Topological order

from quantum loops and nets

Paul Fendley

Experimental and theoretical successes have made us take a close look at quantum physics in two spatial dimensions.

We have now found (mostly theoretically, but also experimentally) behavior unseen in 3+1 dimensions. The mathematics describing the physics also can be quite different.

One potential application of this novel physics is a topological quantum computer.

This idea relies on one particularly unusual property possible only in two dimensions: non-abelian statistics.

There are several possibilities for realizing non-abelian statistics experimentally:

- a two-dimensional electron gas in a large magnetic field (the fractional quantum Hall effect)
- surface of topological insulators
- frustrated magnets

Outline:

1. What are non-abelian statistics?

2. What is topological order?

3. How can one realize this in a spin model?

paper: arXiv:0804.0625

Essential ingredients:

Coupled Potts models: with J. Jacobsen

The Temperley-Lieb algebra and the chromatic polynomial: with V. Krushkal

Quantum Potts nets: with E. Fradkin

The Potts model and the BMW algebra: with N. Read

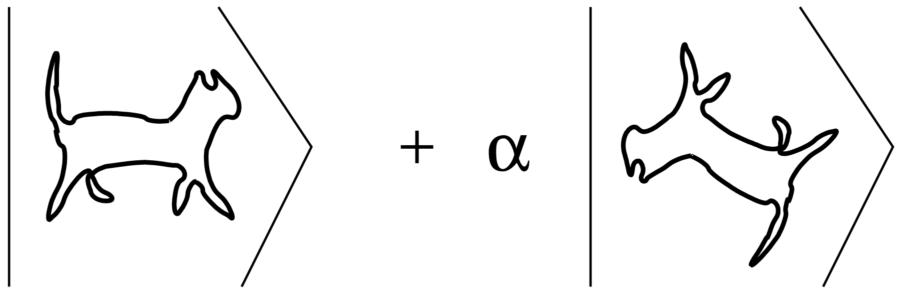
If a large enough quantum computer were built, numbers could be prime-factored in polynomial time.

But with every silver lining.... . .

Errors!

In a classical computer, just be redundant.

In a quantum computer, errors are continuous, not just bit flips. Moreover, measurement destroys the superposition. How do we know errors have occurred?!? Need to preserve



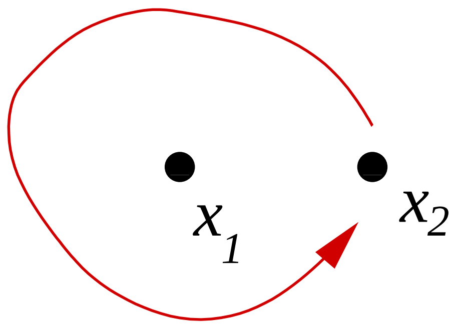
It’s non-trivial to show that error correction is even possible for quantum computers.

The idea behind a topological quantum computer to exploit systems with quantum redundancy.

In three dimensions, the only statistics for identical particles are bosonic and fermionic:

$$\psi(x_1, x_2, \dots) = \pm\psi(x_2, x_1, \dots)$$

Exchanging twice is a closed loop in configuration space.



In three dimensions, this can be adiabatically deformed to the identity.

In two dimensions, it cannot!

Identical particles can pick up an arbitrary phase:

$$\psi(x_1, x_2) = e^{i\alpha}\psi(x_2, x_1)$$

Particles with $\alpha \neq 0, \pi$ are called anyons.

They have been observed in the fractional quantum Hall effect
(these have fractional charge as well).

In 2+1 dimensions, the statistics comes from behavior under braiding. Each particle traces out a worldline in time, and these braid around each other.

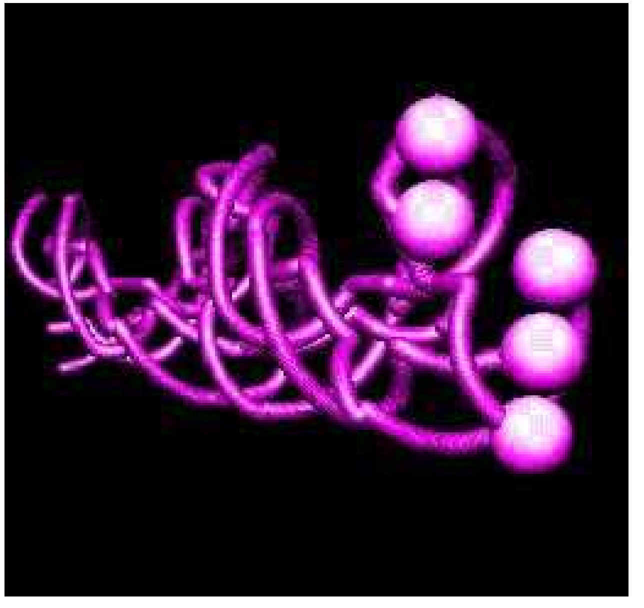


figure from Skjeltorp et al

Braiding is a purely topological property.

Even more spectacular phenomena can occur under braiding if the Hilbert space has degeneracies.

Particles can change quantum numbers!

$$\psi_a(x_1, x_2 \ldots) = \sum B_{ab} \psi_b(x_2, x_1 \ldots)$$

When the matrix B is diagonal with entries ± 1, particles are bosons and fermions.

When B is diagonal with entries $e^{i\alpha}$, particles are anyons.

When B is not diagonal, can get non-abelian statistics.

With non-abelian statistics, the probability amplitude changes depending on the order in which the particles are braided.

Braiding particles with non-abelian statistics entangles them!

This is the fundamental idea behind topological quantum computation.

Kitaev; excellent review in Preskill's lecture notes

We need to understand fusion: what are the statistics of a pair of particles?

This is non-trivial even for abelian anyons. Say we have two identical particles which pick up a phase $e^{i\alpha}$ when exchanged, or equivalently, $e^{2i\alpha}$ under a 2π rotation of one around the other. Then a pair of these picks a phase $e^{8i\alpha}$ when exchanged with different pair.

In the non-abelian case, fusing two particles can give a linear combination of states, analogous to

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \, + \, 1$$

in $SU(2)$.

The simplest kind of non-abelian statistics is in the braiding of Fibonacci anyons.

Fusing two Fibonacci anyons gives a linear combination of a single Fibonacci anyon (denoted ϕ) and a state with trivial statistics (denoted 1). They satisfy the fusion rule

$$\phi \times \phi \sim 1 \,+\, \phi$$

This is like the tensor product of two spin-1 representations if we throw out the spin-2 piece.

There are thus two possible quantum states for two Fibonacci anyons. One has trivial statistics, the other has the braiding properties of a single Fibonacci anyon.

This fusion algebra allows us to count the number of quantum states for $2N$ quasiparticles:

$$\begin{aligned}
1 &: \quad I \\
1 &: \quad \phi \\
2 &: \quad \phi \times \phi = I + \phi \\
3 &: \quad \phi \times \phi \times \phi = I + \phi + \phi \\
5 &: \quad \phi \times \phi \times \phi \times \phi = \phi + 2(I + \phi) \\
8 &: \quad \phi \times \phi \times \phi \times \phi \times \phi = 3I + 5\phi
\end{aligned}$$

The number of states for n particles is the nth Fibonacci number, which grows as $((1+\sqrt{5})/2)^n$. In general, when the dimension of the n-anyon Hilbert space grows as d^n, d is called the quantum dimension of the anyon.

To make a qubit requires 4 Fibonacci anyons. There are two different ways of fusing these four into an overall identity channel – the two states of a qubit!

Namely, when the overall channel is I, when the combination of anyons 1 and 2 fuses to state ϕ, then anyons 3 and 4 must also be in the state ϕ. If anyons 1 and 2 fuse to I, then anyons 3 and 4 must also fuse to I.

The system can be in any linear combination of these two states. We can change the state, by braiding particle 2 around particle 3.

This entanglement is possible even if none of the particles are near each other!

This is why non-abelian statistics allow a fault-tolerant quantum computer.

The braiding of particle 2 with particle 3 allows us to do quantum computation. The result of this braiding depends only on the topological properties of the system: these particles can be very far from each other.

No local perturbation of these quasiparticles will change the results of the braiding.

One can find the braiding and fusing matrices from standard conformal field theory techniques.

For example, for four Fibonacci anyons in overall channel I,

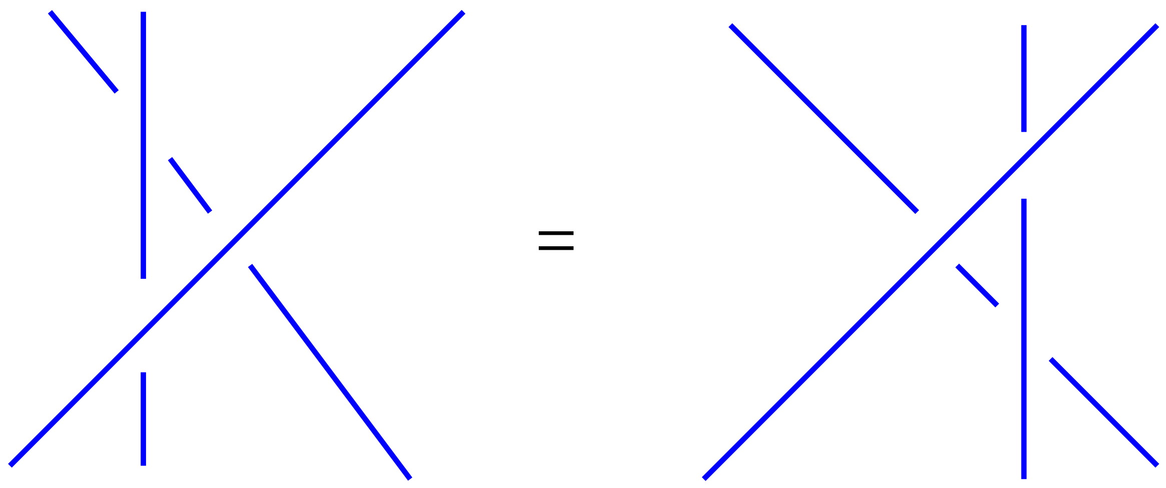
$$B = \frac{1}{\tau} \begin{pmatrix} e^{4\pi i/5} & -e^{2\pi i/5}\sqrt{\tau} \\ -e^{2\pi i/5}\sqrt{\tau} & -1 \end{pmatrix}$$

where $\tau = 2\cos(\pi/5) = (1+\sqrt{5})/2$ is the golden mean.

However, a much more intuitive and instructive way to study non-abelian braiding in depth is to draw pictures!

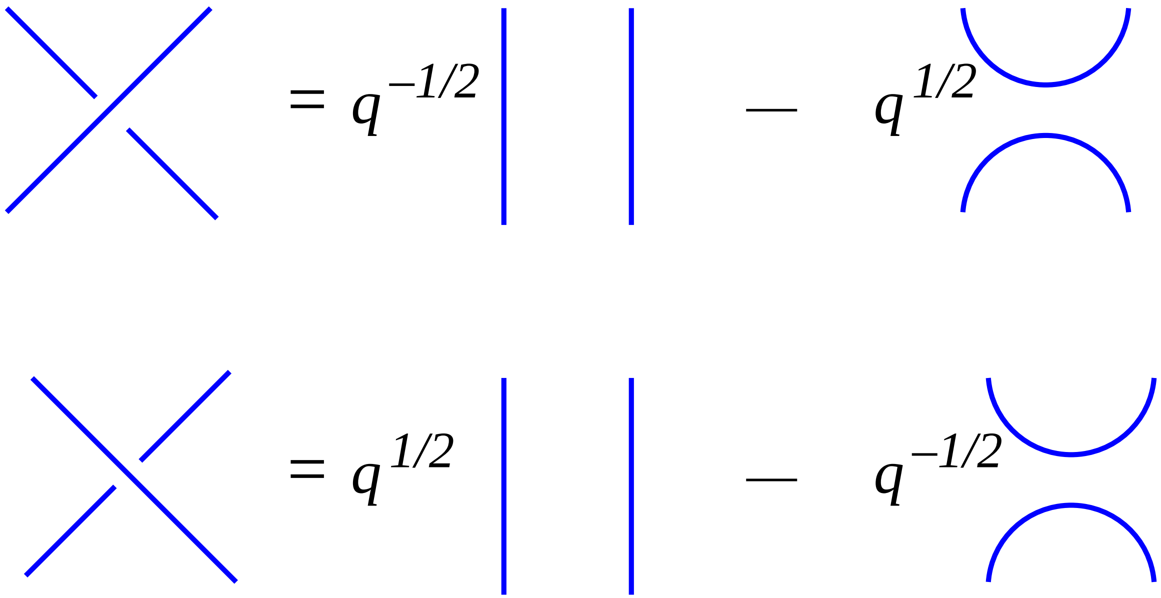
We project the world lines of the particles onto the plane. Then the braids become overcrossings and undercrossings

The braids must satisfy the consistency condition



which in closely related contexts is called the Yang-Baxter equation.

A simple way of satisfying the consistency conditions leads to the Jones polynomial in knot theory. Replace the braid with the linear combination



so that the lines no longer cross. q is a parameter which is a root of unity in the cases of interest: the Fibonacci case corresponds to $q = e^{i\pi/5}$.

This gives a representation of the braid group if the resulting loops satisfy d-isotopy.

- isotopy: Configurations related by deforming without making any lines cross receive the same weight.

- d: A configuration with a closed loop receives weight

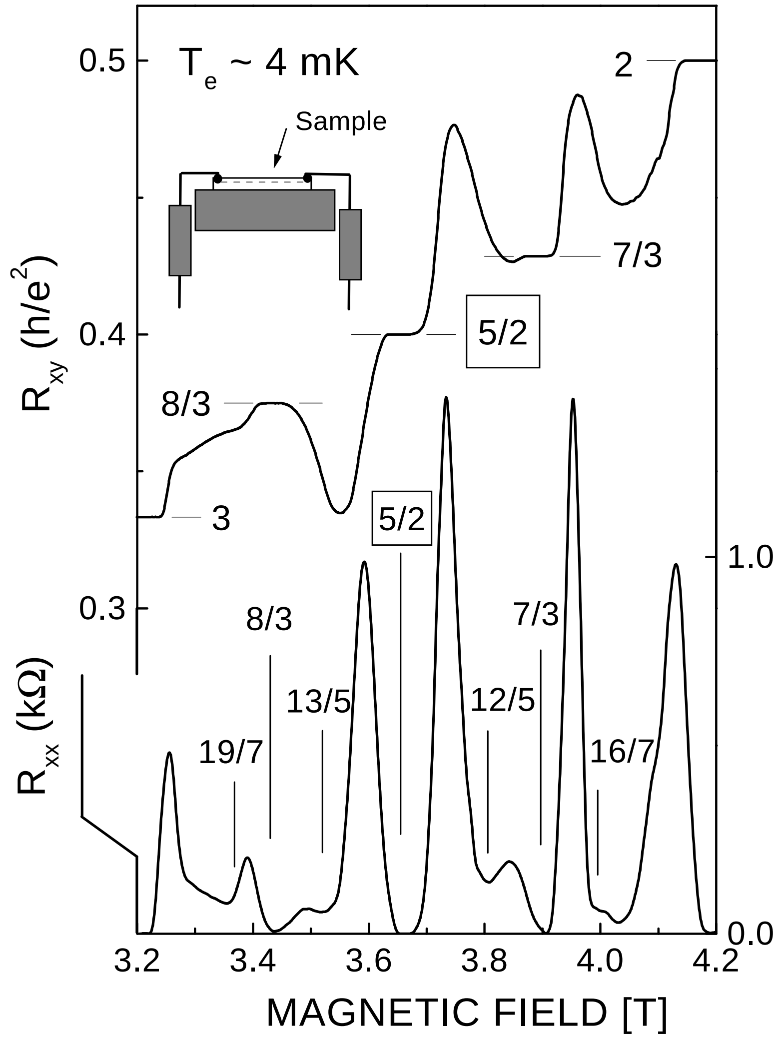
$$d = q + q^{-1}$$

relative to the configuration without the loop.

The problem: find a quantum Hamiltonian acting on a two-dimensional Hilbert space which has quasiparticles with the above properties.

One answer:

Fig. 1, Pan et al



The leading candidate wavefunction for the $\nu = 5/2$ plateau is the Moore-Read Pfaffian, which contains non-abelian anyons with $d = \sqrt{2}$.

A closely-related theory describes the superconductor with $p_x + ip_y$ pairing, believed to be realized in strontium ruthenate. The vortices here are non-abelian anyons.

The task is now to find a lattice model whose quasiparticles have such braiding. In other words, we want a lattice model whose effective field-theory description is (doubled) Chern-Simons theory.

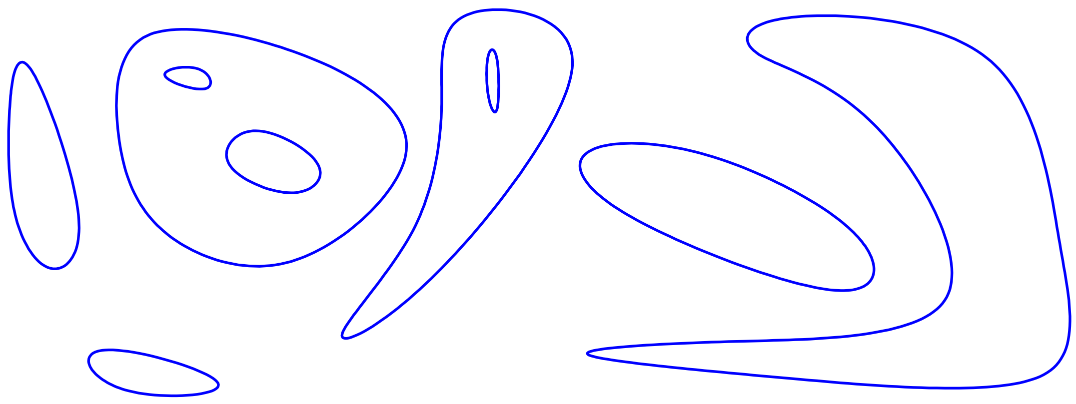
The clever idea of the the quantum loop model is to use these pictures to build the model.

To find a model:

1. find a 2d classical loop model which has a critical point
2. use each loop configuration as a basis element of the quantum Hilbert space
3. find a Hamiltonian whose ground state a sum over loop configurations with the appropriate weighting, so that
4. if you "cut" a loop, you end up with two deconfined anyonic excitations

Kitaev; Moessner and Sondhi; Freedman

In quantum loop models, each loop in the ground state gets a weight d ($= (1+\sqrt{5})/2$ for Fibonacci)

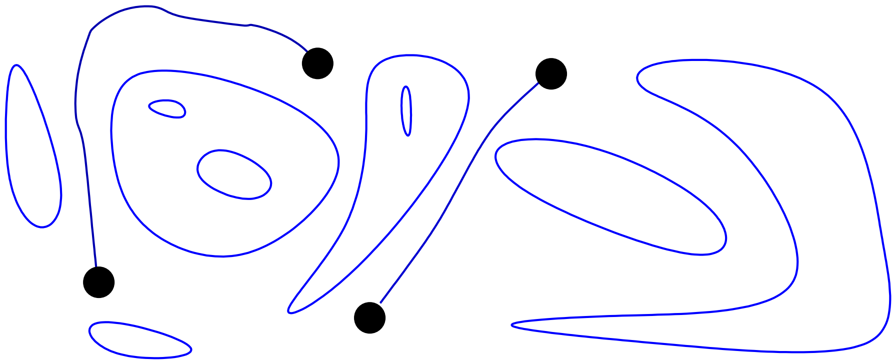


i.e. the ground state Ψ is the sum over all loop configurations

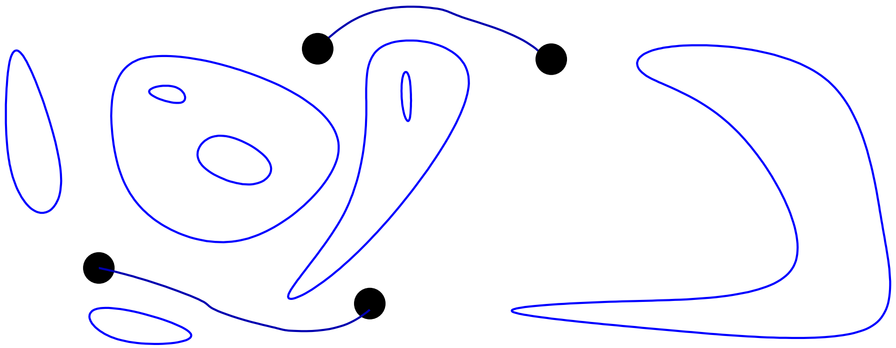
$$|\Psi\rangle = \sum_{\mathcal{L}} d^{n_{\mathcal{L}}} |\mathcal{L}\rangle$$

where $n_{\mathcal{L}}$ is the number of loops in configuration $\mathcal{L}$.

The excitations with non-abelian braiding are defects in the sea of loops.



After braiding, the four quasiparticles can be attached in the other way!



To have non-abelian braiding, the quantum loop models need to be gapped and have topological order.

Topological order is intermediate between order and disorder. It means that e.g. the number of ground states will depend on the genus of the surface on which it's defined.

To get topological order in a quantum loop model, loops of all sizes must appear in the ground state. Otherwise the anyons are confined.

This is easier said than done. To avoid the various complications, we need to:

- Allow the loops to branch, so that they are not really loops, but rather nets.

- Change the inner product in the quantum-mechanical model.

  It turns out that the two are essentially the same!

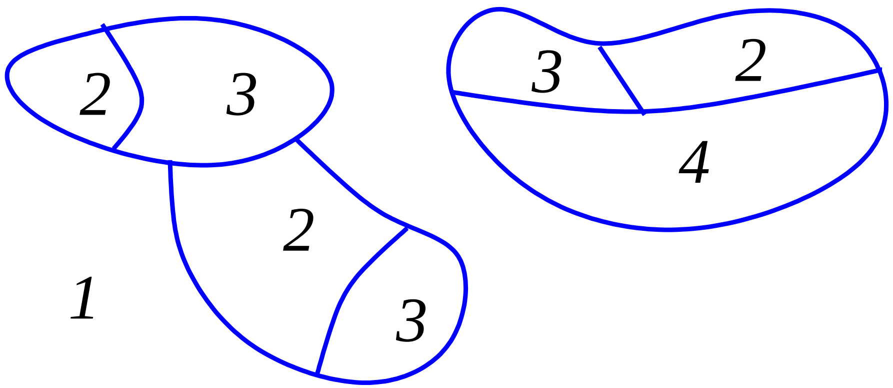
Quantum self-duality allows the construction of a simple(r) Hamiltonian annihilating a topologically-ordered ground state consisting of a sum over nets.

For the square lattice, it requires only four-spin interactions.

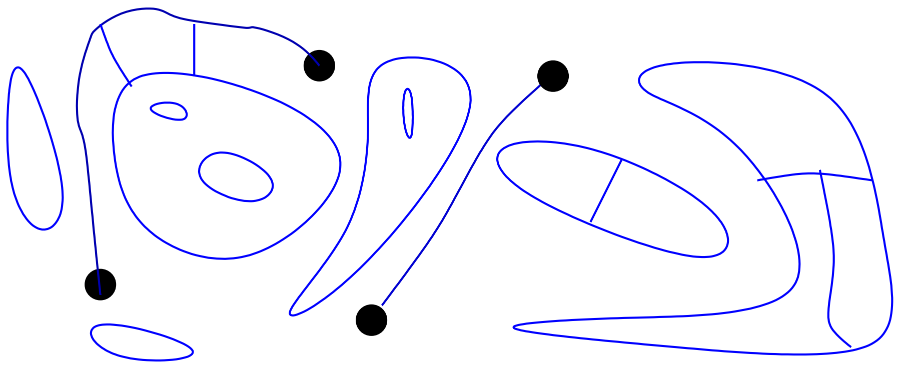
The quantum Potts model uses the 2d classical Potts model for its underlying degrees of freedom.

The domain wall on each link forms a two-state quantum system, i.e. the presence of the domain wall is one state $|1\rangle$, while the absence is the orthogonal state $|0\rangle$.

Each domain-wall configuration in the ground state is a net: by construction, it has no ends.



Configurations with “net ends” are by construction anyonic excited states.



The weight of each net $|N\rangle$ in the ground state $|\Psi\rangle$ of the quantum model is found from the classical model:

$$\langle N|\Psi\rangle = (Q-1)^{-L_N}\chi_{\widehat{N}}(Q)$$

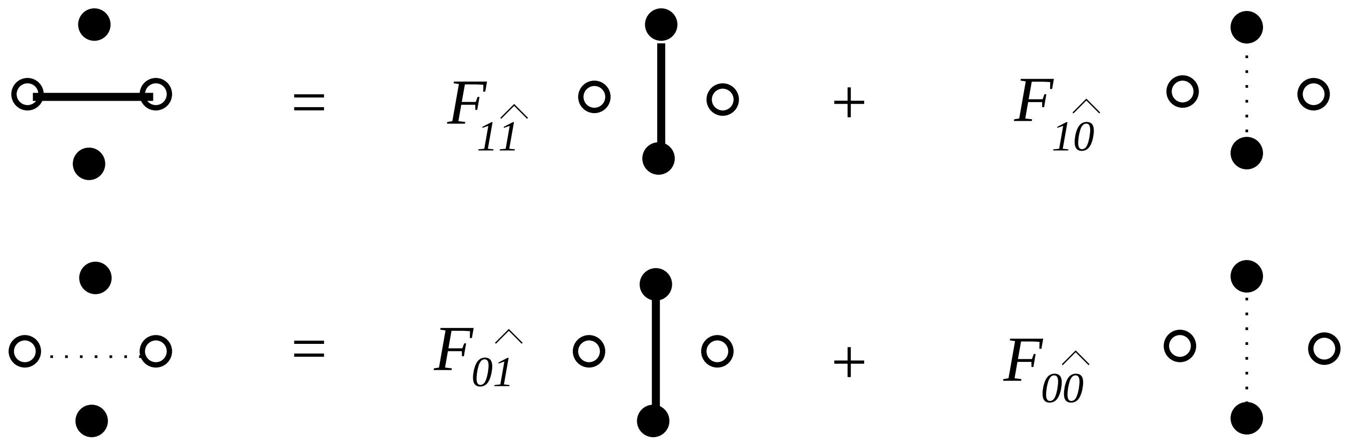
Fendley and Fradkin

$\chi_{\widehat{N}}(Q)$ is the number of spin configurations allowed for each domain-wall configuration N. $\chi_{\widehat{N}}$(Q) is called the chromatic polynomial, and depends only on the topology of N. Its definition can be extended to all Q, not just integers.

L_N is the “length” of each net, the number of $|1\rangle$ states. This factor is put there to ensure that...

The quantum Potts model is quantum self-dual.

This means we can equivalently define the model in terms of dual domain walls, $|\hat{0}\rangle, |\hat{1}\rangle$.



$$F = \begin{pmatrix} \langle\hat{0}|0\rangle & \langle\hat{0}|1\rangle \\ \langle\hat{1}|0\rangle & \langle\hat{1}|1\rangle \end{pmatrix} = \frac{1}{\sqrt{Q}} \begin{pmatrix} 1 & \sqrt{Q-1} \\ \sqrt{Q-1} & -1 \end{pmatrix}$$

The weight of each dual net $|D\rangle$ in the ground state is

$$\langle D|\Psi\rangle = \left(\frac{1}{\sqrt{Q-1}}\right)^{L_D} \chi_{\widehat{D}}(Q)$$

This is the same ground state $|\Psi\rangle$ in a new basis!

This quantum self-duality is highly non-obvious, and extremely useful.

A Hamiltonian H with $|\Psi\rangle$ a ground state can be found simply by demanding that H annihilate all states which are not nets and annihilate all states which are not dual nets.

For the square lattice:

$$
\begin{aligned}
H &= \sum_{+} [P_1 P_0 P_0 P_0 \; + \; \text{rotations}] \\
&\quad + \sum_{\square} [P_{\widehat{1}} P_{\widehat{0}} P_{\widehat{0}} P_{\widehat{0}} \; + \; \text{rotations}]
\end{aligned}
$$

where P_i projects onto the states $|i\rangle$, and $P_{\widehat{i}} = F P_i F$.

Conclusions

- Quantum nets exhibit topological order, and so can have non-abelian anyons.
- Thinking about topology leads to a natural extension of self-duality to two-dimensional quantum systems.
- This in turn leads to a much simpler Hamiltonian annihilating topologically-ordered ground states.