Notes on pure spinors in AdS.

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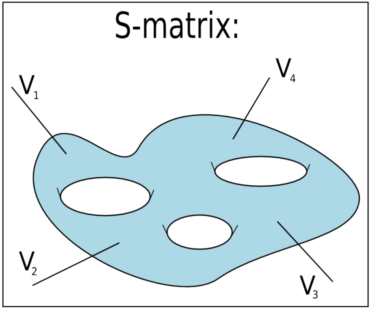
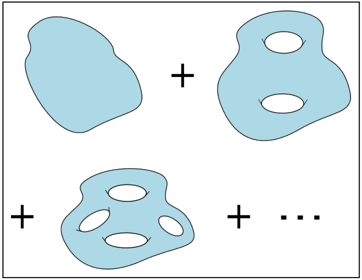
Pure spinor formalism

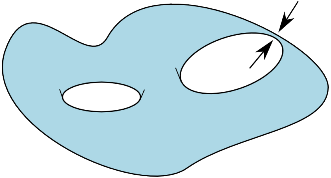
Pure spinor formalism was invented to quantize the wordlsheet theory of the superstring, and for the multiloop calculations.

In string theory, we sum over the Riemann surfaces:





But we face divergencies:



It is one of the basic principles of the string theory, that there should be no divergencies.

The most elegant way to cancel this divergence is to introduce fermions.

The most straightforward way to introduce fermions is called NSR formalism.

NSR superstring

The fermions are introduced in such a way that the conformal symmetry of the worldsheet gets extended to the superconformal symmetry:

$$S = \frac{1}{4\pi\alpha'} \int d\tau^+ d\tau^- \left[\frac{\partial X^\mu}{\partial \tau^+} \frac{\partial X_\mu}{\partial \tau^-} - i\overline{\psi}^\mu \left(\gamma^+ \frac{\partial}{\partial \tau^+} + \gamma^- \frac{\partial}{\partial \tau^-} \right) \psi_\mu \right]$$

This is difficult to deal with, for at least 2 reasons:

- RR field
- summation over spin structures

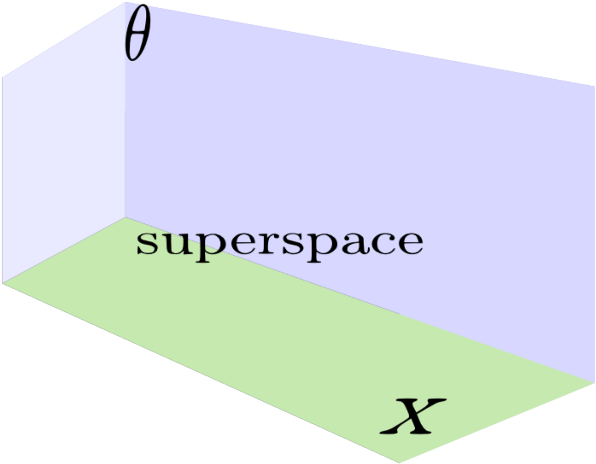
GS superstring

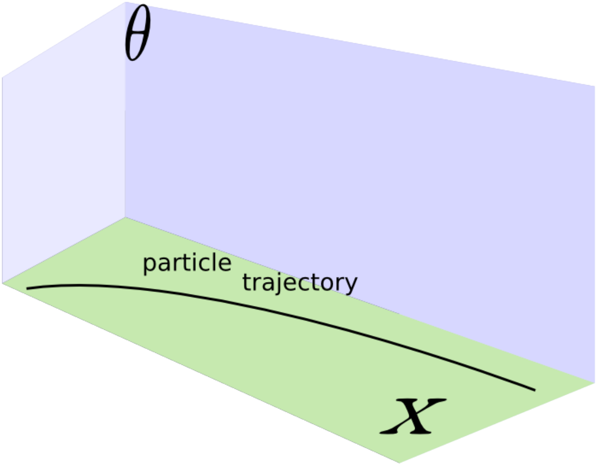
Another approach is the Green-Schwarz approach:

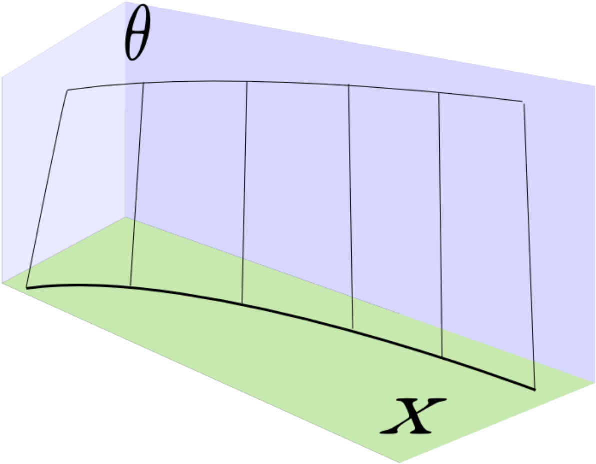
$$S = \frac{1}{4\pi\alpha'} \int d\tau^+ d\tau^- \left[\frac{\partial X^\mu}{\partial \tau^+} \frac{\partial X_\mu}{\partial \tau^-} + \left(\bar{\theta} \frac{\widehat{\partial x}}{\partial \tau^+} \frac{\partial}{\partial \tau^-} \theta \right) + \ldots \right] \quad (1)$$

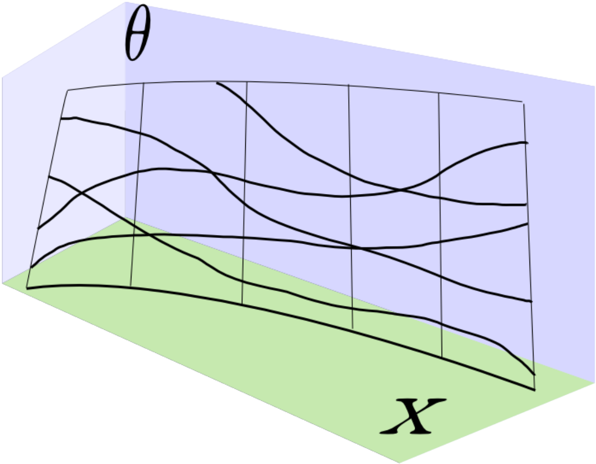
Manifest target space supersymmetry, but difficult to quantize:

- Already the leading (kinetic) term for θ is cubic...
- Difficulty with κ-symmetry


θ
superspace
x


θ
particle trajectory
x


θ
x


θ
x

is: **gauge fix** this κ-symmetry (light cone quantization, usually) and then calculate Feynman diagramms in this gauge fixed theory. But:

- the gauge fixed theory is **ugly**, and already a 2-loop calculation becomes very difficult.
- after fixing κ-symmetry the string ws theory becomes “a generic 2d quantum field theory, not even Lorentz-invariant”. If presented with such a theory, you would never tell (if you didn’t know) that it came from a string theory.

Pure spinor formalism

Starts from the **quadratic** action:

$$S = \frac{1}{4\pi\alpha'}\int d\tau^+ d\tau^- \left[\frac{\partial X^\mu}{\partial\tau^+}\frac{\partial X_\mu}{\partial\tau^-} + p_{\alpha+}\partial_-\theta^\alpha + w_{\alpha+}\partial_-\lambda^\alpha + \right.$$
$$\left. + \tilde{p}_{\alpha-}\partial_+\tilde{\theta}^\alpha + \tilde{w}_{\alpha-}\partial_+\tilde{\lambda}^\alpha\right] \quad (2)$$

In this action λ is restricted to:
$\lambda^\alpha\Gamma^m_{\alpha\beta}\lambda^\beta = 0$ and similar constraint on $\tilde{\lambda}$

But
this is a constraint on coordinates in the **first order action.** This does not make the theory interacting, physically. The quadratic constraints in this context lead to **subtleties**, rather than **difficulties**.

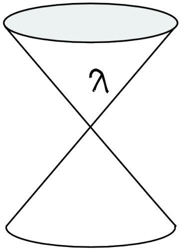
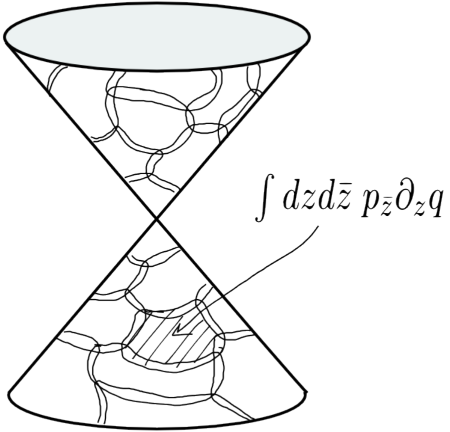


Figure: pure spinor cone

Darboux coordinates



BRST structure

The Berkovits action comes with the BRST structure:

$$Q = Q_L + Q_R \tag{3}$$

$$Q_L = \int d\tau^+ d_{\alpha+} \lambda^\alpha \tag{4}$$

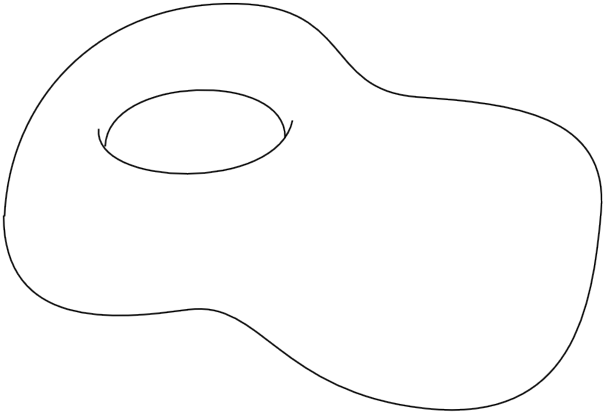
$$Q_R = \int d\tau^- \tilde{d}_{\alpha-} \tilde{\lambda}^\alpha \tag{5}$$

Structure of topological field theory:

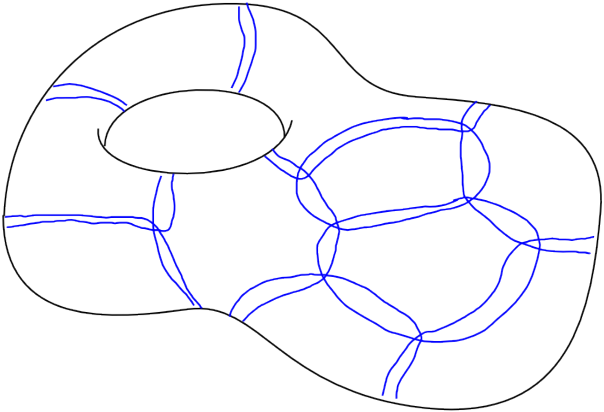
$$T_{++} = \{Q, b_{++}\}$$
$$T_{--} = \{Q, b_{--}\}$$

The *b*-ghost is needed to define the integration measure, needed to integrate over the string worldsheet shapes.

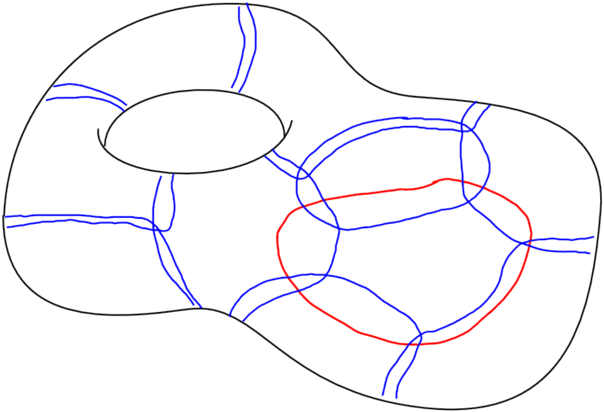
b-ghost



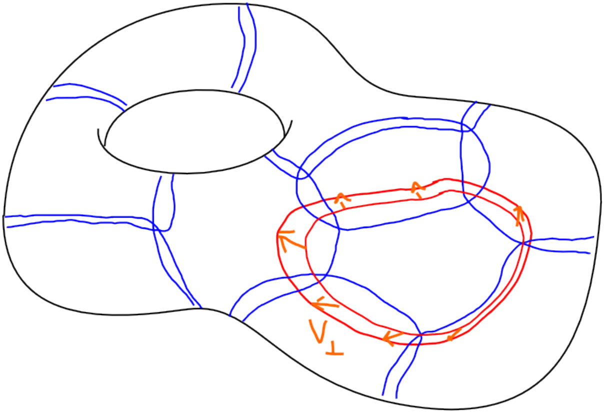
b-ghost



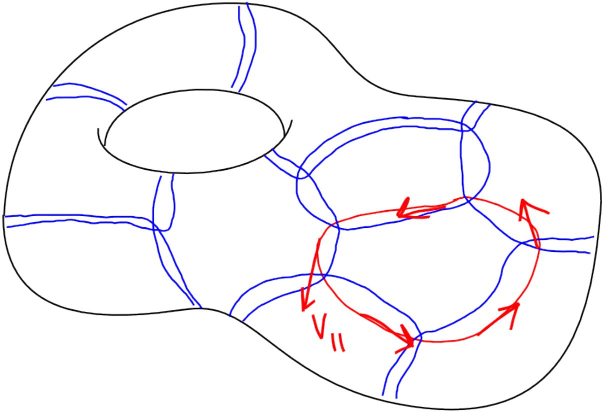
b-ghost



b-ghost

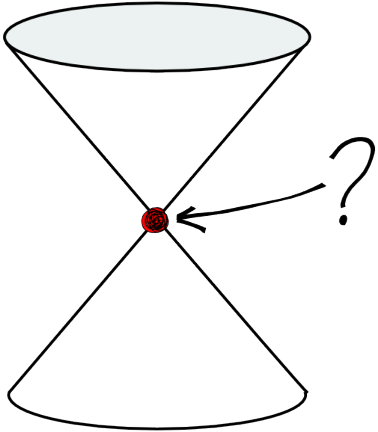


b-ghost



$$\left\langle \cdots \int b_{\alpha\beta} v_{||}^{\alpha} v_{\perp}^{\beta} \cdots \right\rangle$$

Origin of the cone



Applications

multiloop calculations

Progress in the multiloop calculations using the pure spinor approach:

- proof of vanishing of massless N-particle g-loop amplitudes for $N < 4$ and $g > 0$; this implies finiteness of higher loop amplitudes in string theory!
- calculation of $2 \to 2$ scattering at 2 loops for bosonic and fermionic massless states

Applications

AdS/CFT

Pure spinor formalism is good for backgrounds with lots of SUSY. Flat space is a background with the maximal amount of SUSY. Another example is $AdS_5 \times S^5$. The pure spinor formalism so far has not been very efficient in calculating things. But it has been very useful for proving things:

- The worldsheet σ-model is UV finite at all orders of the perturbative expansion in α'
- There are infinitely many nonlocal conserved charges, which are all well-defined in the **quantum** theory perturbatively to all orders in α'

Also, a remarkable equivalent formulation was found, in terms of $N = 2$ **worldsheet supersymmetric** A-model.

But: can we calculate anything useful, using the pure spinor formalism?

Plan

1. Nonlocal conserved charges **a.k.a.** transfer matrix **a.k.a.** Wilson lines
2. Fusion of Wilson lines with defects
3. BRST cohomology; massless vertex operators

Superstring on $AdS_5 \times S^5$

coset space

Bosonic and fermionic degrees of freedom are related by the supersymmetry. The supersymmetry group is $PSU(2,2|4)$. Supersymmetric $AdS_5 \times S^5$ can be identified with the coset space:

$$\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$$

It is natural to build the string worldsheet action based on the left-invariant currents:

$$J = -dgg^{-1}$$

Here $g \in PSU(2,2|4)$ is defined up to a gauge transformation:

$$g \simeq hg \; , \quad h \in SO(1,4) \times SO(5) \tag{6}$$

Superstring on $AdS_5 \times S^5$

$\mathbf{Z}_4$ invariance

For constructing the action it is crucial that $psu(2,2|4)$ has a $\mathbf{Z}_4$ grading:

$$psu(2,2|4) = \mathbf{g} = \mathbf{g}_{\bar{0}} + \mathbf{g}_{\bar{1}} + \mathbf{g}_{\bar{2}} + \mathbf{g}_{\bar{3}}$$

The current splits into the components:

$$J = J_{\bar{0}} + J_{\bar{1}} + J_{\bar{2}} + J_{\bar{3}}$$

The gauge invariance:

$$\delta_\omega J = -d\omega + [\omega, J] \quad , \quad \omega \in \mathbf{g}_{\bar{0}}$$

Pure spinor string in $AdS_5 \times S^5$

pure spinor ghosts

There are two pure spinor ghosts, one taking values in $\mathbf{g}_{\bar{3}}$ and another in $\mathbf{g}_{\bar{1}}$:

$$\lambda_3 \in \mathbf{g}_{\bar{3}} \quad , \quad \lambda_1 \in \mathbf{g}_{\bar{1}}$$

In the flat space limit λ_3 would be left-moving on the worldsheet, and λ_1 would be right-moving.

The "purity" condition is:

$$\{\lambda_1, \lambda_1\} = \{\lambda_3, \lambda_3\} = 0$$

Pure spinor string in $AdS_5 \times S^5$

(N. Berkovits)

The action:

$$S = \frac{R^2}{\pi} \int d^2z \, \mathrm{Str}\Big(\tfrac{1}{2} J_{2+} J_{2-} + \tfrac{3}{4} J_{1+} J_{3-} + \tfrac{1}{4} J_{3+} J_{1-} + w_{1+} \partial_- \lambda_3 + w_{3-} \partial_+ \lambda_1 + N_{0+} J_{0-} + N_{0-} J_{0+} - N_{0+} N_{0-} \Big) ,$$

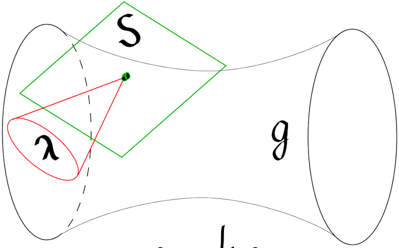
$$N_{0+} = -\{w_{1+}, \lambda_3\} , \qquad N_{0-} = -\{w_{3-}, \lambda_1\} ,$$

The BRST symmetry:

$$Q_{BRST}\, g = (\lambda_3 + \lambda_1) g$$

notice $(\lambda_3 + \lambda_1)^2 = \{\lambda_1, \lambda_3\} \in \mathbf{g_0}$ — gauge transformation

Geometry of pure spinor bundle over AdS



$$g \sim hg$$

$$Qg = \lambda g$$

Classical transfer matrix

(N. Berkovits, B.C. Vallilo)

$$T_\rho[C] = P\exp\left(-\int_C J^a e_a\right) . \tag{7}$$

where e_a are generators of the twisted loop algebra $L\mathbf{psu}(\mathbf{2},\mathbf{2}|\mathbf{4})$ and $J^a e_a$ stands for:

$$\begin{aligned} J_+ &= (J_{0+}^{[\mu\nu]} - N_{0+}^{[\mu\nu]})e^0_{[\mu\nu]} + J^\alpha_{3+}e^{-1}_\alpha + J^\mu_{2+}e^{-2}_\mu + J^{\dot\alpha}_{1+}e^{-3}_{\dot\alpha} + N^{[\mu\nu]}_{0+}e^{-4}_{[\mu\nu]} \\ J_- &= (J_{0-}^{[\mu\nu]} - N_{0-}^{[\mu\nu]})e^0_{[\mu\nu]} + J^\alpha_{1-}e^{1}_{\dot\alpha} + J^\mu_{2-}e^{2}_\mu + J^{\dot\alpha}_{3-}e^{3}_{\alpha} + N^{[\mu\nu]}_{0-}e^{4}_{[\mu\nu]} . \end{aligned}$$

Here $e^m_a = z^m t_a$ where z is the spectral parameter:

$$e^{-3}_\alpha = z^{-3}t^1_\alpha, \quad e^{-2}_\mu = z^{-2}t^2_\mu, \quad e^1_\alpha = zt^1_\alpha \quad etc.$$

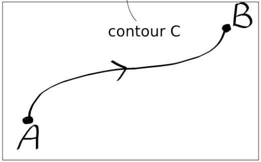
Classical equations of motion are equivalent to the vanishing of the curvature of the Lax connection:

$$[\partial_+ + J_+(z)\ ,\ \partial_- + J_-(z)] = 0 \tag{8}$$

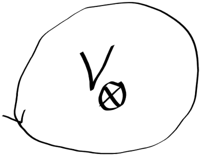
Classical transfer matrix

$$T_\rho[C] = P\exp\left(-\int_C J^a e_a\right)$$

representation of twisted loop algebra

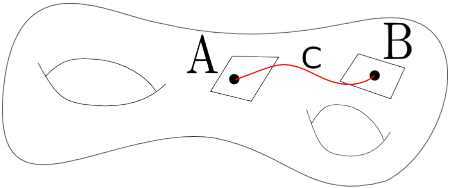


Conserved charges of operator

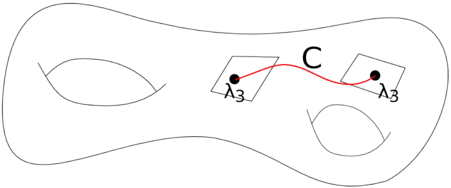


$\mathrm{Tr}\, P \exp \int J^a e_a$ gives conserved charges

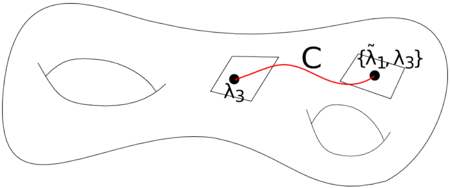
Transfer matrix



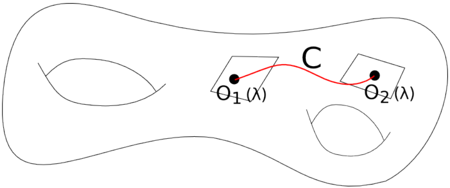
Transfer matrix



Transfer matrix

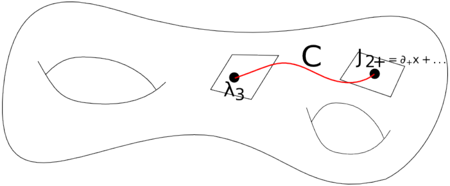


Transfer matrix



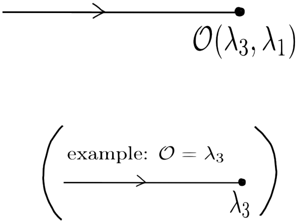
Transfer matrix

We can also introduce matter fields:



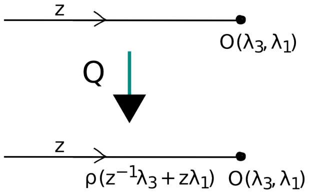
but the symmetries require that they enter only through the derivatives (currents) therefore this would results in operators of higher engineering dimension.

Wilson line with a gauge-invariant endpoint



Endpoint BRST complex

It turns out that the BRST transformations act on the endpoint in the following way:



Endpoint BRST complex

More precisely: we have a representation ρ, of $psu(2,2|4)$. This means that the generators of $psu(2,2|4)$ act on some space $\mathcal{H}_\rho$. This representation can be finite dimensional or infinite dimensional. The BRST complex will be realized on the space of functions (perhaps polynomials) $\mathcal{O}(\lambda_3, \lambda_1)$ invariant under the action of $so(1,4) \oplus so(5) \subset psu(2,2|4)$:

$$\omega \in so(1,4) \oplus so(5) \quad \Rightarrow \quad \rho(\omega)\mathcal{O}(\lambda_3, \lambda_1) = \left.\frac{d}{dt}\right|_{t=0} \mathcal{O}(e^{t\omega}\lambda_3, \lambda_1) + \left.\frac{d}{dt}\right|_{t=0}$$

Wilson line with a gauge-invariant endpoint

Examples of gauge invariant $\mathcal{O}$ in the adjoint representation:

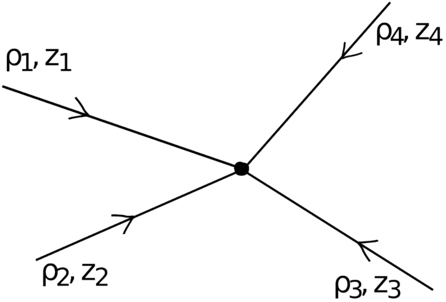
$$\lambda_3$$

$$\{\lambda_3, \lambda_1\}$$

$$C^{\alpha\dot{\alpha}}[\{t^3_\alpha, \lambda_3\}, \{t^1_{\dot{\alpha}}, \lambda_1\}]$$

$$\ldots$$

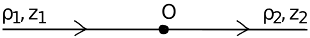
We could also take the ternsor product of several representations:



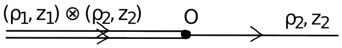
Classically this object is independent of the angles, but quantum mechanically already at the level of one loop it will start depending on the angles. (Somewhat related to "non-ultralocality".) This effect is polynomial in the spectral parameters, but when two legs cross we get rational functions with denominators like $\frac{1}{z_i^4 - z_j^4}$.

A way to understand these denominators is to consider the **fusion**.

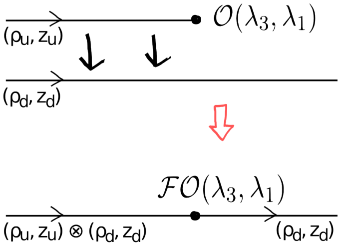
Representation-changing operator



Representation-changing operator



Fusion of endpoint



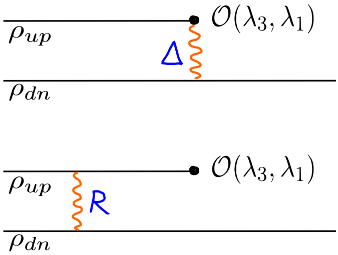
In the classical theory we would have

$$\mathcal{F}\mathcal{O} = \mathcal{O} \otimes \mathbf{1} \in (\rho_{up} \otimes \rho_{dn}) \otimes \rho^*_{dn}$$

Fusion of endpoint

In the quantum theory we get something like this:

$$\mathcal{F}\mathcal{O} = R\Delta\mathcal{O} = \mathcal{O}\otimes\mathbf{1} + \delta(\mathcal{O}) + r_+(\mathcal{O}\otimes\mathbf{1})$$



where the matrix r_+ is:

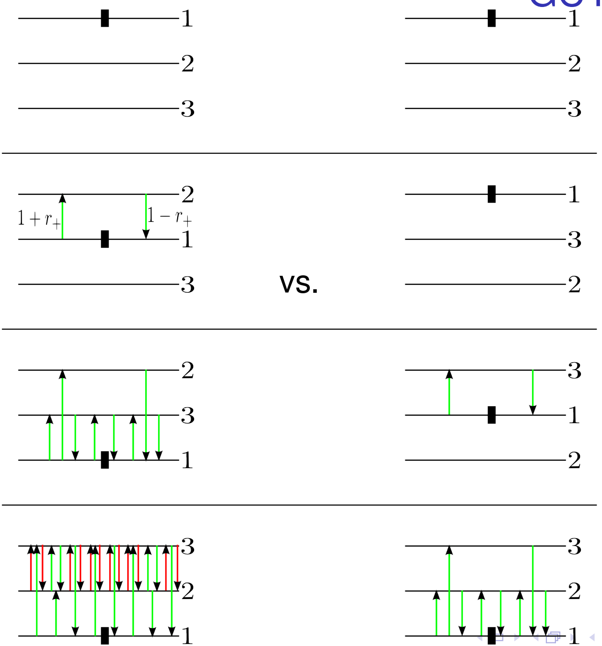
$$\left.\frac{r+s}{2}\right|_{\rho_u\otimes\rho_d} = \frac{1}{z_u^4-z_d^4}\left[\left(z_d^2-\frac{1}{z_d^2}\right)^2\left(z_u z_d^3\, t^1\otimes t^3 + \right.\right.$$

$$+ z_u^2 z_d^2\, t^2\otimes t^2 +$$

$$\left. + z_u^3 z_d\, t^3\otimes t^1 \right) +$$

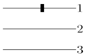
$$\left. +z_u^2 z_d^2\left(z_u^2-\frac{1}{z_u^2}\right)\left(z_d^2-\frac{1}{z_d^2}\right) t^0\otimes t^0\right]$$

GCYBE



GCYBE

$$\Big[\; [r_{23,-}, r_{13,+}] + [r_{13,+}, r_{12,+}] + [r_{23,+}, r_{12,+}] \;,$$



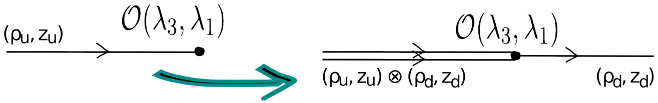
$$\Big] = 0$$

$$[r_{23,-}, r_{13,+}] + [r_{13,+}, r_{12,+}] + [r_{23,+}, r_{12,+}] =$$

$$\begin{aligned} = \quad & t^0 \otimes (z_2^2 - z_2^{-2}) t^2 \otimes (z_3^2 - z_3^{-2}) [t^0, t^2] + \\ & + t^0 \otimes (z_2 - z_2^{-3}) t^1 \otimes (z_3^3 - z_3^{-1}) [t^0, t^3] + \\ & + t^0 \otimes (z_2^3 - z_2^{-1}) t^3 \otimes (z_3 - z_3^{-3}) [t^0, t^1] \end{aligned}$$

Yang-Baxter up to a gauge transformation.

- there is a very simple derivation of the r/s-matrices based on BRST invariance
- fusion is the map of the endpoint BRST complexes:



It is probably true that this map acts as classical on cohomologies.
Q: What are the cohomologies of the endpoint complex?
I do not know the full answer, but I know this cohomology is related to the physical vertex operators in $AdS_5 \times S^5$.

Vertex operators

Vertex operators are what corresponds to the physical states in the pure spinor formalism. The massless vertices are easier to understand for the heterotic string, which has gauge fields in the spectrum:

$$V(x,\theta,\lambda) = (\theta,\Gamma^m\lambda)a_m(x) + (\theta,\Gamma^m\lambda)(\theta\Gamma_m\psi(x)) + \ldots \tag{9}$$

where $\ldots$ denote higher powers in θ. These are annihilated by the BRST operator:

$$Q_L = \lambda^\alpha\left(\frac{\partial}{\partial\theta^\alpha} + \Gamma^m_{\alpha\beta}\theta^\beta\frac{\partial}{\partial x^m}\right) \tag{10}$$

— where the diff. operator is defined so that it commutes with the supersymmetries.

Vertex operators

It turns out that in some sense the vertices in AdS space are easier to understand than the vertices in flat space.

I will explain this using an example.

Example: β-deformation

Pick a constant $B^{ab} \in \Lambda^2 \mathbf{psu}(2,2|4)$:

$$V = B^{ab} \left(g^{-1}(\lambda_3 - \lambda_1)g\right)_a \left(g^{-1}(\lambda_3 - \lambda_1)g\right)_b \tag{11}$$

It turns out:

- it is annihilated by the BRST operator
- if $B^{ab} = f^{ab}{}_c v^c$ then it is BRST-exact, otherwize it is not

This means in other words that the β-deformation lives in the complement to the adjoint representation in the antisymmetric product of two adjoint representations.

Beta-deformation

I can tautologically rewrite V in the following way:

$$V_{ab} = \left(\rho_{\mathrm{ad}\otimes\mathrm{ad}}(g^{-1}) \left((\lambda_3 - \lambda_1) \otimes (\lambda_3 - \lambda_1) \right) \right)_{ab} \quad (12)$$

Observe that $(\lambda_3 - \lambda_1) \otimes (\lambda_3 - \lambda_1)$ is a λ-dependent vector in the tensor product of two adjoint representations of $\mathbf{psu}(2,2|4)$. I can "generalize" this vector to depend on z, just by rescaling:

$$(z^{-1}\lambda_3 - z\lambda_1) \otimes (z^{-1}\lambda_3 - z\lambda_1) \quad (13)$$

Now I can put it at the endpoint of the Wilson line, and I am claiming that (13) is **a nontrivial class of the endpoint cohomology**. We observe:

- matter fields (g) are separated from ghosts (λ)
- vertex operator is related to the endpoint cohomology at the ghost nubmer 2

Covariant vertex

Explanation: It turns out that in $AdS_5 \times S^5$ it is possible to construct the vertex operators so that they transform **covariantly** under $\mathbf{psu}(2,2|4)$.

Vertex operator depends on the state:

$$V_\Psi(x,\theta,\lambda)\ , \quad \Psi \in \mathcal{H}$$

Vertex operators are defined modulo adding something Q-exact, so there is some ambiguity in their definition. Generally speaking, the vertex does transforms covariantly only up to BRST-exact piece:

$$V_{g_0\Psi}(x,\theta,\lambda) = V_\Psi(g_0(x,\theta,\lambda)) + Q(\text{smth})$$

In $AdS_5 \times S^5$ it is possible to define V_Ψ so that:

$$V_{g_0\Psi}(x,\theta,\lambda) = V_\Psi(g_0(x,\theta,\lambda))$$

(while in flat space there is a cohomological obstacle to that).