Wave functions and conductances at the integer quantum Hall transition: conformal invariance and possible theories

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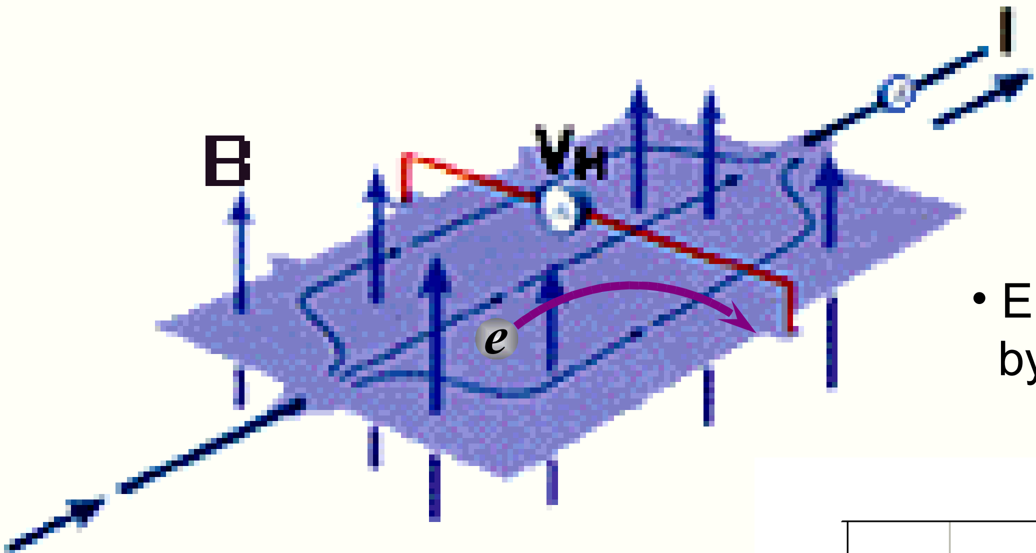
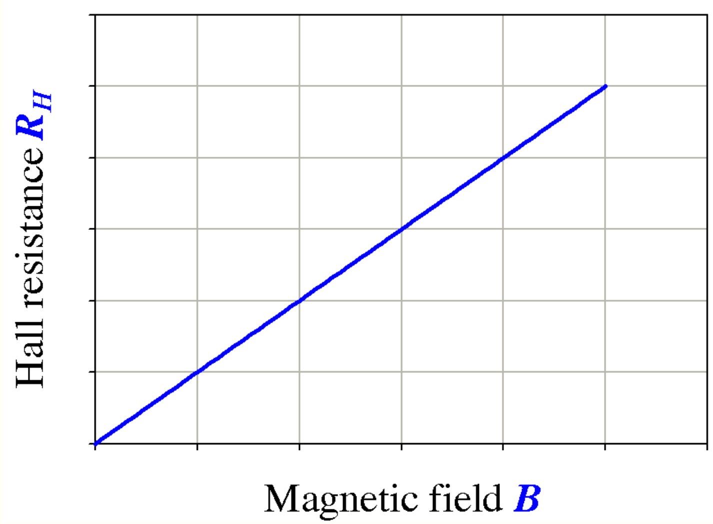
Symposium on Theoretical and Mathematical Physics, EIMI, July 05, 2009

Classical Hall effect



- Electron trajectories bent by magnetic field



- Classical Hall resistance

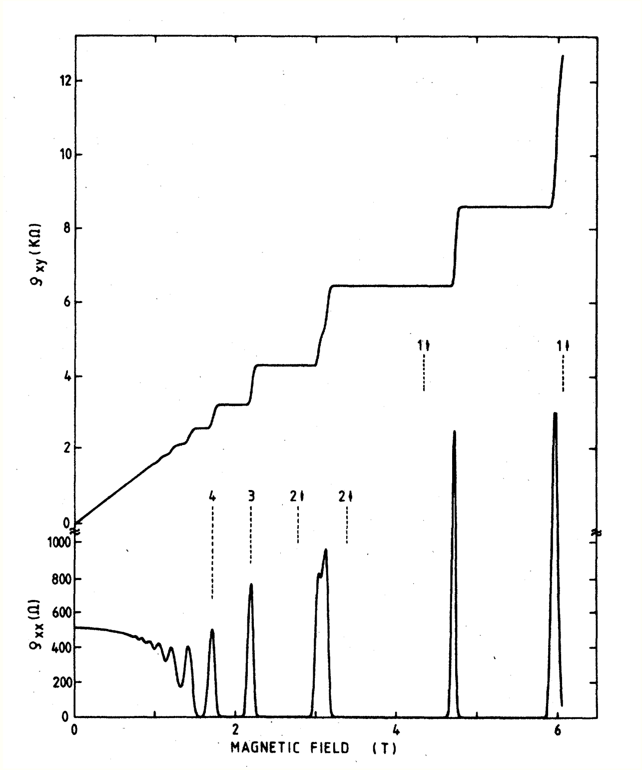
$$R_H = \frac{V_H}{I} \propto B$$

Quantum Hall effect

- Two-dimensional electron gas
- Strong magnetic field, low temperature
- Hall resistance shows plateaus

$$R_H = \frac{1}{n}\frac{h}{e^2}$$

- Longitudinal resistance shows peaks separated by insulating valleys



K. v. Klitzing, Rev. Mod. Phys. 56 (1986)

IQH and localization is strong magnetic field

- Single electron in a magnetic field and a random potential

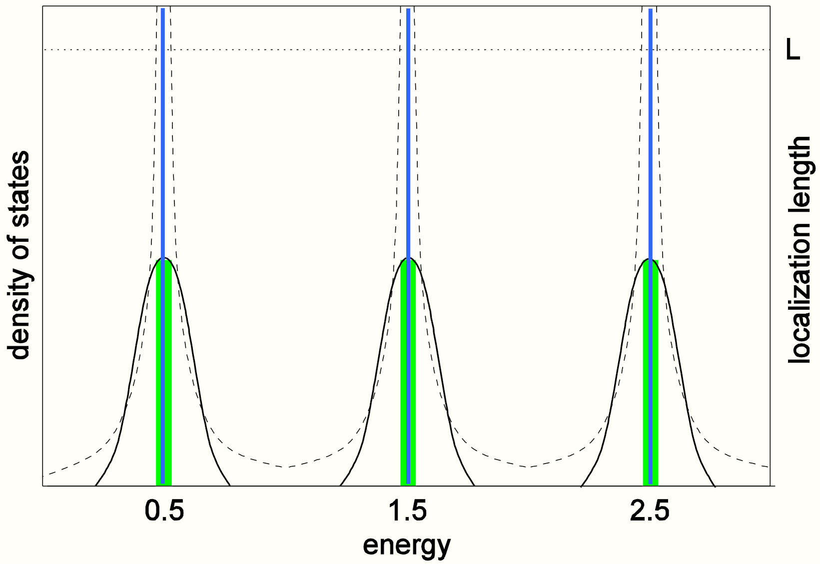
$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 + U(\boldsymbol{r})$$

- Without disorder: Landau levels

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c, \quad \omega_c = \frac{eB}{mc}$$

- Disorder broadens the levels and localizes most states

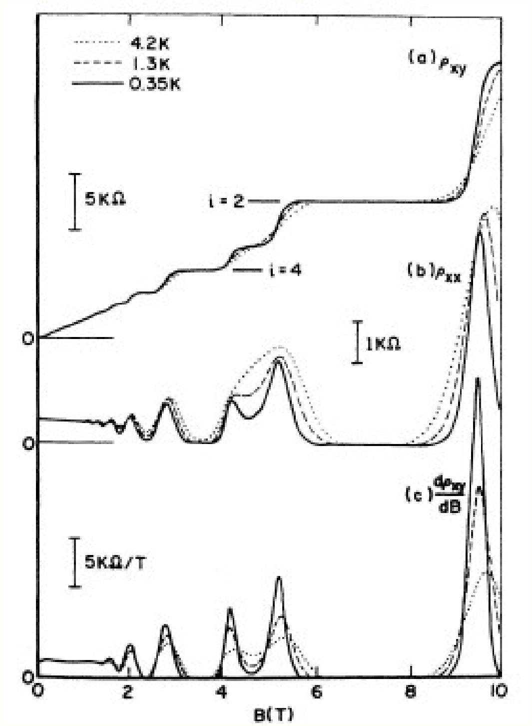
- Extended states near E_n (green)



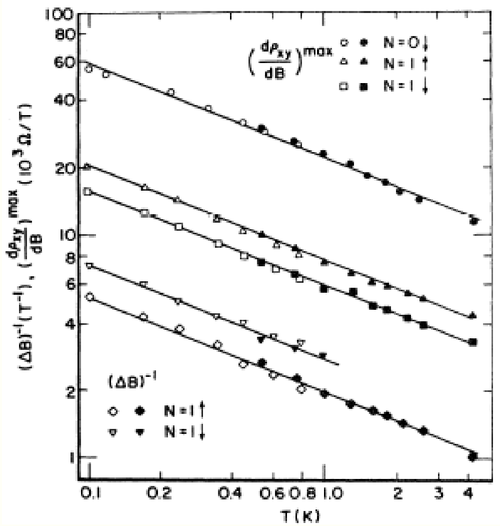
- Transition between QH plateaus upon varying E_F or B
- Density of states is non-singular across the transition

Critical scaling near IQH plateau transition

- Localization length diverges $\xi(E) \propto |E - E_n|^{-\nu}, \quad \nu \approx 2.35$
- Universal scaling with temperature, current, frequency, and system size



$$\sigma_{ij}(B) = \frac{e^2}{h} S_{ij}\left[L_{\text{eff}}^{1/\nu}(B - B_c)\right]$$



H. P. Wei et al., PRL 61 (1988)

- Intense experimental and theoretical research continues
- No analytical description of the critical region

Symmetry classification of disordered electronic systems

A. Altland, M. Zirnbauer '96

Conventional (Wigner-Dyson) classes

	T	spin rot.	chiral	p-h	symbol
GOE	+	+	−	−	AI
GUE	−	+/−	−	−	A
GSE	+	−	−	−	AII

Chiral classes

	T	spin rot.	chiral	p-h	symbol
ChOE	+	+	+	−	BDI
ChUE	−	+/−	+	−	AIII
ChSE	+	−	+	−	CII

Bogoliubov-de Gennes classes

T	spin rot.	chiral	p-h	symbol
+	+	−	+	CI
−	+	−	+	C
+	−	−	+	DIII
−	−	−	+	D

- Integer quantum Hall
- Spin-orbit metal-insulator transition

$$H = \begin{pmatrix} 0 & t \\ t^\dagger & 0 \end{pmatrix}$$

- Spin quantum Hall

$$H = \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^T \end{pmatrix}$$

- Thermal quantum Hall

Anderson transitions

- Metal-insulator transitions (MIT) induced by disorder

- Quantum critical points with highly unconventional properties: no spontaneous symmetry breaking!

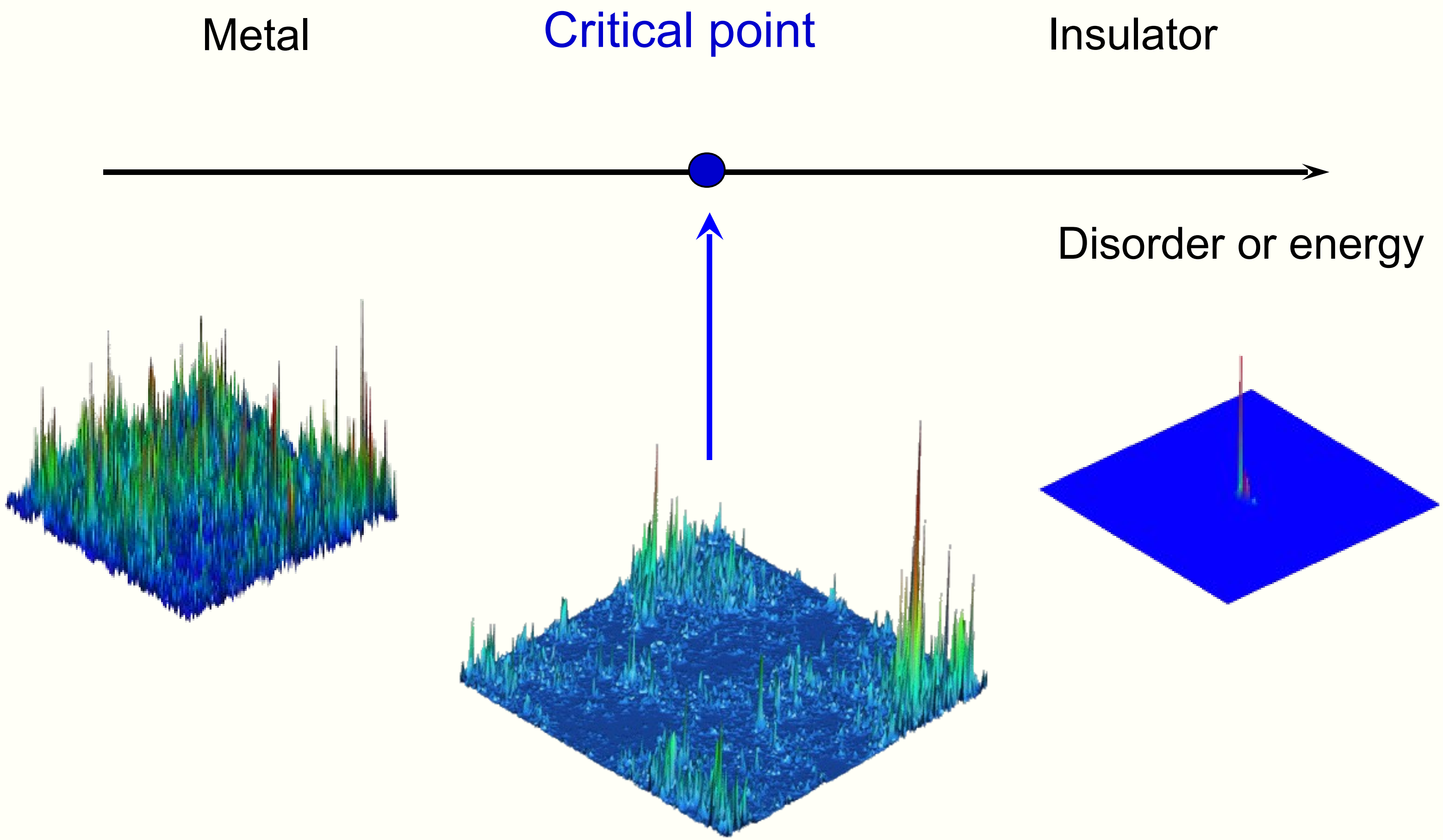
- Critical wave functions are neither localized nor truly extended, but are complicated scale invariant multifractals characterized by an infinite set of exponents

- For most Anderson transitions no analytical results are available, even in 2D where conformal invariance is expected to provide powerful tools of conformal field theory (CFT)

- Field theory: supersymmetric σ -model K. B. Efetov '83

$$S[Q] \propto -\int d^d\boldsymbol{r}\, \mathrm{Str}[D(\nabla Q)^2 + 2i\omega\Lambda Q], \qquad Q^2 = 1$$

Wave functions across Anderson transition

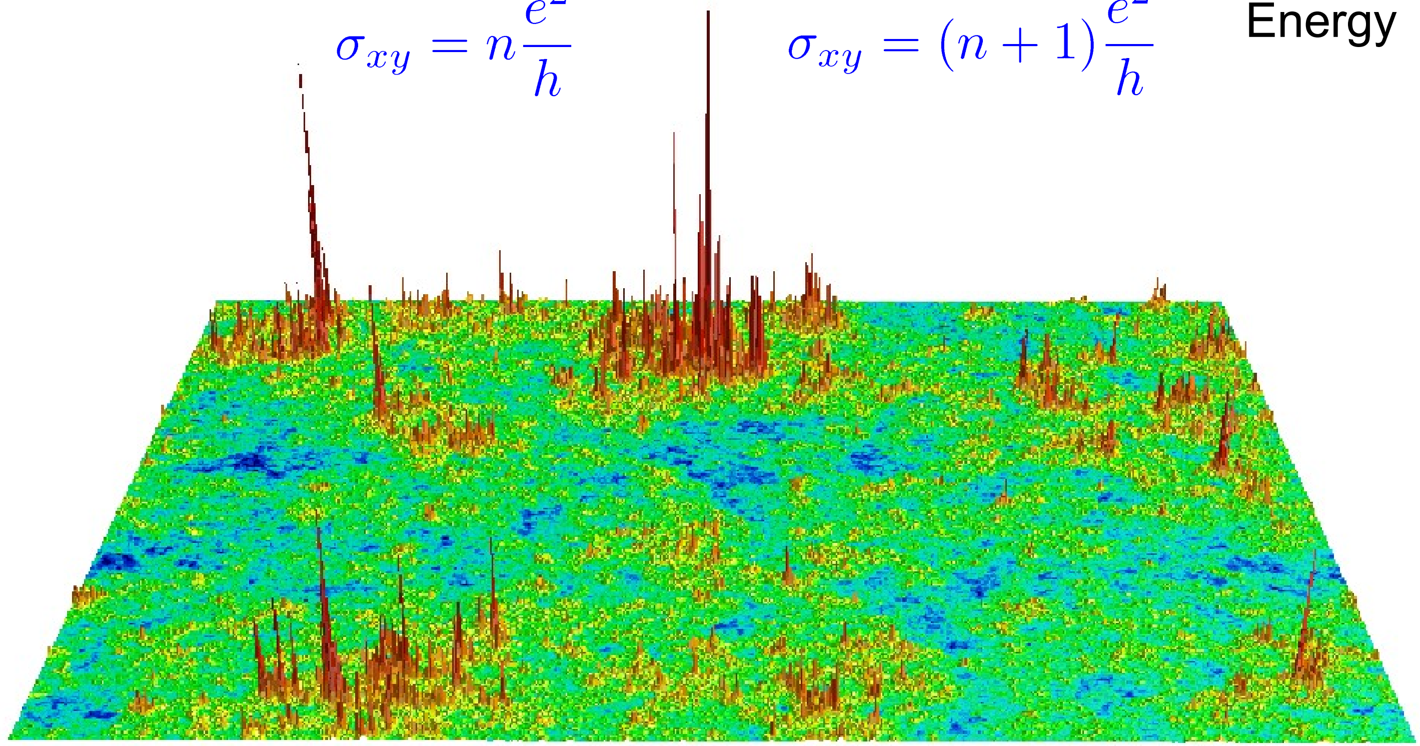


Wave function at IQH plateau transition

Insulator | Critical point | Insulator

Energy

$\sigma_{xy} = n\frac{e^2}{h}$ $\qquad$ $\sigma_{xy} = (n+1)\frac{e^2}{h}$



F. Evers

Multifractal wave functions

F. Wegner `80
C. Castellani, L. Peliti `86

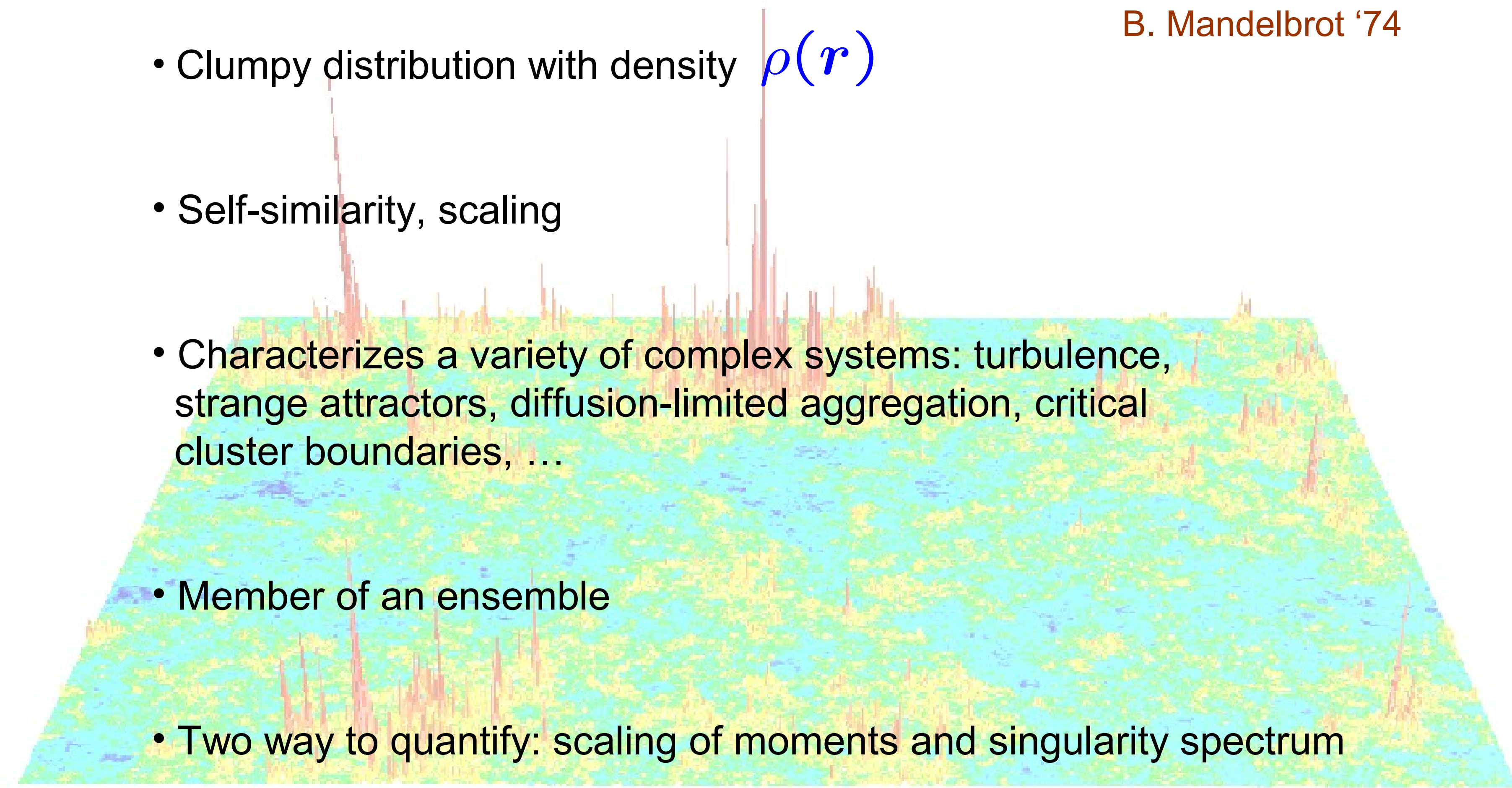
- Moments of the wave function intensity $|\psi(\boldsymbol{r})|^2$

(inverse participation ratios, IPR's)

$$P_q = \int d^d\boldsymbol{r} |\psi(\boldsymbol{r})|^{2q}, \qquad \overline{P_q} \sim \begin{cases} L^0, & \text{insulator} \\ L^{-\tau_q}, & \text{critical} \\ L^{-d(q-1)}, & \text{metal} \end{cases}$$

Multifractals

B. Mandelbrot ‘74



- Clumpy distribution with density $\rho(r)$
- Self-similarity, scaling
- Characterizes a variety of complex systems: turbulence, strange attractors, diffusion-limited aggregation, critical cluster boundaries, …
- Member of an ensemble
- Two way to quantify: scaling of moments and singularity spectrum

Multifractal moments

Density $\rho(\boldsymbol{r}) \geqslant 0$ can be singular but normalized $\int d^d\boldsymbol{r}\,\rho(\boldsymbol{r}) = 1$

Divide the system of size L into N boxes B_i of size a, $N = \left(\frac{L}{a}\right)^d$

Measure of each box $p_i = \int_{B_i} d^d\boldsymbol{r}\,\rho(\boldsymbol{r})$

Moments of the measure scale with L as

$$P_q = \sum_{i=1}^{N} p_i^q \sim \left(\frac{a}{L}\right)^{\tau_q}$$

Multifractal spectrum τ_q. A priori known

$\tau_0 = -d$ (dimension of support), $\tau_1 = 0$ (normalization)

Multifractal moments

- Extreme cases

  - Uniform distribution

    $$p_i = \frac{1}{N} = \left(\frac{a}{L}\right)^d, \qquad P_q = N p_i^q = \left(\frac{a}{L}\right)^{d(q-1)}$$

    "Gap scaling" $\tau_q = d(q-1)$, linear in q

  - Mass localized in a volume ξ^d, where $a < \xi < L$

    $\left(\frac{\xi}{a}\right)^d$ boxes are filled with $p_i = \left(\frac{a}{\xi}\right)^d$

    $P_q = \left(\frac{a}{\xi}\right)^{d(q-1)}$ is L-independent $\Rightarrow \tau_q = 0$

- In general $\quad P_q(E) \sim L^{-\tau_q} F_q\left[(E - E_c) L^{1/\nu}\right]$

Multifractality and field theory

- Anomalous exponents: $\tau_q = dq + \Delta_q - d$
- In $d = 2 + \epsilon, \quad \Delta_q = -\epsilon q(q-1) + O(\epsilon^4)$ F. Wegner '80
- Δ_q - scaling dimensions of operators $\mathcal{O}_q$ in a field theory

B. Duplantier, A. Ludwig '91

- Infinity of operators with negative dimensions
- $\Delta_1 = 0$ corresponds to non-critical DOS
- Connection trough Green's functions and local DOS

Multifractality and field theory

- Green's functions $$G_{R/A}(r,r';E) = \sum_n \frac{\psi_n(r)\psi_n^*(r')}{E - E_n \pm i\eta}$$

- Local DOS $$\rho_E(r) = \lim_{\eta\to 0}\frac{1}{2\pi i}\left[G_A(r,r;E) - G_R(r,r;E)\right]$$
$$= \sum_n |\psi_n(r)|^2 \delta(E - E_n)$$

- Global DOS $$\rho_E = \int_{L^d} d^d r\, \rho_E(r) = \sum_n \delta(E - E_n)$$

- Wave functions (statistically identical at a given *E*)

$$|\psi_E(r)|^2 \sim \frac{\sum_n |\psi_n(r)|^2 \delta(E - E_n)}{\sum_n \delta(E - E_n)} = \frac{\rho_E(r)}{\rho_E}$$

Multifractality and field theory

- Disorder averages of Green's functions described by a field theory

$$\overline{[L^d \rho_E(r)]^q} \sim \langle \mathcal{O}_q(r) \rangle$$

F. Wegner '80

- Wave function moments are given by field theory expectation values of certain operators and their scaling dimensions

$$\overline{[L^d |\psi_E(r)|^2]^q} \sim \langle \mathcal{O}_q(r) \rangle \sim \frac{1}{L^{x_q}}$$

- Hence: $\tau_q = dq + x_q - d$ or $\Delta_q = x_q$

- Note: $x_0 = 0$ and $x_1 = 0$ (normalization of wave function)

Boundary properties at Anderson transitions

- Surface critical behavior has been extensively studied at ordinary critical points, but not for Anderson transitions

- Why are boundaries interesting?

  - practical interest: leads attach to the surface

  - can impose various boundary conditions: richer behavior

  - in 2D boundaries provide access to elusive CFT description: we use them to provide direct numerical evidence for presence of conformal invariance

Boundary multifractality

A. Subramaniam et al., PRL **96**, 126802 (2006)

- By analogy with critical phenomena expect different scaling behavior near a boundary

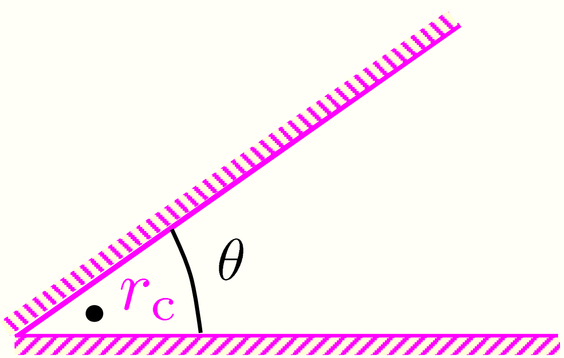
$$\langle \mathcal{O}_q(r_{\mathrm{s}}) \rangle \sim \frac{1}{L^{x_q^{\mathrm{s}}}} \Rightarrow \overline{|\psi_E(r_{\mathrm{s}})|^{2q}} \sim \frac{1}{L^{qd+x_q^{\mathrm{s}}}}, \qquad x_q^{\mathrm{s}} \neq x_q^{\mathrm{b}}$$

- "Surface" (boundary) contribution to IPR

$$\int_{L^{d-1}} d^{d-1} r_{\mathrm{s}} \overline{|\psi(r_{\mathrm{s}})|^{2q}} \sim L^{-\tau_q^{\mathrm{s}}}, \quad \tau_q^{\mathrm{s}} = dq + \Delta_q^{\mathrm{s}} - (d-1), \quad \Delta_q^{\mathrm{s}} = x_q^{\mathrm{s}}$$

- τ_q^s, Δ_q^s surface multifractal exponents

- Analogously, corner mulftifractality



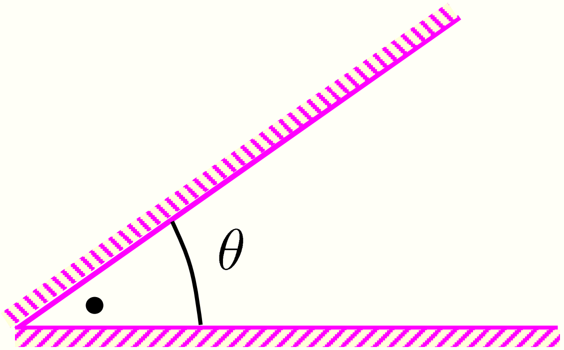
Multifractality and conformal invariance

H. Obuse et al., PRL **98**, 156802 (2007); in preparation

- Assumption: $\overline{|\psi(r)|^{2q}}$ corresponds to a primary operator

- Boundary CFT relates corner and surface dimensions:

J. Cardy '84



$$\Delta_q^{\theta} = \frac{\pi}{\theta} \Delta_q^{\mathrm{s}}$$

- Predictions for quasi-1D localization length

- Numerically tested for a number of Anderson transitions

Models for the IQH plateau transition

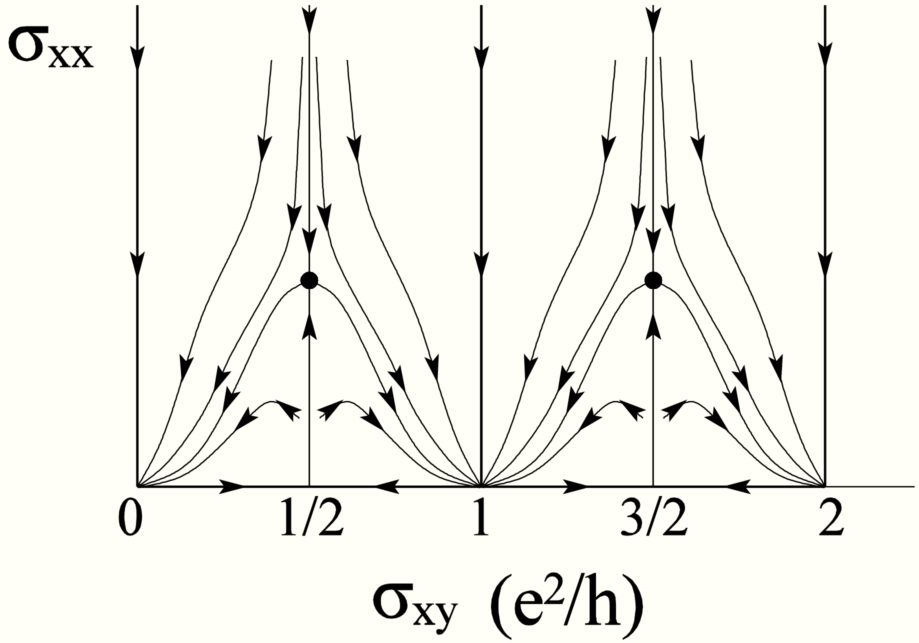
- Sigma model

H. Levine, S. B. Libby, A. M. M. Pruisken `83
H. Weidenmuller `87

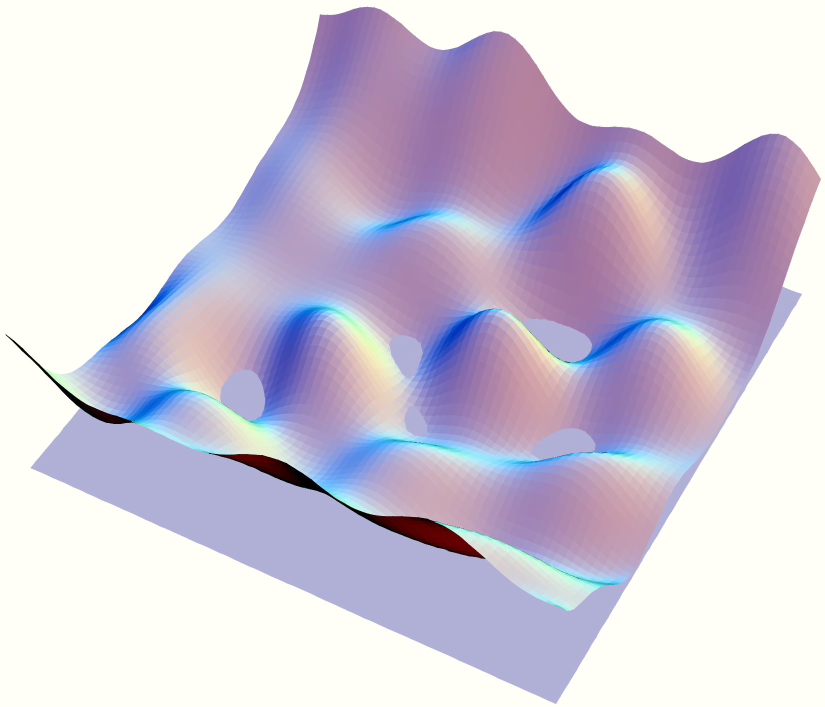
$$S[Q] = \frac{1}{8}\int d^2\boldsymbol{r}\,\mathrm{Str}[-\sigma_{xx}(\nabla Q)^2 + 2\sigma_{xy} Q\partial_x Q \partial_y Q], \qquad Q^2 = 1$$

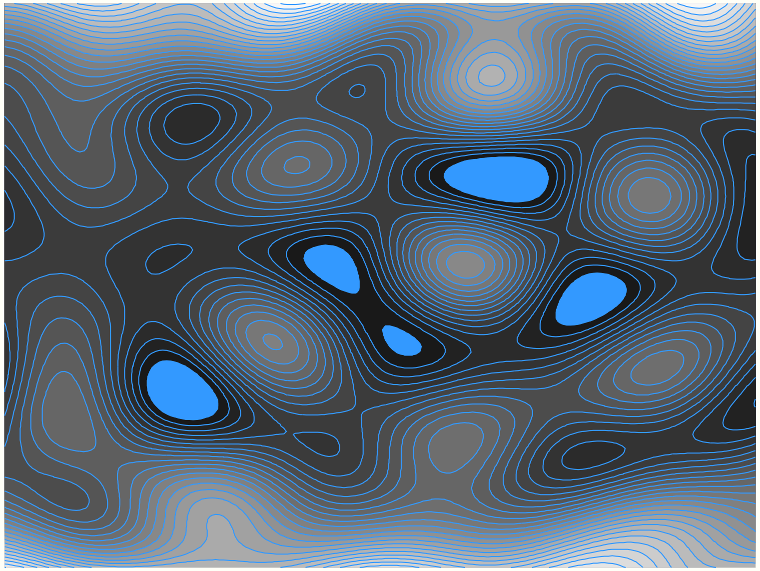
- Two-parameter flow diagram

D. E. Khmelnitskii `83



Electrons in smooth random potential





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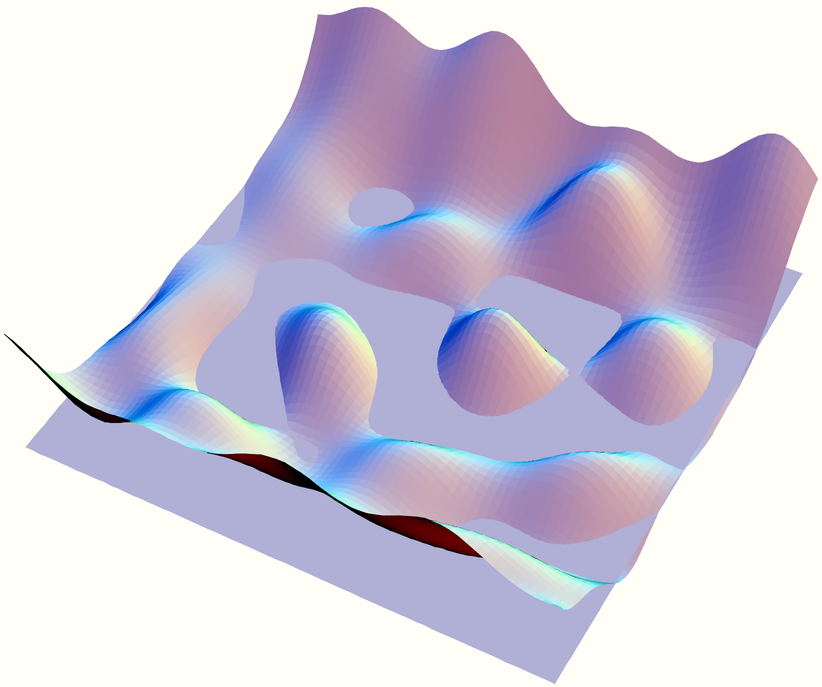
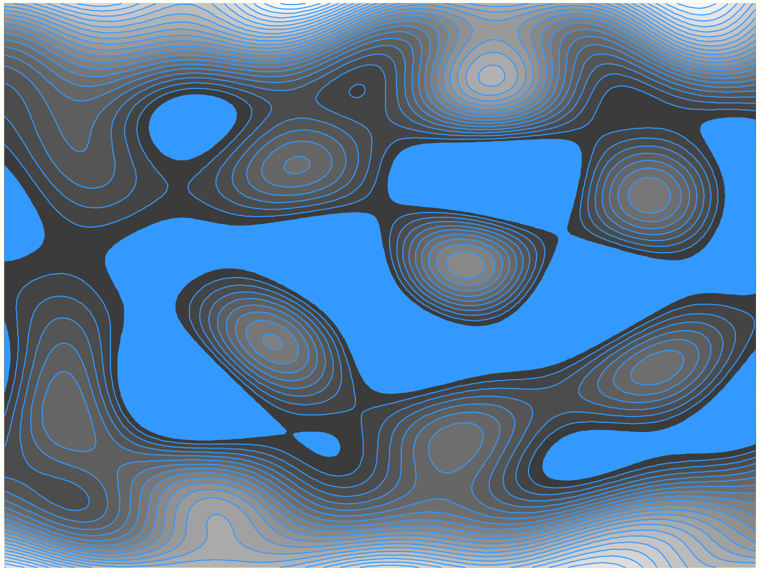
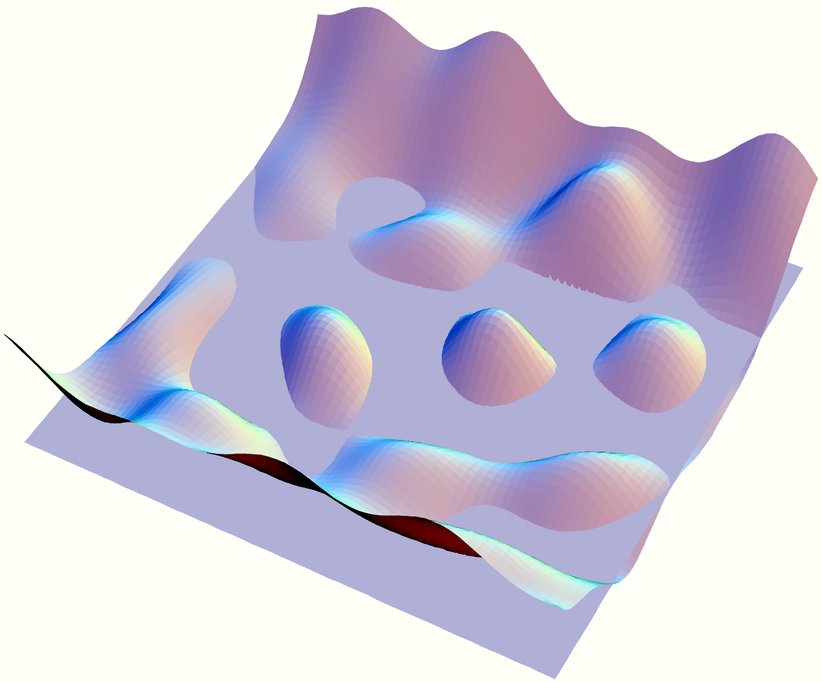
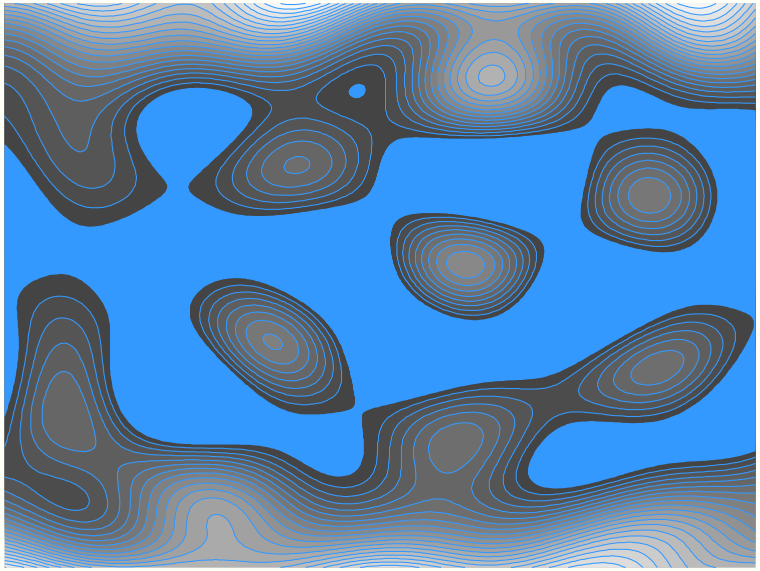


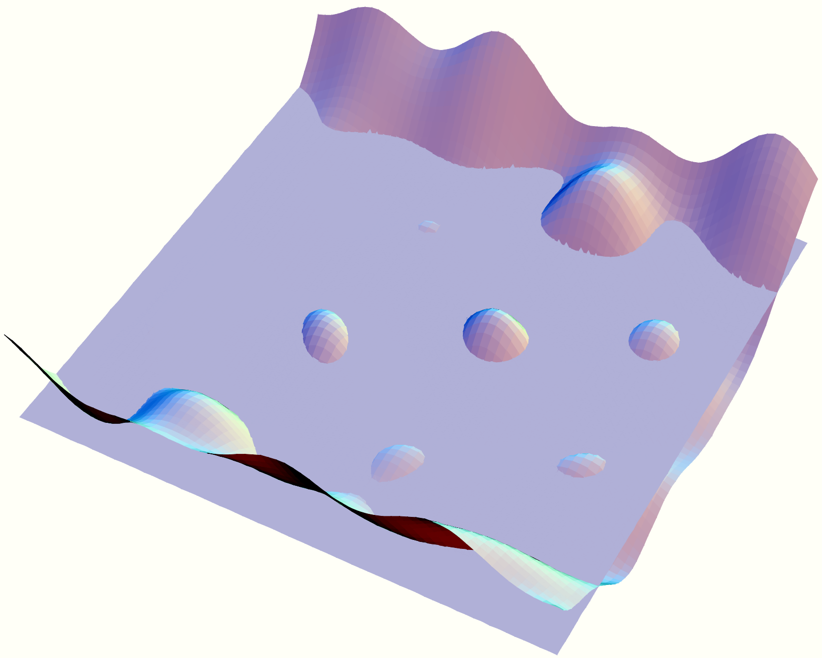


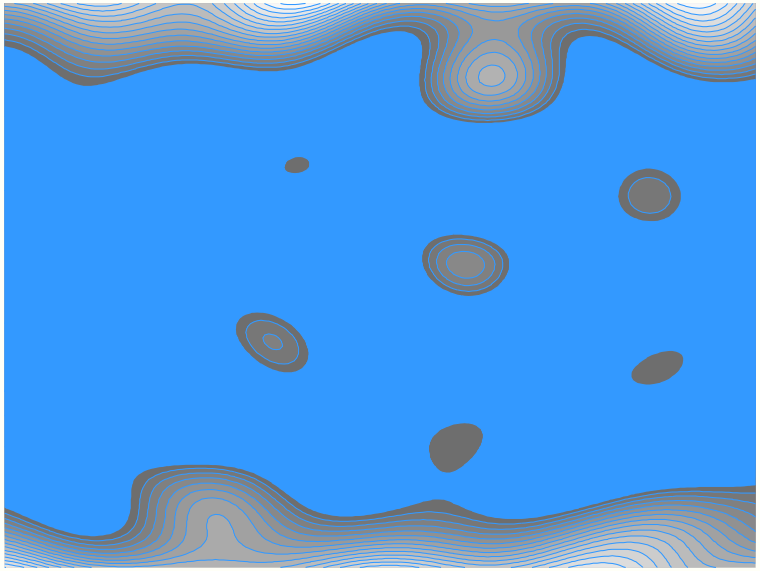
Electrons in smooth random potential





Electrons in smooth random potential

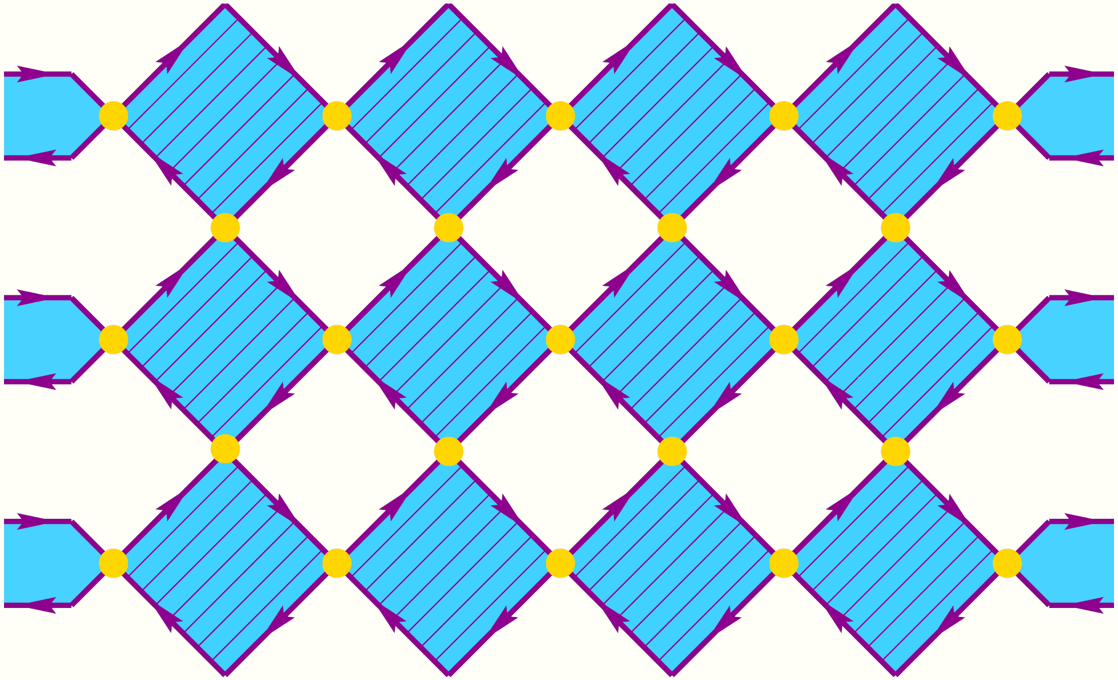




Models for the IQH plateau transition

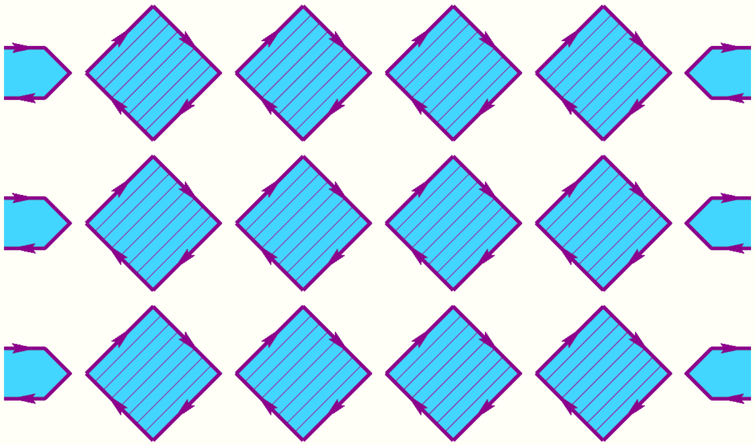
- Chalker-Coddington network model J. T. Chalker, P. D. Coddington `88

- Obtained from semi-classical drifting orbits in smooth potential

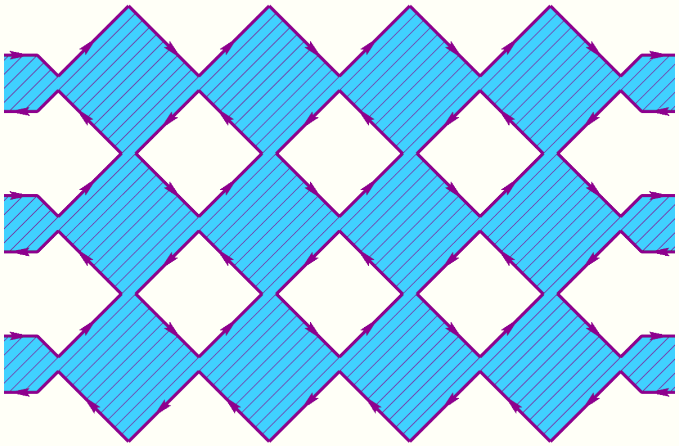


Models for the IQH plateau transition

- Chalker-Coddington network model J. T. Chalker, P. D. Coddington `88
- Extreme limits



Insulator



Quantum Hall

Models for the IQH plateau transition

- Chalker-Coddington (CC) network model — J. Chalker, P. Coddington `88
- SUSY for CC model: spin chain (a variant of Efetov's SUSY method) — N. Read `91
- Reduction of sigma model to spin chain — M. Zirnbauer `94
- Dirac fermions in random potentials — A. Ludwig et al. `94
- Dirac fermions from network models — C.-M. Ho, J. Chalker`96
- Can we "solve" ay of these?
- Idea: analogy with SU(2) AF spin-1/2 chain for which a solvable critical field theory is known – a Wess-Zumino model

Theory proposals for the IQH transition

- Variants of Wess-Zumino (WZ) models on a supergroup G

$$S[g] = \frac{1}{2\pi f^2}\int d^2 r\, \mathrm{str}\, \partial_\mu g^{-1}\partial_\mu g + ik S_{\mathrm{WZ}}[g], \quad g \in G$$

M. Zirnbauer `99
M. Bhaseen, I. Kogan, O. Soloviev, N. Taniguchi, A. Tsvelik `00
A. Tsvelik `07
A. LeClair `07

- Different target spaces G (same Lie superalgebra $\mathrm{psl}(2|2)$)

- Different proposals for the value of the “level” k

What is known about the WZ model

- This model describes transport properties in the *chiral* unitary Gade-Wegner (AIII) class

S. Guruswamy, A. LeClair, A. Ludwig `00

- Has a line of fixed points with continuously varying critical properties parametrized by f

- Has current algebra (Kac-Moody) symmetry at $f^{-2} = k$

- Operators are classified by representations λ of $\mathrm{psl}(2|2)$

- Dimensions at KM point are known in terms of quadratic Casimir

$$x_\lambda = C_\lambda^{(2)}/k$$

Conjectures about the WZ model

1. Describes spectral and transport properties at the IQH critical point

1. Dimensions away from KM point are given by quadratic Casimir

$$x_\lambda = f^2 C_\lambda^{(2)}$$

- Conjectured link with IQH transition goes through moments of point contact conductance (PCC)

Point contact conductance

M. Janssen, M. Metzler, M. Zirnbauer `99

- Recall definition

- Moments of PCC within CC model

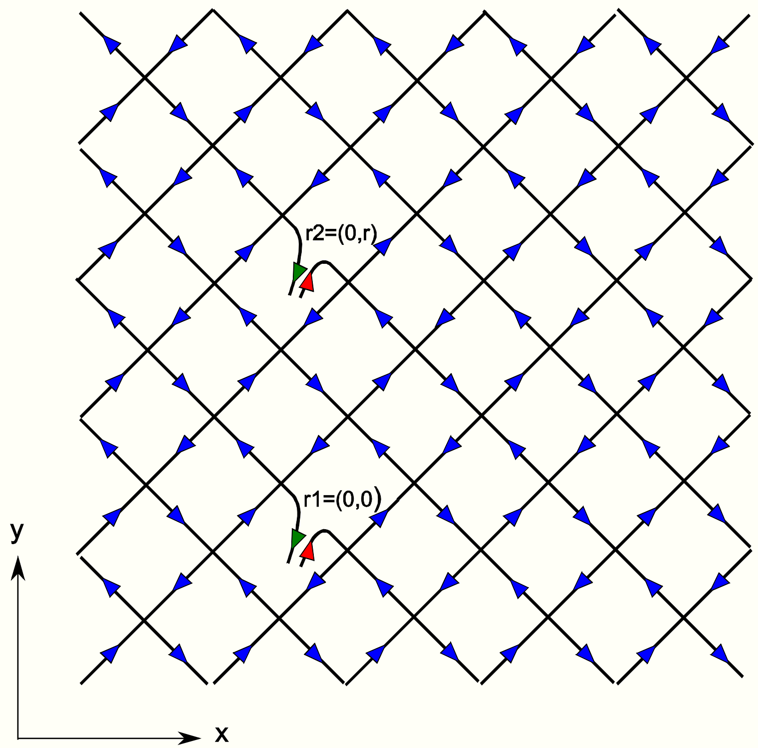
  carry a certain representation λ_q

  of the global SUSY $\mathrm{psl}(2|2)$

- Dimensions X_q of moments of PCC

  at the IQH transition are known to be

$$X_q = 2\Delta_q \quad (q < 1/2)$$



R. Klesse, M. Zirnbauer `01
F. Evers, A. Mildenberger, A. Mirlin `01

Conjectured multifractality at the IQH transition

- From conjecture 2, dimension of operator in representation λ_q

$$x_q = 2\gamma q(1-q)$$

- From conjecture 1 it follows $x_q = X_q$

- This gives exact parabolicity: $\Delta_q^{\mathrm{b}} = \gamma^{\mathrm{b}} q(1-q)$ M. Zirnbauer `99

- Reminiscent of exact spectra for Dirac fermions in random gauge potentials (and one loop result in sigma model)

A. Ludwig, M. P. A. Fisher, R. Shankar, G. Grinstein `94
C. Mudry, C. Chamon, X.-G. Wen `96

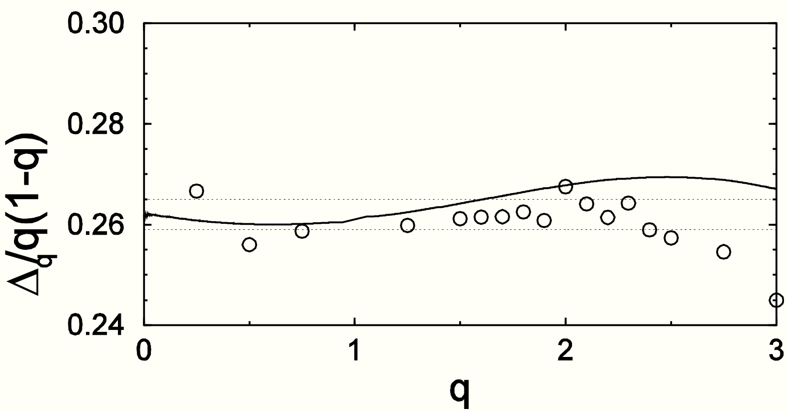
- Same reasoning would give parabolic $\Delta_q^{\mathrm{s}} = \gamma^{\mathrm{s}} q(1-q)$

with $\gamma^{\mathrm{s}} = 2\gamma^{\mathrm{b}}$ A. Subramaniam, A. Ludwig, IAG, in preparation

Actual multifractality at the IQH plateau transition

- Only accessible by numerical analysis

- Previous numerical study of CC model F. Evers et al. `01



- Δ_q^{b} are parabolic within 1%

$$\Delta_q^{\mathrm{b}} = \gamma^{\mathrm{b}} q(1-q), \quad \gamma = 0.262 \pm 0.003$$

Actual multifractality at the IQH plateau transition

H. Obuse et al., PRL **101**, 116802 (2008)

- Our numerics

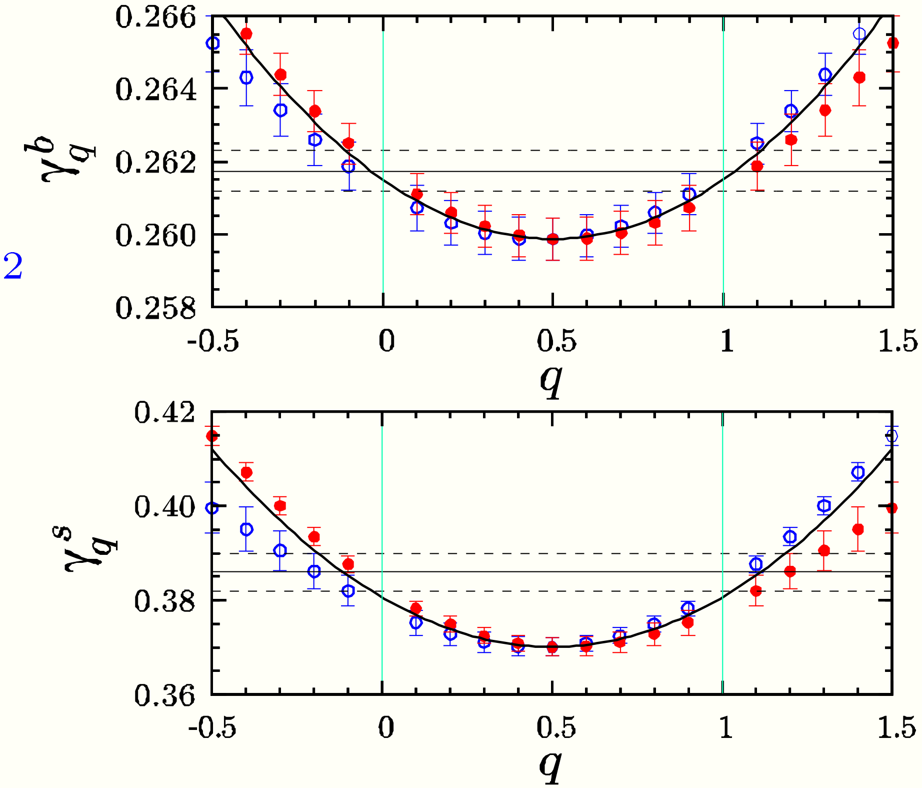
$$\gamma_q^{\mathrm{x}} = \frac{\Delta_q^{\mathrm{x}}}{q(1-q)}$$

- Fit $\gamma_q^{\mathrm{x}} = c_1^{\mathrm{x}} + c_2^{\mathrm{x}}(q-1/2)^2$

$$c_1^{\mathrm{b}} = 0.2560, \quad c_2^{\mathrm{b}} = 0.0065,$$
$$c_1^{\mathrm{s}} = 0.370, \quad c_2^{\mathrm{s}} = 0.042$$

- Δ_q^{x} are not parabolic,

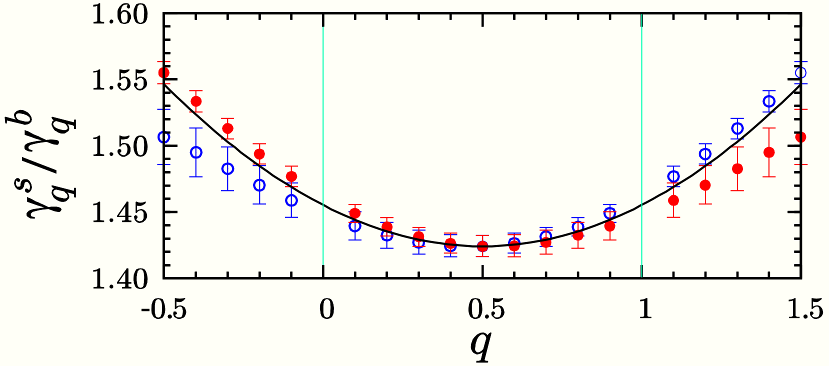
  ruling out WZ-type theories!



- Surface exponents Δ_q^{s} have stronger q-dependence

Actual multifractality at the IQH plateau transition

- Ratio of bulk and surface exponents



- The ratio $\gamma_q^{\mathrm{s}}/\gamma_q^{\mathrm{b}} \neq 2$ (the case for free field theories)
- Our results strongly constrain any candidate theory for IQH transition
- Practically identical results and conclusions from Karlsruhe group

Conclusions

- Scaling of critical wave functions contains a lot of useful information
- Boundary and corner multifractal spectra provide direct numerical evidence of conformal invariance at 2D Anderson transitions
- Multifractal spectrum at the IQH plateau transition is *not parabolic*, ruling out existing theoretical proposals
- Stochastic geometry (conformal restriction): a new approach to quantum Hall transitions