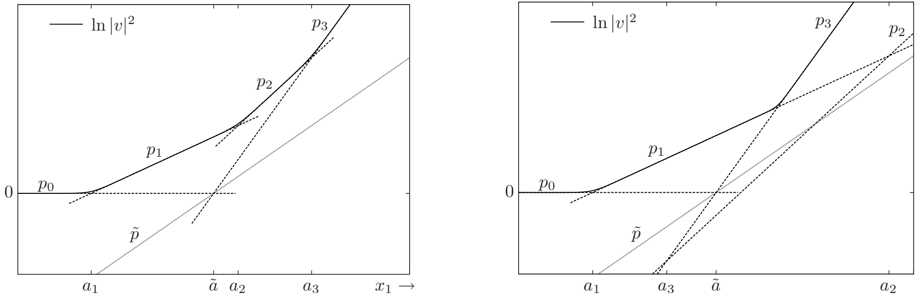
- $v_j \sim \exp(..\,\mathrm{Re}\, z)$, in top. density $q \sim \Delta \ln|\vec{v}|$:
  generically one exponential term dominates q
  which one depends on the Re z-region and the parameters



$\Rightarrow \ln|\vec{v}|$ piecewise linear, q piecewise 0
up to corrections at transitions = constituent locations

- constituents top. fraction $\sim$ difference of slopes (cf. ∂_x^2 at cusps)
  total Q: from the asymptotic slopes (here $p_0,\ p_3$) ✓

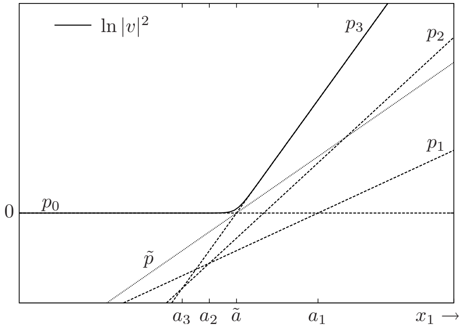
- constituents join to a bigger one:



Q from sum of slopes ✓

constituents cannot go through each other <geometric argument>

- $n+1$ constituents join: all their p's 'low'
  the only time-dependent slope $\tilde{p}$ becomes relevant



$\Rightarrow$ time-dependent lump

$\leftrightharpoons$ data in Nahm transformation

Fermionic zero modes

- minimal coupling via the gauge field

$$S = \int d^2x \Big[\big| D_\mu \vec{n} \big|^2 + i\bar{\psi}\gamma^\mu D_\mu \psi \Big]$$

expect index theorem: $\leftrightharpoons$ Atiyah-Singer index theorem

$$Q = n_R - n_L$$

for right-handed and left-handed zero modes of $i\gamma^\mu D_\mu$

instantons with $Q > 0$:

$$n_L = 0, \quad n_R = Q$$

- phase bc.s for fermions:

$$\psi(z+i\beta) = e^{2\pi i\zeta}\psi(z)$$

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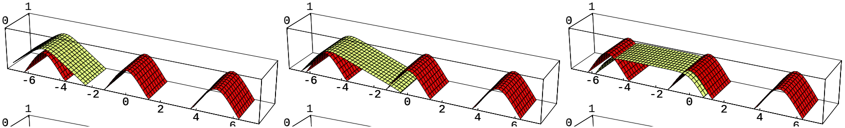
we find explicitly

$$\begin{aligned} n_L=0,\quad n_R &= Q \\ \text{unless } \zeta=0: \; n_R &= Q-1 \end{aligned}$$

$\zeta\in\{0.17, 0.25, 0.33\}$:



= exp. localized to constituents depending on phase ζ: hopping $\leftrightharpoons \checkmark$

and 'bridges' $\leftrightharpoons$ alg. localised

$\zeta=0$: 'bridge' to infinity = non-normalisable $\leftrightharpoons$ normalisable chiral representation:

$$\psi = \left(\begin{array}{c} 0 \\ \frac{1}{|v|}\, e^{2\pi(s+\zeta)z} \end{array} \right) \qquad \text{with } 0 < s + \zeta < Q$$

again considerations about these exponential terms and the old ones from v (bosonic) explain the localisation behaviour

'bridges': two slopes parallel

similar on cooled lattice configurations

Summary

CP(N) on $R^1 \times S^1$:

instantons consist of N constituents (per charge Q)

when twisted boundary conditions

= part of global symmetry

YM on $R^3 \times S^1$: calorons

= part of local symmetry $\Rightarrow$ holonomy

fermions hop with their phase boundary conditions

realisation in condensed matter?!

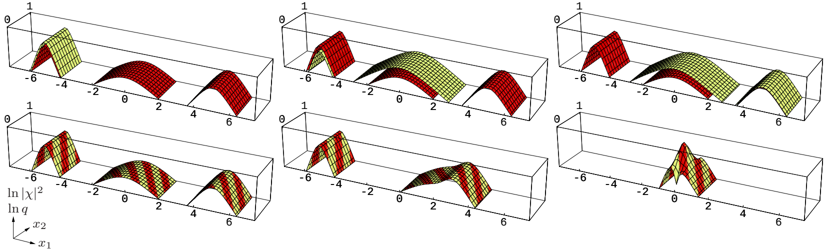
outlook: semiclassics, phase diagram, large N, ...

Supersymmetric coupling to fermions

linearised Dirac equation: $n_L = 0$

right-handed zero modes: exponential again

SUSY conservation restricts phases



localised on two (or one) consituent

'half-BPS state': profile = $q(x)$