

# Integrability of high energy scattering amplitudes in $N = 4$ supersymmetric theory

Lev N. Lipatov

Petersburg Nuclear Physics Institute

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References: J. Bartels, L. N. Lipatov, A. Sabio Vera,  
arXiv:0802.2065; 0807.0894; L.N.Lipatov, arXiv: 0902.1444

# 1 Gluon reggeization

Regge kinematics of scattering amplitudes

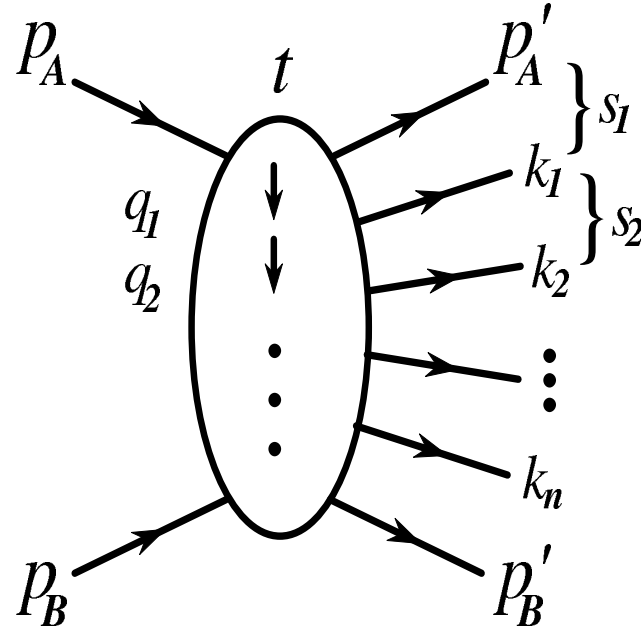
$$s = 4E^2 \gg -t = |q|^2 \approx E^2 \theta^2$$

Elastic amplitude in LLA of QCD

$$M_{AB}^{A'B'}(s, t) = s g T_{A'A}^c \delta_{\lambda_{A'} \lambda_A} \frac{s^{\omega(t)} + (-s)^{\omega(t)}}{t} g T_{B'B}^c \delta_{\lambda_{B'} \lambda_B}$$

Gluon Regge trajectory in the leading order

$$\begin{aligned} \omega(-|q|^2) &= -\frac{\alpha_s N_c}{4\pi^2} \int d^2k \frac{|q|^2}{|k|^2 |q-k|^2} \\ &= -\frac{\alpha_s N_c}{2\pi} \ln \frac{|q^2|}{\lambda^2} = -\frac{\alpha_s N_c}{2\pi} \left( \ln \frac{|q^2|}{\mu^2} - \frac{1}{\epsilon} \right) \end{aligned}$$



Multi-Regge amplitudes in LLA (F.,K.,L. (1975))

$$M_{2 \rightarrow 2+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2} g T_{c_2 c_1}^{d_1} C(q_2, q_1) \frac{s_2^{\omega_2}}{|q_2|^2} \dots g T_{c_{n+1} c_n}^{d_n} C(q_{n+1}, q_n) \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2},$$

$$C(q_2, q_1) = \frac{q_2 q_1^*}{q_2^* - q_1^*}, \quad \sigma_t = \sum_n \int d\Gamma_n |M_{2 \rightarrow 2+n}|^2$$

## 2 Analyticity, unitarity and bootstrap

Steinmann relations for overlapping channels

$$\Delta_{s_r} \Delta_{s_{r+1}} M_{2 \rightarrow 2+n} = 0$$

Dispersion representation for  $M_{2 \rightarrow 3}$  in the Regge ansatz

$$M_{2 \rightarrow 3} = c_1 (-s)^{j(t_2)} (-s_1)^{j(t_1) - j(t_2)} + c_2 (-s)^{j(t_1)} (-s_2)^{j(t_2) - j(t_1)}$$

Dispersion representation for  $M_{2 \rightarrow 4}$  in the Regge ansatz

$$\begin{aligned} M_{2 \rightarrow 4} = & d_1 (-s)^{j_3} (-s_{012})^{j_2 - j_3} (-s_1)^{j_1 - j_2} + d_2 (-s)^{j_1} (-s_{123})^{j_2 - j_1} (-s_3)^{j_3 - j_2} \\ & + d_3 (-s)^{j_3} (-s_{012})^{j_1 - j_3} (-s_2)^{j_2 - j_1} + d_4 (-s)^{j_1} (-s_{123})^{j_3 - j_1} (-s_2)^{j_2 - j_3} \\ & + d_5 (-s)^{j_2} (-s_1)^{j_1 - j_2} (-s_3)^{j_3 - j_2}, \quad j_r = j(t_r) \end{aligned}$$

Bootstrap relation in LLA

$$-i\pi\omega(t_1) M_{2 \rightarrow 2+n} = \sum_{r=(0,1,2,\dots)} \Delta_{s_r} M_{2 \rightarrow 2+n} = \sum_t M_{2 \rightarrow 2+t} M_{2+t \rightarrow 2+n}^*$$

### 3 BFKL equation (1975)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E \Psi(\vec{\rho}_1, \vec{\rho}_2) = H_{12} \Psi(\vec{\rho}_1, \vec{\rho}_2), \quad \sigma_t \sim s^\Delta, \quad \Delta = -\frac{\alpha_s N_c}{2\pi} E$$

BFKL Hamiltonian

$$H_{12} = \ln |p_1 p_2|^2 + \frac{1}{p_1 p_2^*} (\ln |\rho_{12}|^2) p_1 p_2^* \\ + \frac{1}{p_1^* p_2} (\ln |\rho_{12}|^2) p_1^* p_2 - 4\psi(1), \quad \rho_{12} = \rho_1 - \rho_2$$

Möbius invariance and conformal weights (L. (1986))

$$\rho_k \rightarrow \frac{a\rho_k + b}{c\rho_k + d},$$

$$m = \gamma + n/2, \quad \tilde{m} = \gamma - n/2, \quad \gamma = 1/2 + i\nu$$

## 4 BKP equation (1980)

Bartels-Kwiecinski-Praszalowicz equation

$$E \Psi(\vec{\rho}_1, \dots, \vec{\rho}_n) = H \Psi(\vec{\rho}_1, \dots, \vec{\rho}_n), \quad H = \sum_{k < l} \frac{\vec{T}_k \vec{T}_l}{-N_c} H_{kl}$$

Holomorphic separability at large  $N_c$  (L. (1988))

$$H = h + h^*, \quad h_{12} = \ln p_1 + \ln p_2 + \frac{1}{p_1} (\ln \rho_{12}) p_1 + \frac{1}{p_2} (\ln \rho_{12}) p_2 - 2\psi(1)$$

Holomorphic factorization of wave functions

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, \dots, \vec{\rho}_n) = \sum_{r,s} a_{r,s} \Psi_r(\rho_1, \dots, \rho_n) \Psi_s(\rho_1^*, \dots, \rho_n^*)$$

Möbius invariance

$$M_a^2 \Psi_r = m(m-1) \Psi_r, \quad M_a^{*2} \Psi_s = \tilde{m}(\tilde{m}-1) \Psi_s$$

## 5 Integrability at $N_c \rightarrow \infty$

Monodromy and transfer matrices (L. (1993))

$$t(u) = L_1 L_2 \dots L_n = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \quad T(u) = A(u) + D(u),$$

$$L_k = \begin{pmatrix} u + \rho_k p_k & p_k \\ -\rho_k^2 p_k & u - \rho_k p_k \end{pmatrix}, \quad [T(u), h] = 0$$

Yang-Baxter equation (L. (1993))

$$t_{r'_1}^{s_1}(u) t_{r'_2}^{s_2}(v) l_{r_1 r_2}^{r'_1 r'_2}(v - u) = l_{s'_1 s'_2}^{s_1 s_2}(v - u) t_{r_2}^{s'_2}(v) t_{r_1}^{s'_1}(u), \quad \hat{l} = u \hat{1} + i \hat{P}$$

Duality symmetry (L. (1999))

$$p_r \rightarrow \rho_{r+1, r} \rightarrow p_{r+1}$$

Heisenberg spin model (L. (1994); F., K.(1995))

## 6 Pomeron in $N = 4$ SUSY

BFKL kernel in two loops (F., L. (1998))

$$\omega = 4\hat{a} \chi(n, \gamma) + 4\hat{a}^2 \Delta(n, \gamma), \quad \hat{a} = g^2 N_c / (16\pi^2),$$

Hermitian separability in  $N = 4$  SUSY (K.,L. (2000))

$$\Delta(n, \gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}, \quad M = \gamma + \frac{|n|}{2},$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2), \quad \beta'(z) = \frac{1}{4} \left[ \Psi'\left(\frac{z+1}{2}\right) - \Psi'\left(\frac{z}{2}\right) \right]$$

Maximal transcendentality (2002) and integrability for  $\gamma$  (L (1997))

$$\phi(M) = 3\zeta(3) + \Psi''(M) - 2\Phi(M) + 2\beta'(M) \left( \Psi(1) - \Psi(M) \right),$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M} \left( \Psi'(k+1) - \frac{\Psi(k+1) - \Psi(1)}{k+M} \right)$$



## 7 Elastic BDS amplitude

Regge asymptotics at  $s/t \rightarrow \infty$

$$M_{2 \rightarrow 2} = \Gamma(t) \left( \frac{-s}{\mu^2} \right)^{\omega(t)} \Gamma(t)$$

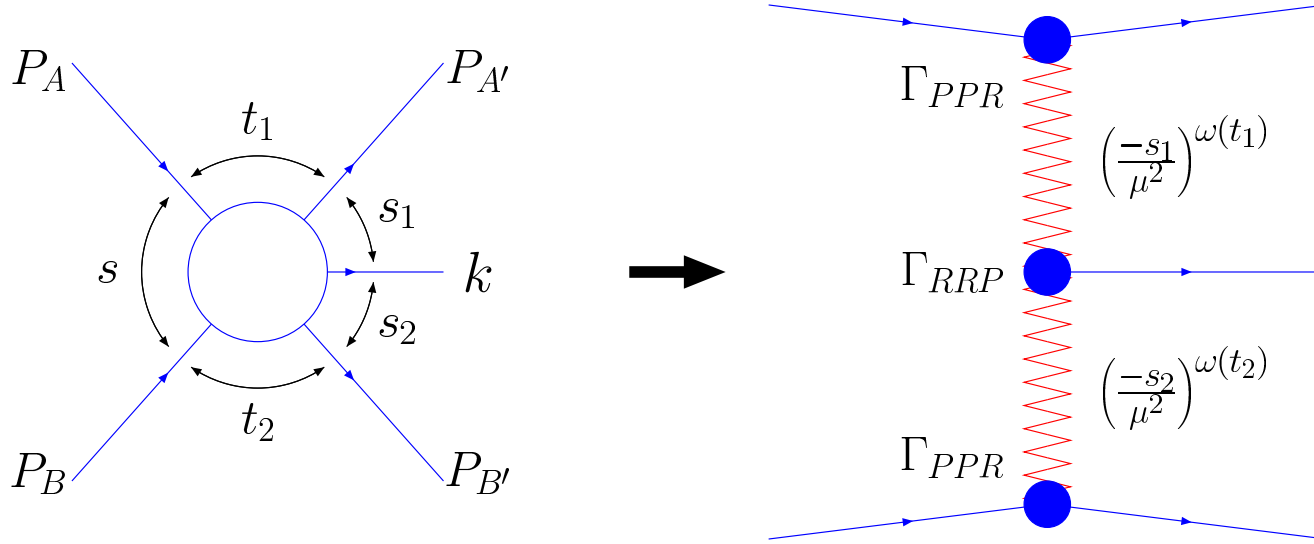
Reggeized gluon trajectory

$$\omega(t) = -\frac{\gamma_K(a)}{4} \ln \frac{-t}{\mu^2} + \int_0^a \frac{da'}{a'} \left( \frac{\gamma_K(a')}{4\epsilon} + \beta(a') \right)$$

Reggeon residues

$$\begin{aligned} \ln \Gamma(t) = & \ln \frac{-t}{\mu^2} \int_0^a \frac{da'}{a'} \left( \frac{\gamma_K(a')}{8\epsilon} + \frac{\beta(a')}{2} \right) + \frac{C(a)}{2} + \frac{\gamma_K(a)}{2} \zeta_2 \\ & - \int_0^a \frac{da'}{a'} \ln \frac{a}{a'} \left( \frac{\gamma_K(a')}{4\epsilon^2} + \frac{\beta(a')}{\epsilon} + \delta(a') \right) \end{aligned}$$

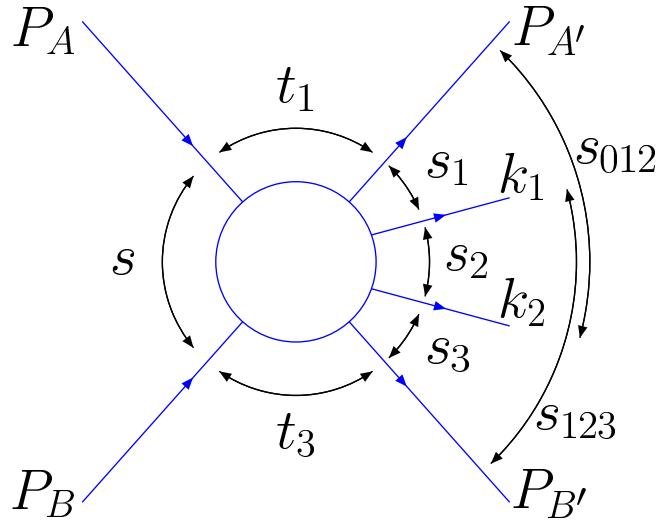
## 8 One particle production (BLS)



$$\ln \Gamma_{\kappa=s_1 s_2/s} = -\frac{1}{2} \left( \omega(t_1) + \omega(t_2) - \int_0^a \frac{da'}{a'} \left( \frac{\gamma_K(a')}{4\epsilon} + \beta(a') \right) \right) \ln \frac{-\kappa}{\mu^2} -$$

$$\frac{\gamma_K(a)}{16} \left( \ln^2 \frac{-\kappa}{\mu^2} - \ln^2 \frac{-t_1}{-t_2} - \zeta_2 \right) - \frac{1}{2} \int_0^a \frac{da'}{a'} \ln \frac{a}{a'} \left( \frac{\gamma_K(a')}{4\epsilon^2} + \frac{\beta(a')}{\epsilon} + \delta(a') \right)$$

## 9 Regge factorization violation



$$M_{2 \rightarrow 4} |_{s, s_2 > 0; s_1, s_3 < 0} = \exp \left[ \frac{\gamma_K(a)}{4} i\pi \left( \ln \frac{t_1 t_2}{(\vec{k}_1 + \vec{k}_2)^2 \mu^2} - \frac{1}{\epsilon} \right) \right]$$

$$\times \Gamma(t_1) \left( \frac{-s_1}{\mu^2} \right)^{\omega(t_1)} \Gamma(t_2, t_1) \left( \frac{-s_2}{\mu^2} \right)^{\omega(t_2)} \Gamma(t_3, t_2) \left( \frac{-s_3}{\mu^2} \right)^{\omega(t_3)} \Gamma(t_3)$$

# 10 Mandelstam cuts in $j_2$ -plane

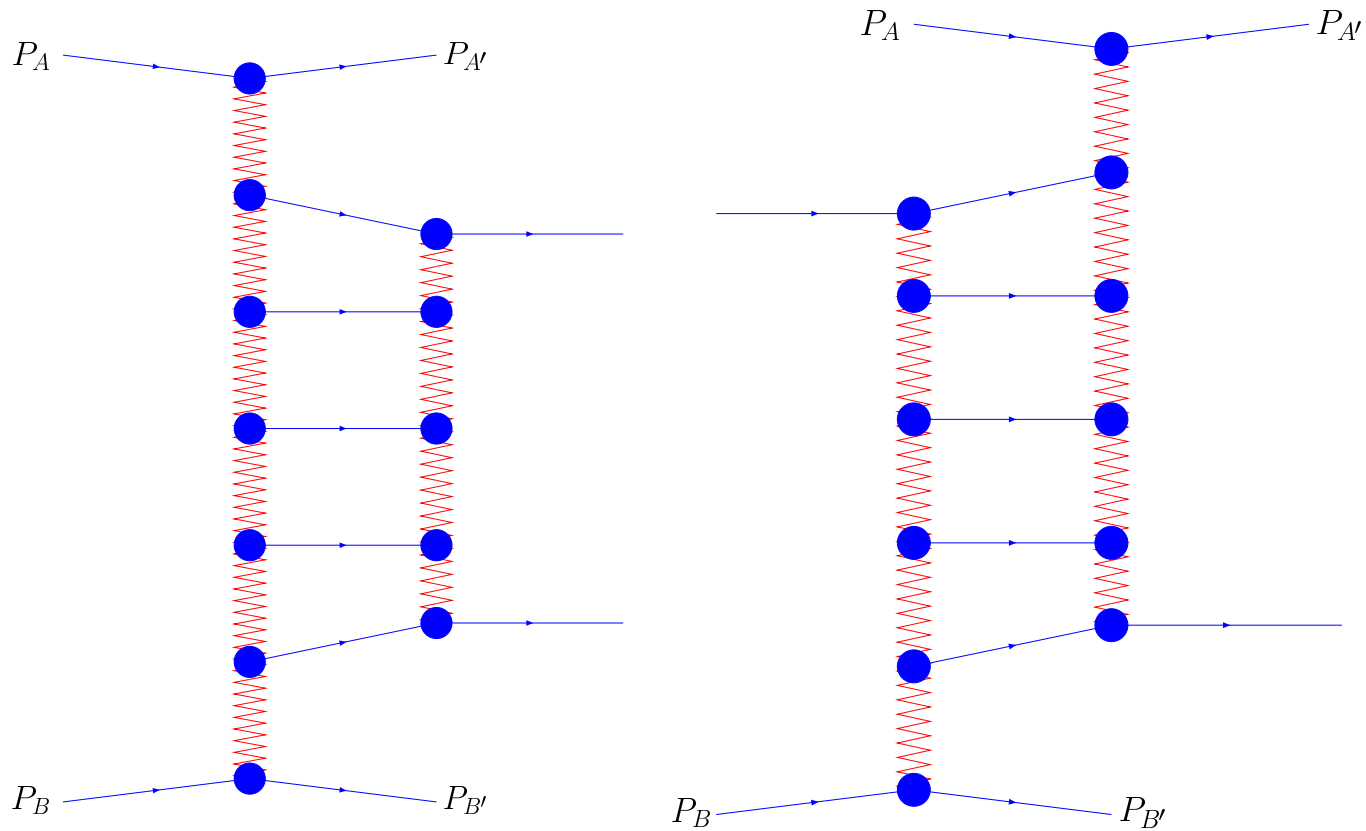


Figure 1: BFKL ladders in  $M_{2 \rightarrow 4}$  and  $M_{3 \rightarrow 3}$

# 11 BFKL equation for octets (BLS)

Regge singularities in the  $t_2$ -channel

$$\omega(t_2) = -a \left( E + \ln \frac{-t_2}{\mu^2} - \frac{1}{\epsilon} \right)$$

BFKL hamiltonian for the partial wave  $f_{j_2}$

$$H = \ln \frac{|p_1 p_2|^2}{|p_1 + p_2|^2} + \frac{1}{2} p_1 p_2^* \ln |\rho_{12}|^2 \frac{1}{p_1 p_2^*} + \frac{1}{2} p_1^* p_2 \ln |\rho_{12}|^2 \frac{1}{p_1^* p_2} + 2\gamma$$

Eigenfunctions and eigenvalues

$$\Psi_{n,\nu} = \left( \frac{p_1}{p_2} \right)^{i\nu+n/2} \left( \frac{p_1^*}{p_2^*} \right)^{i\nu-n/2}, \quad E_{n,\nu} = 2\text{Re} \psi(i\nu + \frac{|n|}{2}) - 2\psi(1)$$

Factorization of infrared divergencies in LLA

$$M_{|s,s_2>0;s_1,s_3<0}^{2\rightarrow 4} = (1 + i\Delta_{2\rightarrow 4}) M_{2\rightarrow 4}^{BDS},$$

## 12 Möbius and conformal invariances

Analytic result in LLA in the region  $a \ln s_2 \sim 1$

$$\Delta_{2 \rightarrow 4} = \frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} (V^*)^{i\nu - \frac{n}{2}} V^{i\nu + \frac{n}{2}} \left( s_2^{\omega(\nu, n)} - 1 \right)$$

Duality transformation to the Möbius representation

$$V = \frac{q_3 k_1}{k_2 q_1} \rightarrow \frac{z_{03} z_{0'1}}{z_{0'3} z_{01}}$$

Perturbation theory expansion

$$i\Delta_{2 \rightarrow 4} = -2i\pi a^2 \ln s_2 \ln \frac{|k_1 + k_2||q_2|}{|k_2||q_1|} \ln \frac{|k_1 + k_2||q_2|}{|k_1||q_3|} + \dots$$

Functions of 4-dimensional anharmonic ratios

$$i\Delta_{2 \rightarrow 4} = \frac{a^2}{4} Li_2(\chi) \ln \frac{\chi t_2 s_{13}}{s_3 t_1} \ln \frac{\chi t_2 s_{02}}{t_3 s_1} + \dots, \quad \chi = 1 - \frac{s s_2}{s_{012} s_{123}}$$

# 13 Multi-gluon states in octet channels

Channels with the Mandelstam cuts constructed from  $n$  gluons

$$s_1, s_2, \dots, s_{n-1}, s_{n+1}, \dots, s_{2n} < 0, s, s_n > 0$$

Schrödinger equation for octet composite states

$$H\Psi = E\Psi, \quad \omega(t) = a \left( -\ln \frac{-t}{\mu^2} + \frac{1}{\epsilon} \right) - \frac{a}{2}E, \quad a = \frac{g^2 N_c}{8\pi^2}$$

Holomorphic separability

$$H = h + h^*, \quad h = \ln \frac{p_1 p_n}{q^2} + \sum_{r=1}^{n-1} h_{r,r+1}^t$$

Helpful operator relation

$$\ln \partial = -\ln x + \frac{1}{2} (\psi(x\partial + 1) + \psi(-x\partial))$$

# 14 Möbius invariance in the $p$ -space

Duality transformation

$$p_k = Z_{k-1,k}, \quad \rho_{k,k+1} = i \frac{\partial}{\partial Z_k} = i \partial_k$$

Consequence of the Möbius invariance in  $Z$ -space

$$h = \ln(Z_1^2 \partial_1) - 2\psi(1) + \ln \partial_{n-1} + \sum_{k=1}^{n-2} h_{k,k+1}, \quad Z_0 = 0, \quad Z_n = \infty,$$

$$h_{1,2} = \ln(Z_{12}^2 \partial_1) + \ln(Z_{12}^2 \partial_2) - 2 \ln Z_{12} - 2\psi(1)$$

First integral of motion

$$A' = Z_1 \prod_{s=1}^{n-2} Z_{s,s+1} \prod_{r=1}^{n-1} \partial_r, \quad [h, A'] = 0$$



# 15 Integrable Heisenberg spin chain

Helpful identity

$$[L_k(u)L_{k+1}(u), h_{k,k+1}] = -i(L_k(u) - L_{k+1}(u))$$

Integrals of motion:  $[D, h] = 0$

$$D(u) = \sum_{k=0}^{n-1} u^{n-1-k} q'_k, \quad q'_k = - \sum_{0 < r_1 < \dots < r_k < n} Z_{r_1} \prod_{s=1}^{k-1} Z_{r_s, r_{s+1}} \prod_{t=1}^k i \partial_{r_t}$$

Sklyanin ansatz and Baxter equation

$$\Omega = \prod_k Q(\hat{u}_k) \Omega_0, \quad \Omega_0 = \prod_{l=1}^{n-1} \frac{1}{|Z_l|^4},$$

$$D(u)Q(u) = (u + i)^{n-1} Q(u + i)$$

# 16 Three-gluon composite state

Wave function in the coordinate representation

$$\Psi = Z_2^{a_1+a_2} (Z_2^*)^{\tilde{a}_1+\tilde{a}_2} \int \frac{d^2 y}{|y|^2} y^{-a_2} (y^*)^{\tilde{a}_2} \left( \frac{y-1}{y-Z_2/Z_1} \right)^{a_1} \left( \frac{y^*-1}{y^*-Z_2^*/Z_1^*} \right)^{\tilde{a}_1}$$

Fourier transformation

$$\Psi(\vec{Z}_1, \vec{Z}_2) = \int d^2 p_1 d^2 p_2 \exp(i\vec{p}_1 \vec{Z}_1) \exp(i\vec{p}_2 \vec{Z}_2) \Psi(\vec{p}_1, \vec{p}_2), \quad E = E(a_1) + E(a_2)$$

Baxter-Sklyanin representation

$$\Psi^t(\vec{p}_1, \vec{p}_2) = P^{-a_1-a_2} (P^*)^{-\tilde{a}_1-\tilde{a}_2} \int d^2 u u \tilde{u} Q(u, \tilde{u}) \left( \frac{p_1}{p_2} \right)^u \left( \frac{p_1^*}{p_2^*} \right)^{u^*}$$

Baxter function

$$Q(u, \tilde{u}) = \frac{\Gamma(-u) \Gamma(-\tilde{u})}{\Gamma(1+u) \Gamma(1+\tilde{u})} \frac{\Gamma(u-a_1) \Gamma(u-a_2)}{\Gamma(1-\tilde{u}+\tilde{a}_1) \Gamma(1-\tilde{u}+\tilde{a}_2)}, \quad \int d^2 u = \int d\nu \sum_n$$

# 17 Discussion

1. Multi-Regge amplitudes and bootstrap relations.
2. Integrability of BFKL dynamics in LLA.
3. Remarkable properties of NLLA in  $N = 4$  SUSY.
4. BDS amplitudes in the multi-Regge kinematics.
5. Breakdown of the Regge factorization.
6. Mandelstam cuts in the planar amplitudes  $M_n$  for  $n > 5$ .
7. Integrable open spin chain for scattering amplitudes.