Statistical Mechanics and Extreme Value Statistics in models with logarithmic correlations

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Problem:

- Given an instance of the $2D$ **Gaussian free field**:

$$\mathcal{P}\left[V(\mathbf{x})\right] \propto \exp -\frac{1}{g^2}\int \left[\nabla V(\mathbf{x})\right]^2 \, d^2\mathbf{x}$$

characterized by the covariance

$$\langle V(\mathbf{x_1})V(\mathbf{x_2})\rangle = -2g^2 \ln |\mathbf{x_1} - \mathbf{x_2}|$$

we wish to understand the statistics of its **minima/maxima** along various curves in the plane, and ultimately in various planar domains.

- The problem turns out to be intimately connected to the mechanism of **freezing transitions** in disordered systems theory (Random Energy Models, Dirac fermions in random magnetic field). It has also interesting relations to **Liouville Quantum Gravity** $\&$ conformal field theory, to **multifractal** random measures, $1/f$ **noises**, and processes arising in turbulence and mathematical finance, as well as to various aspects of **Random Matrix Theory**.

Idea of the method: We concentrate on considering samples of the Gaussian Free Field (**GFF**) along **planar curves** $\mathcal{C}$ parametrised by $\mathbf{x}(t) = (x(t), y(t))$ with real $t \in [a, b]$.
Given a measure $d\mu_\rho(t) = \rho(t)\, dt$, we consider the integral

$$Z_\beta = \epsilon^{\beta^2 g^2} \int_a^b e^{-\beta V_\epsilon(\mathbf{x}(t))} d\mu_\rho(t), \quad \beta > 0$$

where $V_\epsilon(\mathbf{x})$ is the regularized version of the **GFF** with a short scale cutoff $\epsilon \ll 1$, i.e. zero mean and the covariance

$$\langle V_\epsilon(\mathbf{x}) V_\epsilon(\mathbf{x}') \rangle = -2g^2 \ln |\mathbf{x} - \mathbf{x}'|_\epsilon = \begin{cases} -2g^2 \ln |\mathbf{x} - \mathbf{x}'|, & |\mathbf{x} - \mathbf{x}'| > \epsilon \\ 2g^2 \ln(1/\epsilon), & |\mathbf{x} - \mathbf{x}'| < \epsilon \end{cases}$$

The integral is to be interpreted as the **partition function** of the associated **Random Energy Model** at the temperature $T = \beta^{-1}$. This is to be studied in the limit $\epsilon \to 0$.

Note:

- For $\rho(t) = 1$ the **Gibbs measure** of the disordered system identifies with the random **Liouville measure** (cf. **Duplantier** $\&$ **Sheffield** 2008), and the partition function Z_β can be interpreted as the (fluctuating) length of a curve in critical Liouville quantum gravity.

For example, the generating function (Laplace transform of the partition function probability density) is given by:

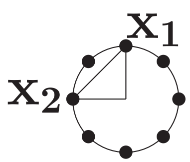
$$\left\langle e^{-pZ_\beta}\right\rangle \propto \int DV e^{-\frac{1}{g^2}\int[\nabla V(\mathbf{x})]^2\,d^2\mathbf{x}-p\int_{\mathcal{C}} e^{-\beta V(\mathbf{x}(t))}dl}$$

which resembles the Liouville partition function, see e.g. **Fateev, Zamolodchikov & Zamolodchikov** .

Guiding example: CIRCULAR LOGARITHMIC MODEL:

Let the contour $\mathcal{C}$ be the unit circle: $x(t) = \cos t, y(t) = \sin t$, with $t \in [0, 2\pi)$. Sample the Gaussian Free Field at M equidistant points along the circle with $t_k = \frac{2\pi}{M}(k-1),\ k = 1, \ldots, M$

As the distance $|\mathbf{x}_1 - \mathbf{x}_2|$ between a pair of points is simply $2|\sin\frac{t_1-t_2}{2}|$, we deal with the collection of normally distributed variables with covariances



$$\langle V_k V_m \rangle = -2g^2 \ln |2 \sin \frac{2\pi}{M}(k-m)|, \quad \text{for} \quad k \neq m$$

We have to choose the variance accordingly:

$$\langle V_k^2 \rangle = 2g^2 \ln M + W, \quad \text{with any} \quad W > 0$$

Equivalently, we consider $2\pi-$ periodic Gaussian $\frac{1}{f}$ **noise**:

$$V(t) = \sum_{n=1}^{\infty} (v_n e^{int} + \overline{v}_n e^{-int}) \quad \text{with i.i.d. coefficients } \langle v_n \overline{v}_k \rangle = \frac{g^2}{n} \delta_{n,k}$$

Observation: The positive integer moments $\langle Z^n(\beta)\rangle$, $n = 1, 2, \ldots$ of the partition function $Z(\beta) = \sum_{i=1}^{M} e^{-\beta V_i}$ for the circular model in the high-temperature phase $\gamma = \beta^2 g^2 < 1$ turn out to be given in the thermodynamic limit $M \gg 1$ by

$$\langle Z^n_{circ}(\beta)\rangle = \begin{cases} M^{1+\gamma n^2} O(1) & n > 1/\gamma \\ M^{(1+\gamma)n} D_n(\gamma) & n < 1/\gamma \end{cases}$$

where $D_n(\gamma)$ is the **Dyson-Morris** Integral

$$D_n(\gamma) = \frac{1}{(2\pi)^n} \int_0^{2\pi} d\theta_1 \ldots \int_0^{2\pi} d\theta_n \prod_{a<b} |e^{i\theta_a} - e^{i\theta_b}|^{-2\gamma} = \frac{\Gamma(1-n\gamma)}{\Gamma^n(1-\gamma)}$$

Aim: to reconstruct the distribution of the partition function $P(Z)$ from its moments in the high temperature phase $\gamma \leq 1$.

Outcome of the analysis:

The probability density $\mathcal{P}(Z)$ of the partition function $Z_{circ}(\beta) \equiv Z$ in the high-temperature phase $\gamma = \beta^2 g^2 < 1$ consists of two pieces. The **"body"** of the distribution is given by:

$$\mathcal{P}(Z) = \frac{1}{\gamma}\frac{1}{Z}\left(\frac{Z_e}{Z}\right)^{\frac{1}{\gamma}} e^{-\left(\frac{Z_e}{Z}\right)^{\frac{1}{\gamma}}}, \quad Z \ll M^2$$

which has a pronounced maximum at $Z \sim Z_e = \frac{M^{1+\gamma}}{\Gamma(1-\gamma)} \ll M^2$, and the powerlaw decay at $Z_e \ll Z \ll M^2$.

At $Z \gg M^2$ the above expression is replaced by the **lognormal tail**:

$$\mathcal{P}(Z) = \frac{M}{\sqrt{4\pi\gamma \ln M}}\frac{1}{Z} f\left(\frac{1}{2}\frac{\ln Z}{\ln M}\right) e^{-\frac{1}{4\ln M\gamma}\ln^2 Z} \quad \text{where } f(x) \sim O(1) \text{ for } x \sim O(1)$$

Define $z = \Gamma(1-\gamma)Z = e^{-\beta f}$, the free energy $f = -\beta^{-1}\ln z$ is distributed according to the **Gumbel law**: $\mathcal{P}(f) = Ae^{Af - e^{Af}}$ where $A = T/T_c^2$ for $T \geq T_c = g$. From now on we put $g = 1$.

Freezing scenario: Consider the generating function

$$g_\beta(x) = \left\langle \exp(-e^{\beta x} z) \right\rangle_{M \gg 1}, \quad \beta = 1/T$$

In the high-temperature phase $\beta < \beta_c = 1$ the function turns out to satisfy a remarkable **duality relation**:

$$g_{\beta}(x) = g_{\frac{1}{\beta}}(x) \,.$$

This however does not allow to continue to $\beta > \beta_c$ regime. The phase transition at $\beta = \beta_c$ is believed to be described by the following **freezing scenario**: $g_\beta(x)$ **freezes** to the **temperature independent** profile $g_{\beta_c}(x)$ in the "glassy" phase $T \le T_c$. The scenario is supported by
(i) a heuristic **real-space renormalization group arguments** for the logarithmic models (**Carpentier, Le Doussal ’01**) revealing an analogy to the **travelling wave** analysis of polymers on disordered trees (**Derrida, Spohn 1989**)
(ii) **duality** which implies

$$\partial_\beta g_\beta(x)|_{\beta=\beta_c^-} = 0 \quad , \quad \text{for all } \ x$$

showing that the "temperature flow" of this function vanishes at the critical point $\beta = \beta_c = 1$
(iii) our **numerics**.

Assuming validity of such scenario for the problem in hand, one finds the frozen profile for the circular model:

$$g_{\beta_c}^{circ}(x) = 2e^{x/2} K_1(2e^{x/2})$$

where $K_1(z)$ is the Macdonald function. This allows to reconstruct the **distribution of the free energy** $f = -\beta^{-1} \ln z$ for any $T < T_c$. The corresponding formula takes a form of an infinite series:

$$\mathcal{P}_{\beta > \beta_c}^{CLM}(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isf} \frac{1}{\Gamma(1 + \frac{is}{\beta})} \Gamma^2 \left(1 + \frac{is}{\beta_c}\right) ds$$

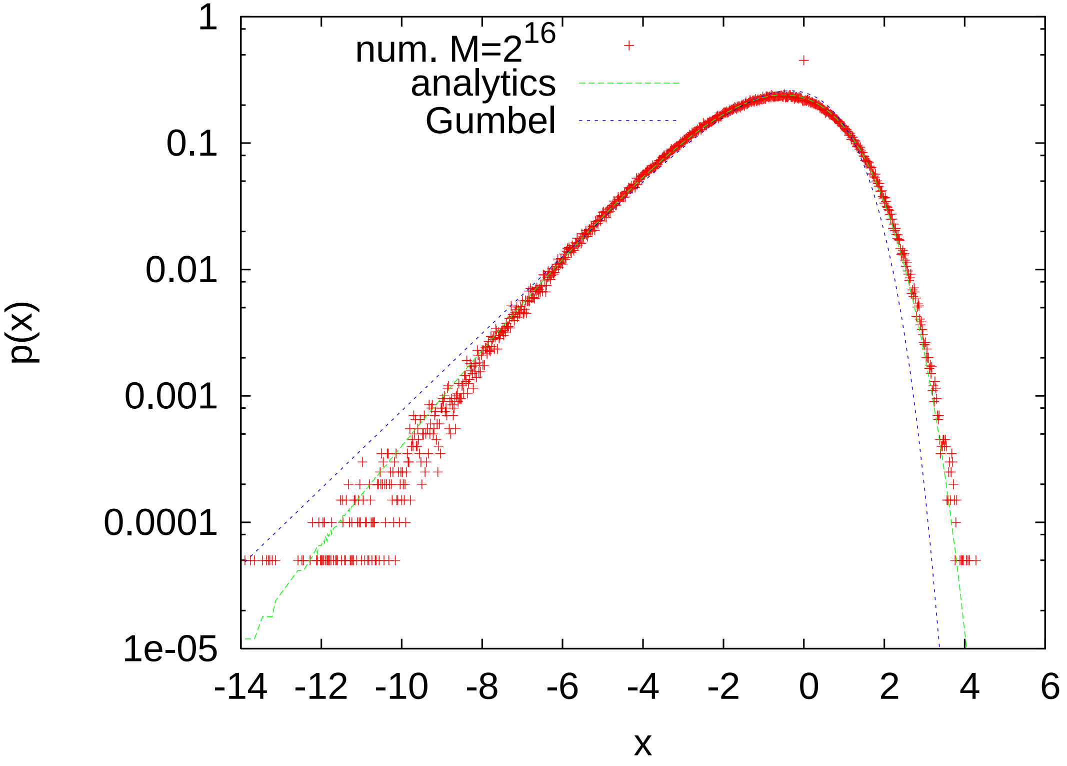
$$= -\frac{d}{df} \left[1 + \sum_{n=1}^{\infty} \frac{e^{n\beta_c f}}{n!(n-1)!\Gamma\left(1 - n\frac{\beta_c}{\beta}\right)} \left(\beta_c f + \frac{1}{n} - 2\psi(n+1) + \frac{\beta_c}{\beta} \psi\left(1 - n\frac{\beta_c}{\beta}\right)\right)\right]$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. In the zero temperature limit $\beta \to \infty$ the free energy distribution yields the **extreme value probability density.**

The minimum of the random potential is simply given by $V_{min} = -\lim_{T\to 0} f = const + x$, with known $const$ and the probability density of x related to the frozen profile $g_{\beta_c}(x)$ by

$$p(x) = -g'_{\beta_c}(x) = -\frac{d}{dx}\left[2e^{x/2}K_1(2e^{x/2})\right] \quad (1)$$

This is different from **Gumbel** distribution $p_{Gum}(x) = -\frac{d}{dx}\left[\exp -Be^{Ax}\right]$.



Distribution of extremes: we compare three distributions: (i) the histrogram for ensemble of 10^6 realizations of the Gaussian free field sampled at $M = 2^{16}$ points equispaced along the unit circle, (ii) the analytical prediction (1), and (iii) the Gumbel distribution for the mean $\&$ variance given by (1)

Numerics by A.Rosso

From circles to intervals:

Unfortunately, the direct methods which work for the circular case fail for problem of the **interval** $x \in [0, 1]$ of the real axis with the density $\rho(x) = x^a(1-x)^b$ when

$$\left\langle Z_{[0,1]}^n \right\rangle = \int_0^1 \cdots \int_0^1 \prod_{1 \le i < j \le n} |x_i - x_j|_\epsilon^{-2\gamma} \prod_{i=1}^n x_i^a (1-x_i)^b \, dx_i,$$

For a fixed $n = 1, 2, ..,$ a well defined and universal $\epsilon \to 0$ limits exists whenever the integrals are convergent, that is for $\gamma = \beta^2 g^2 < 1/n$, in which case they are given by celebrated **Selberg integral** formula. Defining $z = \Gamma(1-\gamma)Z = e^{-\beta f}$ we obtain the moments

$$z_n = \left\langle z_{[0,1]}^n \right\rangle = \prod_{j=1}^{j=n} \frac{\Gamma[1+a-(j-1)\gamma]\Gamma[1+b-(j-1)\gamma]\Gamma(1-j\gamma)}{\Gamma[2+a+b-(n+j-2)\gamma]}$$

Exploiting the recursion relations we perform a **continuation to negative integer moments:**

$$z_{-k} = \prod_{j=1}^{k} \frac{\Gamma[2+a+b+(k+j+1)\gamma]}{\Gamma[1+(j-1)\gamma]\,\Gamma[1+a+j\gamma]\Gamma[1+b+j\gamma]}$$

For $a = b = 0$ the negative moments were announced independently in **Ostrovsky D** *2008, Lett. Math. Phys* **83** *265*.

To restore the corresponding **probability density** we define the generic moments

$$M_{\beta;a,b}(s) = \left\langle z^{1-s} \right\rangle, \quad M_{\beta;a,b}(1) = 1$$

for any **complex** s, at fixed inverse temperature β and parameters a, b.

The **probability density** $P(z)$ of the scaled partition function z and the **generation function** $g_\beta(x)$ can be related to the function $M_{\beta;a,b}(s)$ via the contour integrals:

$$P\left(e^{-t}\right) = e^{2t} \frac{1}{2i\pi} \int_{s_0-i\infty}^{s_0+i\infty} e^{-st} M_{\beta;a,b}(s) ds$$

$$g_\beta(x) = e^x \frac{1}{2i\pi} \int_{s_0-i\infty}^{s_0+i\infty} e^{-sx} M_{\beta;a,b}(s) \Gamma(s-1)\, ds$$

where the integration goes parallel to the imaginary axis to the right of all singularities of the integrand.

For a somewhat simpler case of the **critical** temperature $\beta = \beta_c = 1$ the moments $M_{\beta;a,b}(s) \equiv M_{a,b}(s)$ can be expressed via the **Barnes function** $G(s)$ satisfying the functional equation

$$G(s+1) = G(s)\Gamma(s), \quad \text{with} \quad G(1) = 1.$$

as

$$M_{a,b}(s) = 2^{2s^2+s(1+2(a+b))-3-2(a+b)}\pi^{1-s}\frac{G(2+a)G(2+b)G(4+a+b)}{\Gamma(2+\frac{a+b}{2})G(2+\frac{a+b}{2})^2G(\frac{5}{2}+\frac{a+b}{2})^2}$$
$$\times\frac{\Gamma(1+\frac{a+b}{2}+s)G(1+\frac{a+b}{2}+s)^2G(\frac{3}{2}+\frac{a+b}{2}+s)^2}{G(s)G(1+a+s)G(1+b+s)G(3+a+b+s)}\,.$$

For example,

$$M_{0,0}(s) = \frac{2^{2s^2+s-2}}{G(5/2)^2\pi^{s-1}}\frac{\Gamma(s+\frac{1}{2})^2}{\Gamma(s)\Gamma(s+2)}\frac{G(s+\frac{1}{2})^2}{G(s)^2}, \quad M_{-1/2,-1/2}(s) = 2^{2s^2-s-1}\pi^{1-s}\frac{\Gamma(\frac{1}{2}+s)}{s\Gamma(3/2)}$$

To guarantee that this is the correct continuation, we have checked

(i) positivity: $M_{a,b}(s)$ given above is finite and positive on the interval $s \in [0, +\infty[$

that is all real moments $n = 1 - s < 1$ exist.

(ii) convexity: on this interval $\partial_s^2 \ln M_{a,b}(s) > 0$.

(iii) For integer values of s gives back positive/negative moments.

Deforming the integration contour one obtains the **frozen profile** $g_{\beta_c}(x)$. For the general case the expression can be obtained as expansion in powers of e^x for $x \to -\infty$. For example, for $a = b = 0$

$$g_{\beta_c}(x \to -\infty) = 1 + (x + A')e^x + (A + By + Cx^2 + \frac{1}{6}x^3)e^{2x} + \dots \tag{2}$$

with $A' = 2\gamma_E + \ln(2\pi) - 1$ and $C = -0.253846$, $B = 1.25388$, $A = -5.09728$. For the special case $a = b = -1/2$ we obtain the closed form expression:

$$g_{\beta_c}(x) = \frac{\pi}{4} \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{2\pi}} e^{-\frac{t^2}{2} - 2\sqrt{\ln 2}t} \int_{e^x}^{\infty} \left(1 - \frac{e^x}{u}\right) e^{-\sqrt{\pi u/2}\, e^{-\sqrt{\ln 2}t}} du$$

Although these expressions are different from the circle case, the universal **Carpentier-Le Doussal tail** for the probability density of **extreme values**

$$p(x \to -\infty) = -g'_{\beta_c}(x \to -\infty) \sim -xe^x$$

is shared by all these distributions. It has its origin in the characteristic tail of the partition function density $P(z \gg 1) \propto 1/z^2$ developed at criticality, with the first moment $< z >$ becoming **infinite**.

High temperature phase for $[0, 1]$ interval:

Let $\beta < \beta_c$. Define the function $G_\beta(x)$ for $\Re(x) > 0$ by :

$$\ln G_\beta(x) = \frac{x - Q/2}{2}\ln(2\pi) + \int_0^\infty \frac{dt}{t}\left(\frac{e^{-\frac{Q}{2}t} - e^{-xt}}{(1-e^{-\beta t})(1-e^{-t/\beta})} + \frac{e^{-t}}{2}(Q/2 - x)^2 + \frac{Q/2 - x}{t}\right)$$

where $Q = \beta + 1/\beta$. This function is self-dual: $G_\beta(x) = G_{1/\beta}(x)$ and satisfies the functional relation

$$G_\beta(x+\beta) = \beta^{1/2-\beta x}(2\pi)^{\frac{\beta-1}{2}}\Gamma(\beta x)\, G_\beta(x)$$

see e.g. [Fateev,Zamolodchikov,Zamolodchikov 2000]. With these definitions $G_{\beta=1}(s) = G(s)$ is the Barnes function used by us to perform the analytical continuation of moments at the critical temperature. The function $G_\beta(s)$ provides us with a natural tool which can be used to perform the required analytical continuation of moments for any temperature above critical.

We find

$$M_\beta(s) = A_\beta 2^{(s-1)(2+\beta^2(2s+1))} \pi^{1-s}$$
$$\times \frac{\Gamma(1+\beta^2(s-1))G_\beta(\frac{\beta}{2}+\frac{1}{\beta}+\beta s)G_\beta(\frac{3}{2\beta}+\beta s)G_\beta(\frac{\beta}{2}+\frac{3}{2\beta}+\beta s)}{G_\beta(\beta+\frac{2}{\beta}+\beta s)G_\beta(\frac{1}{\beta}+\beta s)^2}$$

with $A_\beta = \frac{G_\beta(\frac{1}{\beta}+\beta)^2 G(2\beta+\frac{2}{\beta})}{G_\beta(\frac{3\beta}{2}+\frac{1}{\beta})G_\beta(\frac{3}{2\beta}+\beta)G_\beta(\frac{3\beta}{2}+\frac{3}{2\beta}))}$.

Further exploiting the relation between the moments $M_\beta(s)$ and the generating function $g_\beta(x)$

$$\ln\left[-\int_{-\infty}^{\infty} g'_\beta(x)\, e^{xs}\, dx\right] = \ln M_\beta(1+\frac{s}{\beta}) + \ln\Gamma(1+\frac{s}{\beta})$$

we can use the above expression to verify the **duality relation**:

$$g_\beta(x) = g_{\frac{1}{\beta}}(x)\,.$$

thus extending its validity to the case of the interval $[0,1]$.

Conclusions & Discussions:

- **Using the methods of statistical mechanics we were able to extract the explicit expressions for distributions of extrema of the Gaussian Free Field sampled along (i) circles of unit radius and (ii) intervals of unit length. The distributions are manifestly non-Gumbel and show universal backward tail.** **The results are expected to describe extreme value statistics for $1/f$ signals, and in this way could be relevant for spectral fluctuations of quantum chaotic systems and Riemann zeta-function (cf. e.g. Relaño *et al* PRL 89 (2002) 244102).**
- **We revealed a "duality relation" satisfied by specific generating function of scaled free energies everywhere in the high temperature phase. The same object is expected to show freezing of its shape at critical temperature. It is tempting to conjecture relation between freezing and self-duality.**

- Our method is based on a few assumptions, most importantly **(i)** freezing scenario for REM-type models, and **(ii)** ability to continue analytically moments given by Selberg integrals away from positive integers.

  It remains a challenge:

  – to verify/justify/extend the assumptions/steps of the derivation; e.g. the continuation fails for the Gaussian density $\rho(t) = e^{-\frac{t^2}{2}}, t \in [-\infty, \infty]$

  – to understand universality of the results for other $1d$ curves

  – access extreme value statistics of GFF in 2D domains.

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