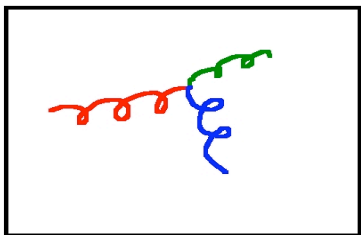
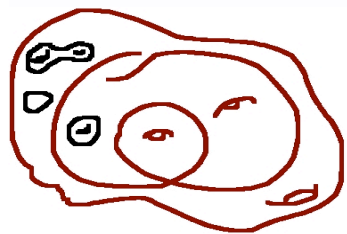
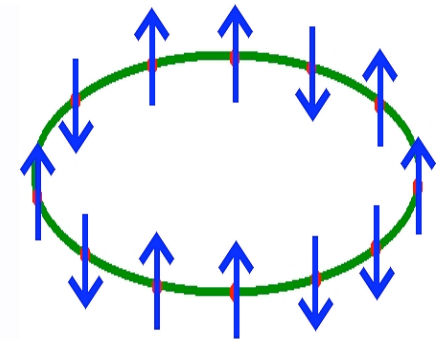
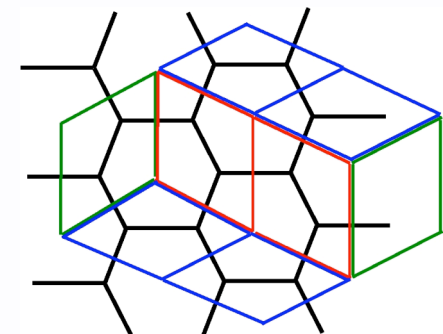


Supersymmetric Gauge Theories and Quantization of Integrable Systems



Nikita Nekrasov
(IHES, ITEP)

Euler Mathematical Institute
symposium
July 5 2009



Based on

NN, S.Shatashvili,

arXiv:0901.4744, arXiv:0901.4748 + to appear ;

Earlier work

G.Moore, NN, S.Shatashvili.,

arXiv:hep-th/9712241 ;

A.Gerasimov, S.Shatashvili.

arXiv:0711.1472, arXiv:hep-th/0609024

In the past few years
a connection
between two seemingly
different subjects
was found:

*The supersymmetric
gauge
(and perhaps more exotic)
theories*

*With as little as 4 supersymmetries
($N=1$ $d=4$)*

on the one hand

and

Quantum integrable systems

on the other hand

and

Quantum integrable systems
soluble by Bethe Ansatz

on the other hand

The more precise slogan is:

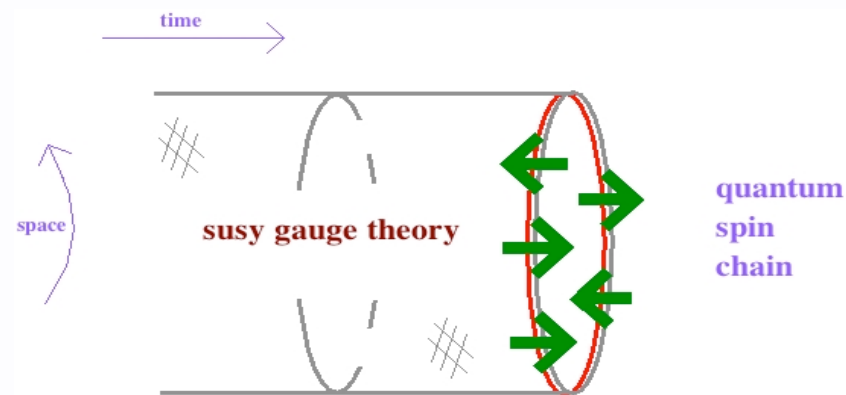
The **supersymmetric vacua** of
gauge theory

(possibly, in finite volume)

are **the eigenstates** of a
quantum integrable system

The correspondence

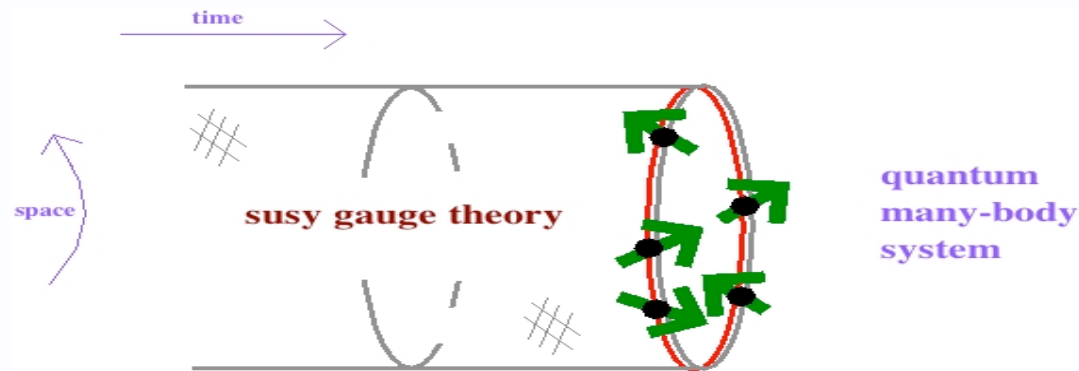
The **supersymmetric vacua** of
gauge theory



are **the eigenstates** of a
quantum integrable system

The correspondence

The **supersymmetric vacua** of
gauge theory



are **the eigenstates** of a
quantum integrable system

The correspondence

The « **twisted chiral ring** » operators

$$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \dots, \mathcal{O}_n$$

map to **quantum Hamiltonians**

$$H_1, H_2, H_3, \dots, H_n$$

More precise correspondence

The vacuum expectation values of the **twisted chiral ring** operators

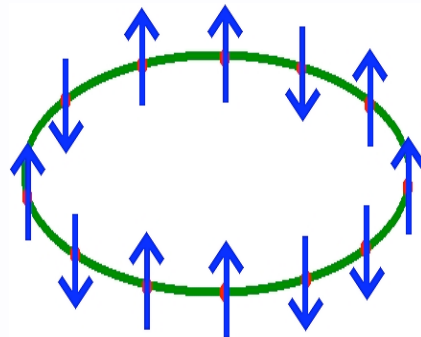
$$E_k(\lambda) = \langle \lambda | \mathcal{O}_k | \lambda \rangle$$

map to the **energies** and other **eigenvalues** on the integrable side

$$H_k \Psi_\lambda = E_k(\lambda) \Psi_\lambda$$

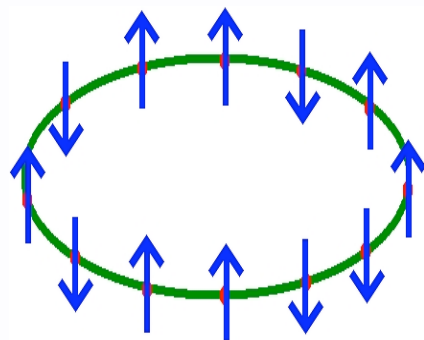
For example,

the XXX Heisenberg magnet:
the spin $1/2$ $SU(2)$
isotropic spin chain of
length L

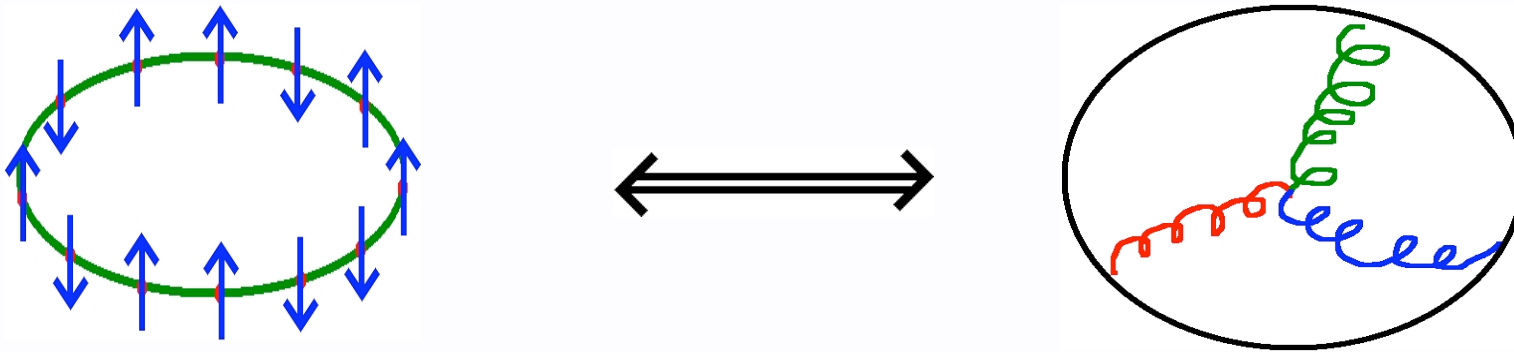


For example,

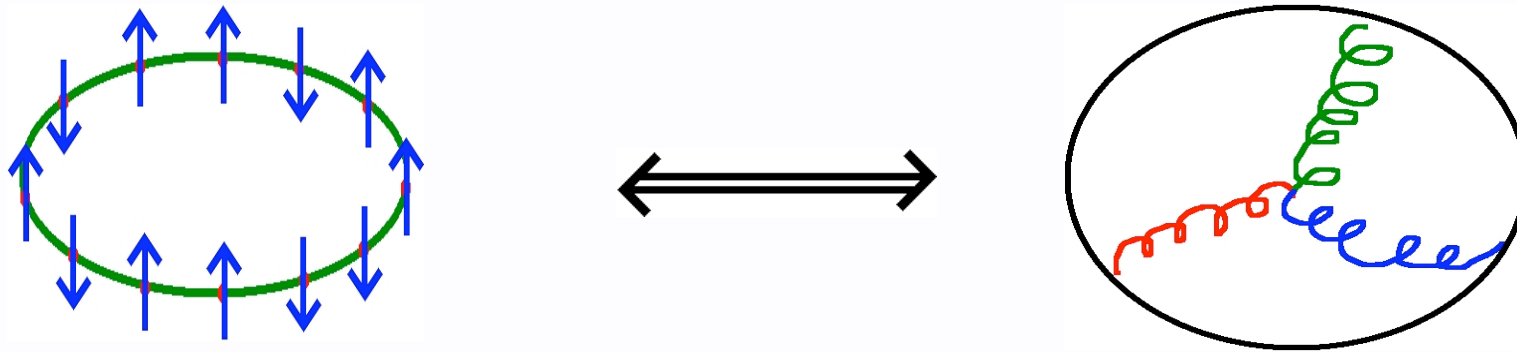
the XXX Heisenberg magnet
in the sector with N spins
up



Maps to the
 $d=2$ $U(N)$
 $N=4$ gauge theory with L
fundamental
hypermultiplets



With the $N=4$ susy softly
broken down to $N=2$
by the generic
twisted mass terms



Reminder on
 $N=2$ supersymmetry in two
dimensions: Lagrangians

There are two types of special
Potential terms:

The F-terms

And the Twisted F-terms

Reminder on $N=2$ supersymmetry in two dimensions: Lagrangians

Fayet-Iliopoulos and theta terms

$$\tau = ir + \theta/2\pi.$$

Give an example of
the twisted superpotential coupling

Reminder on
 $N=2$ supersymmetry in two
dimensions: Lagrangians

Twisted superpotential terms

$$= \int d^2y \left(\sqrt{2} \widetilde{W}'(\sigma) (D - i v_{01}) + 2 \widetilde{W}''(\sigma) \bar{\lambda}_+ \lambda_- \right) + h.c.$$

Reminder on $N=2$ supersymmetry in two dimensions: Lagrangians

Twisted mass terms

$$\mathcal{L}_{\text{mass}} = \int d^4\theta \operatorname{tr} \left(\Phi^\dagger e^{\tilde{V}} \otimes \operatorname{Id}_{\text{color space}} \Phi \right)$$

$$\tilde{V} = \tilde{m} \theta_+ \bar{\theta}_-$$

- Background vector field for global symmetry

Reminder on $N=2$ supersymmetry in two dimensions: Lagrangians

Cf. the ordinary mass terms

$$\mathcal{L}_{\text{mass}} = \sum_{i, \tilde{j}} \int d^2\theta \, m_i^{\tilde{j}} \tilde{Q}_{\tilde{j}} Q^i + \text{h.c.},$$

Which are just the superpotential terms

Unbroken global symmetries

Allow the deformations by the

Twisted masses

Unbroken global symmetries

allow the deformations by the

Twisted masses

Alvarez-Gaume, Freedman
Gates, Hull, Rocek
Hanany-Hori

**Unbroken global
symmetries,
not the R-symmetry**
*allow the susy-preserving
deformations by the*

Twisted masses

Twisted masses

Background vector fields for global symmetry

$$\mathcal{L}_{\widetilde{\text{mass}}} = \int d^4\theta \text{tr} \left(\Phi^\dagger e^{\widetilde{V}} \otimes \text{Id}_{\text{color space}} \Phi \right)$$

where $\widetilde{V} = \widetilde{m} \theta_+ \bar{\theta}_-$, \widetilde{m} acts in a flavour space, and, to preserve susy:

$$[\widetilde{m}, \widetilde{m}^*] = 0$$

General strategy

Take an $\mathcal{N}=2$ $d=2$ gauge theory with
matter,

in some representations \mathbf{R}_f
of the gauge group \mathbf{G}

integrate out the massive matter
fields

Study the effective theory

General strategy

*compute
the effective*

twisted super-potential
on the Coulomb branch

$$W_{\text{eff}} = \sum_f \text{Tr}_{R_f} (\sigma + m_f) (\log (\sigma + m_f) - 1) + 2\pi i \tau \text{Tr} \sigma$$

General strategy

*compute
the effective*

twisted super-potential
on the Coulomb branch

$$W_{\text{eff}} = \sum_f \text{Tr}_{R_f} (\sigma + m_f) (\log (\sigma + m_f) - 1) + 2\pi i \tau \text{Tr} \sigma$$

Plus the non-perturbative corrections

General strategy

*compute
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twisted super-potential
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$$W_{\text{eff}} = \sum_f \text{Tr}_{R_f} (\sigma + m_f) (\log (\sigma + m_f) - 1) + 2\pi i \tau \text{Tr} \sigma$$

*Plus the non-perturbative corrections
If the theory is secretly **d>2***

Vacua of the gauge theory

For $\mathbf{G} = \mathbf{U}(N)$

$$\sigma \rightarrow \text{diag}(\sigma_1, \dots, \sigma_N)$$

$$\frac{\partial W_{\text{eff}}}{\partial \sigma_i} = \lambda_i - i + \frac{1}{2}(N+1)$$

Due to quantization of the gauge flux

$$\lambda_i \in \mathbf{Z}$$

A typical example

Field content

Gauge group: $G=U(N)$ $\sigma \rightarrow \text{diag}(\sigma_1, \dots, \sigma_N)$

Matter *chiral multiplets*:

1	<u>Adjoint</u>	twisted mass	M
N_f	<u>fundamentals</u>	... mass	m_f
N_f	<u>anti-fundamentals</u>	... mass	$m_{\bar{f}}$

Effective twisted superpotential:

$$\begin{aligned} & N_f (\sigma_i + m_f) (\log (\sigma_i + m_f) - 1) + \\ & N_{\bar{f}} (-\sigma_i + m_{\bar{f}}) (\log (-\sigma_i + m_{\bar{f}}) - 1) + \\ & \sum_{i,j} (\sigma_i - \sigma_j + M) (\log (\sigma_i - \sigma_j + M) - 1) \\ & + (N_{\bar{f}} - N_f) \log \Lambda \sum_i \sigma_i \end{aligned}$$

Equations for vacua

$$\Lambda^{N_{\bar{f}}-N_f} \frac{\left(\sigma_i + m_f\right)^{N_f}}{\left(-\sigma_i + m_{\bar{f}}\right)^{N_{\bar{f}}}} = \prod_{j \neq i} \frac{\sigma_i - \sigma_j + M}{\sigma_i - \sigma_j + M}$$

Non-anomalous, UN finite case

$$N_f = N_{\overline{f}} = L$$

Non-anomalous, UV finite case

*Can be related to the
Sigma model on $T^*\text{Gr}(N, L)$*

The twisted chiral ring
=
equivariant quantum cohomology

Non-anomalous, UV finite case

$$\sigma_j = \frac{1}{2} \left(m_{\bar{f}} - m_f \right) - iM\lambda_j$$

Redefine:

$$\frac{1}{2} \left(m_{\bar{f}} + m_f \right) = Ms$$

Vacua of gauge theory

$$\left(\frac{\lambda_i + is}{\lambda_i - is} \right)^L = \prod_{j \neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}$$

Vacua of gauge theory

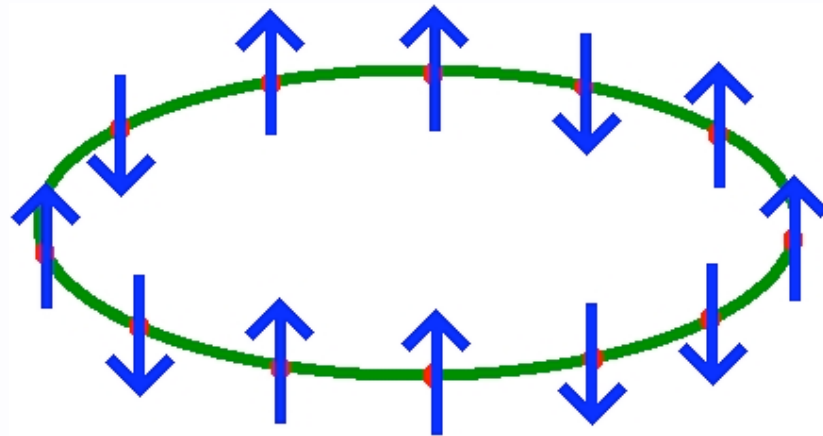
$$\left(\frac{\lambda_i + is}{\lambda_i - is} \right)^L = \prod_{j \neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i} \times e^t$$

$$t = r + i\vartheta$$

Gauge theory - spin chain

Identical
to the
Bethe
equations
for spin s
XXX magnet

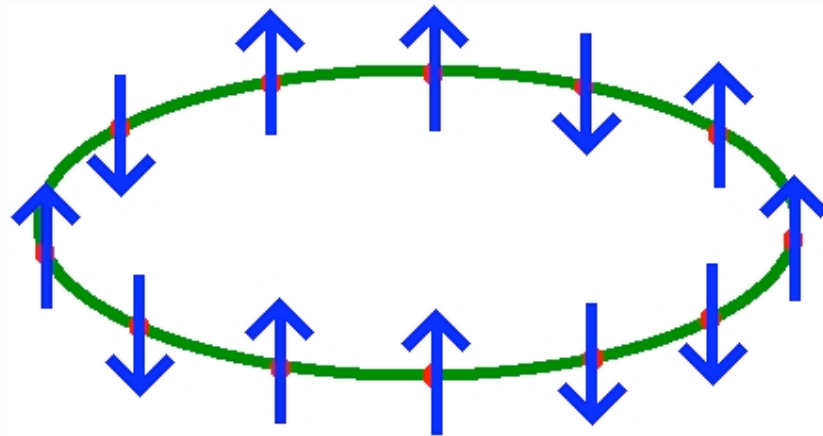
$$\left(\frac{\lambda_i + is}{\lambda_i - is} \right)^L = \prod_{j \neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}$$



Gauge theory - spin chain

Identical to
the Bethe
equations for
spin s XXX
magnet
with twisted
boundary
conditions

$$\left(\frac{\lambda_i + is}{\lambda_i - is} \right)^L = \prod_{j \neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i} \times e^t$$



Gauge theory - spin chain

Gauge theory
vacua -
eigenstates of the
spin Hamiltonian
(transfer-matrix)

$$H = \sum_{n=1}^L \sigma_n^a \otimes \sigma_{n+1}^a$$

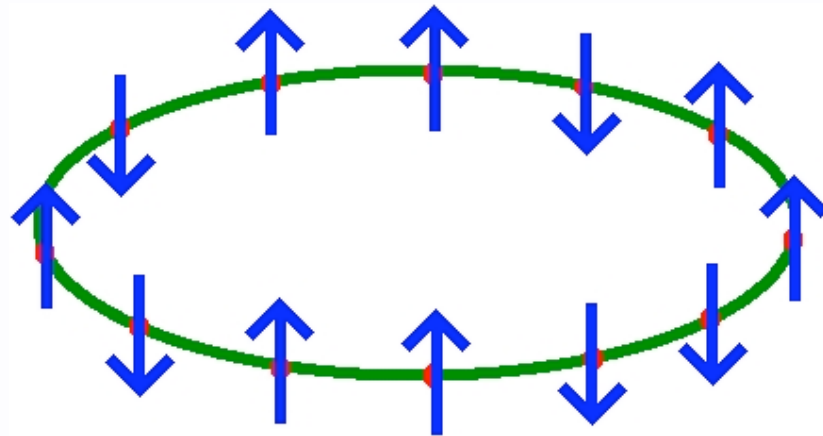


Table of dualities

XXX spin chain

$SU(2)$

L spins

N excitations

$$\left(\frac{\lambda_i + is}{\lambda_i - is}\right)^L = \prod_{j \neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}$$

$U(N)$ $d=2$ $N=2$

Chiral

multiplets:

1 adjoint

L fundamentals

L anti-fund.

Higher dimensional theories

Consider a $d=2+k$ gauge theory
With $N=2$ $d=2$ supersymmetry

For example, $N=1,2$ $d=4$ would do

Higher dimensional theories

Viewed as the $N=2$ $d=2$ theory it has a global symmetry group:

Poincare(k)

Higher dimensional theories

*One has several possibilities for
the twisted masses,
corresponding to the*

abelian subalgebras in $\text{Poincare}(\mathfrak{k})$

Higher dimensional theories

*One has several possibilities for
the twisted masses,
abelian subalgebras in Poincare(k)*

Translations in \mathbf{R}^1

Rotations in \mathbf{R}^{k-1}

Higher dimensional theories

The resulting theories are:

Translations in \mathbf{R}^1 : compactification on \mathbf{T}^1

Rotations in \mathbf{R}^{k-1} :

the theory in the Ω -background

Table of dualities

XXZ spin chain

$SU(2)$

L spins

N excitations

$U(N)$ $d=3$ $N=2$

Compactified on a circle

Chiral multiplets:

1 adjoint

L fundamentals

L anti-fund.

$$\left(\frac{\sinh(\lambda_i + i s \gamma)}{\sinh(\lambda_i - i s \gamma)} \right)^L = \prod_{j \neq i} \frac{\sinh(\lambda_i - \lambda_j + i \gamma)}{\sinh(\lambda_i - \lambda_j - i \gamma)}$$

Table of dualities

XYZ spin chain

$SU(2)$, $L = 2N$

spins

N excitations

$$H = \sum_{n=1}^L J_x \sigma_n^x \otimes \sigma_{n+1}^x + J_y \sigma_n^y \otimes \sigma_{n+1}^y + J_z \sigma_n^z \otimes \sigma_{n+1}^z$$

$U(N)$ $d=4$ $N=1$

Compactified on a

2-torus = elliptic curve E

Chiral multiplets:

1 adjoint

$L = 2N$ fundamentals

$L = 2N$ anti-fund.

**Masses = wilson loops
of the flavour group**

Table of dualities

It is remarkable that the spin chain has precisely those generalizations: rational (XXX), trigonometric (XXZ) and elliptic (XYZ) that can be matched to the 2, 3, and 4 dim cases.

$$H = \sum_{n=1}^L J_x \sigma_n^x \otimes \sigma_{n+1}^x + J_y \sigma_n^y \otimes \sigma_{n+1}^y + J_z \sigma_n^z \otimes \sigma_{n+1}^z$$

$$J_x = 1 + k \operatorname{sn}^2 2\eta, \quad J_y = 1 - k \operatorname{sn}^2 2\eta, \quad J_z = \operatorname{cn} 2\eta \operatorname{dn} 2\eta$$

Table of dualities

Yang-Yang counting function =
effective twisted superpotential

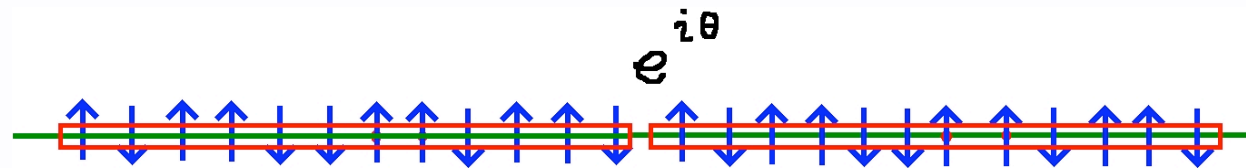
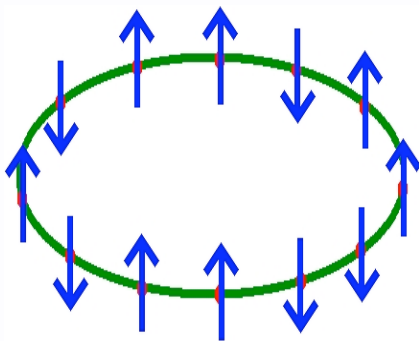
Table of dualities

Commuting hamiltonians (expansion of
transfer matrix) =
the twisted chiral ring generators, e.g.

$$\mathcal{O}_M = \text{Tr } \sigma^M$$

Table of dualities

Gauge theory theta angle (complexified)
is mapped to the spin chain theta angle
(twisted boundary conditions)



Algebraic Bethe Ansatz

Faddeev et al.

The spin chains are solved
algebraically using certain
operators,

$$A(\lambda), B(\lambda), C(\lambda), D(\lambda)$$

obeying exchange commutation
relations

Algebraic Bethe Ansatz

$$A(\lambda), B(\lambda), C(\lambda), D(\lambda)$$

Faddeev, Takhtajan,
Reshetikhin,
Jimbo, Drinfeld,
Sklyanin,
Lusztig

Yangian, quantum affine $U_q(\mathfrak{sl}_2)$,
Elliptic quantum group,

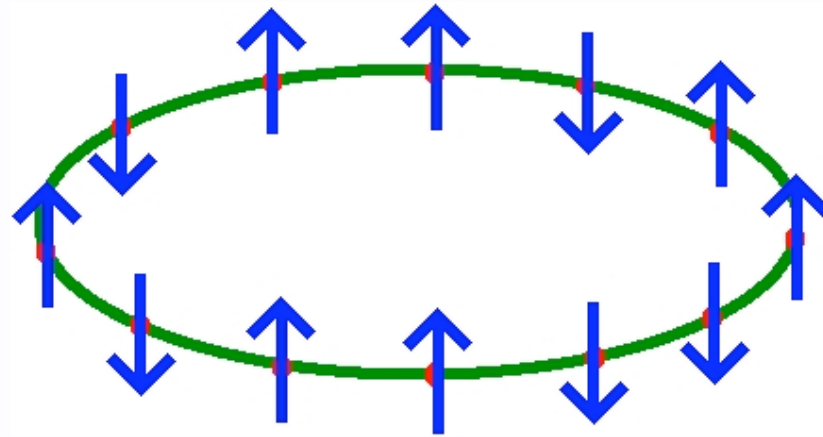
Algebraic Bethe Ansatz

The eigenvectors, Bethe vectors, are obtained by applying these operators to the (pseudo)vacuum.

$$\Psi_{\vec{\lambda}} = B(\lambda_1)B(\lambda_2)\dots B(\lambda_N)\Omega$$

Algebraic Bethe Ansatz VS GAUGE THEORY

For the spin chain it is natural to fix L =
total number of spins
and consider various N = excitation levels

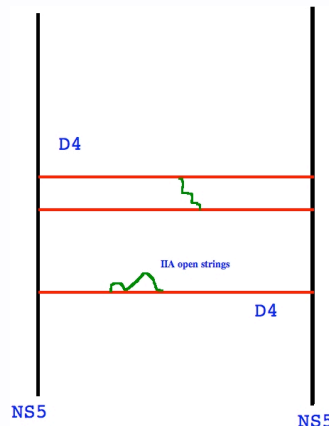


In the gauge theory context N is fixed.

Algebraic Bethe Ansatz VS STRING THEORY

However, if the theory is embedded
into string theory via brane
realization

then changing N is easy:
bring in an extra brane.



One might use the constructions
of Witten'96, Hanany-Hori'02

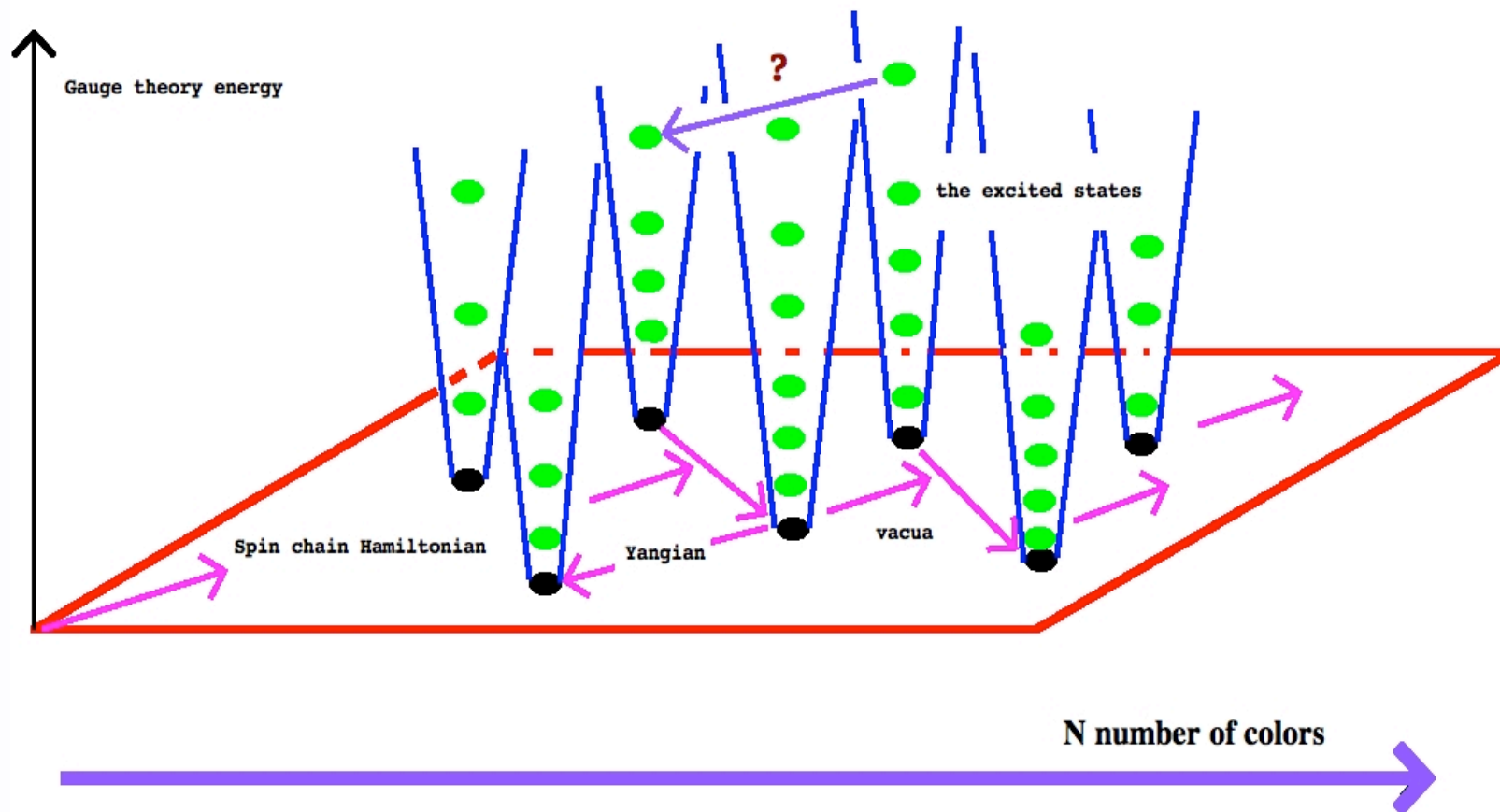
Algebraic Bethe Ansatz vs STRING THEORY

THUS:

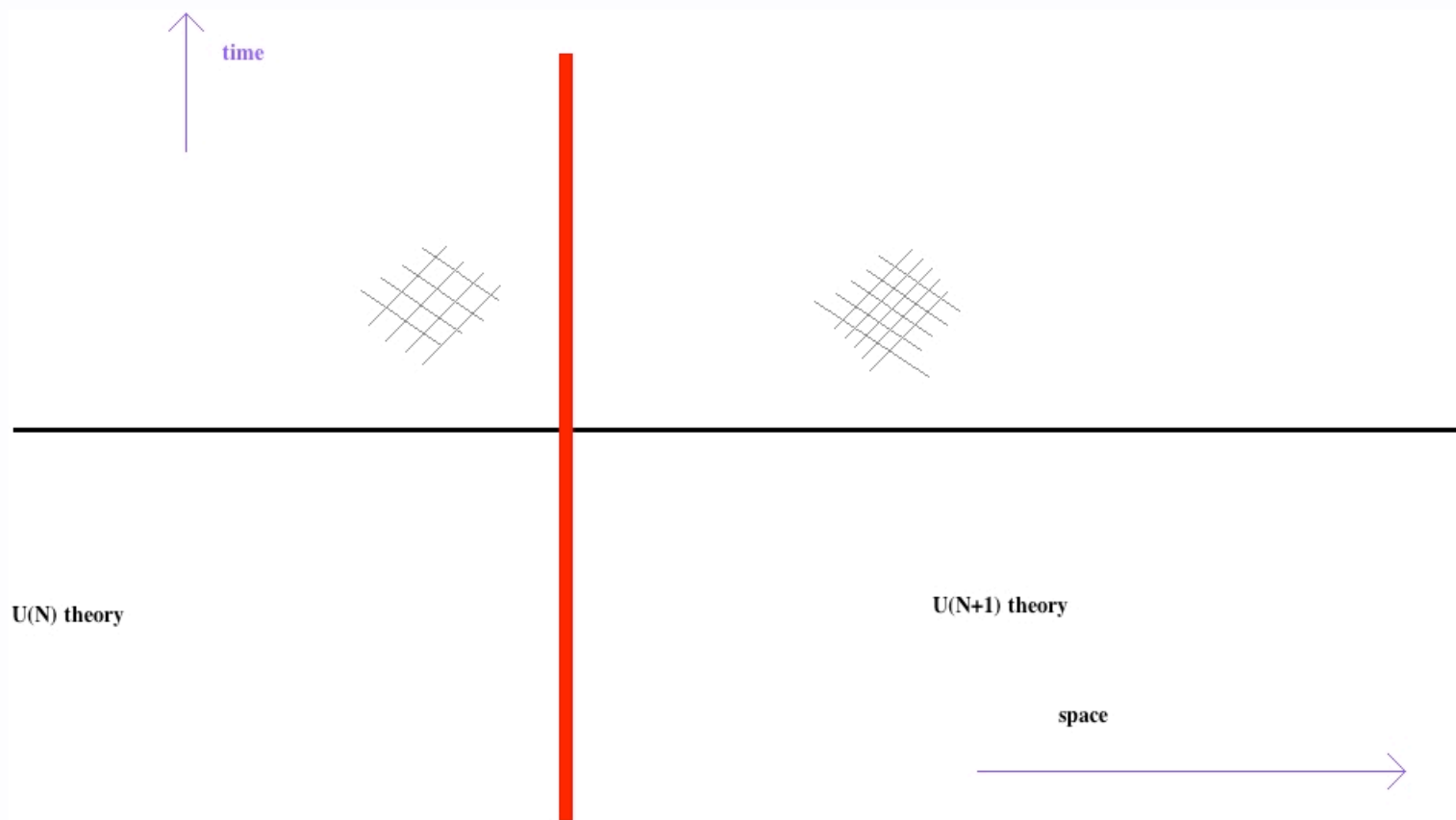
$B(\lambda)$ is for BRANE!

λ is for location!

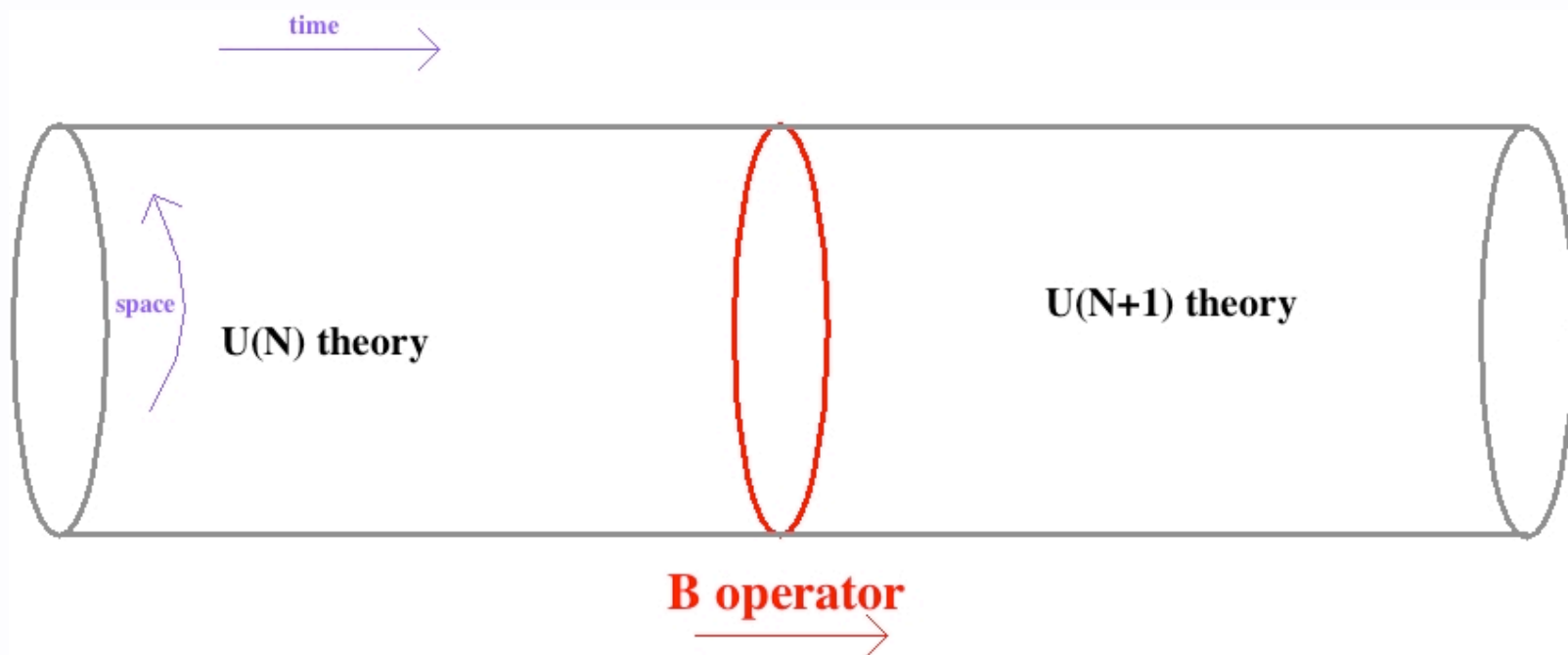
*Gauge theoretically
this is a miracle*



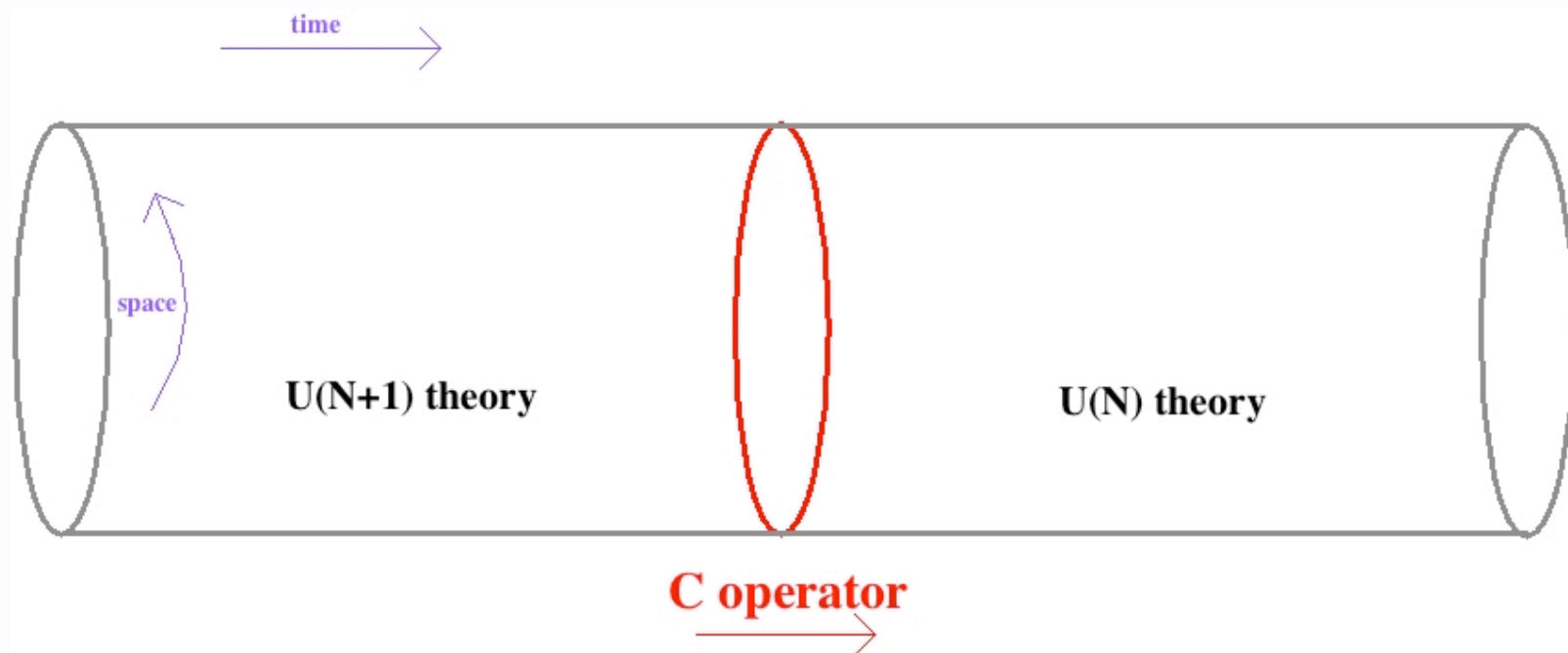
Non-local operators from domain walls



Non-local operators from domain walls



Non-local operators from domain walls

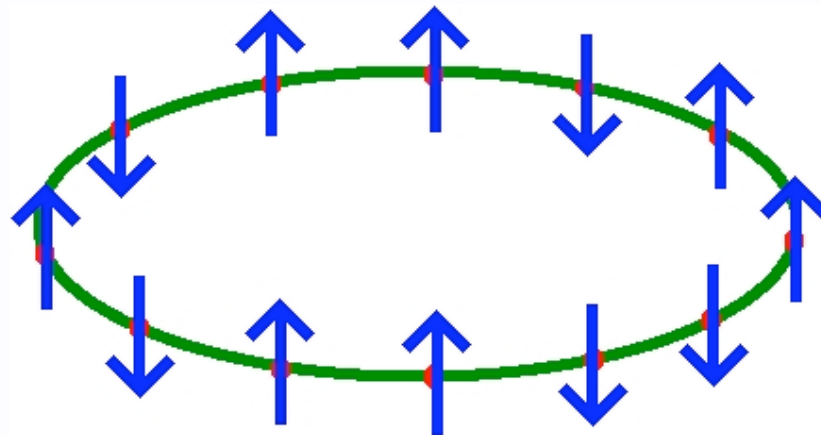


Are these models
too special, or the
gauge theory/integrable
lattice model
correspondence is
more general?

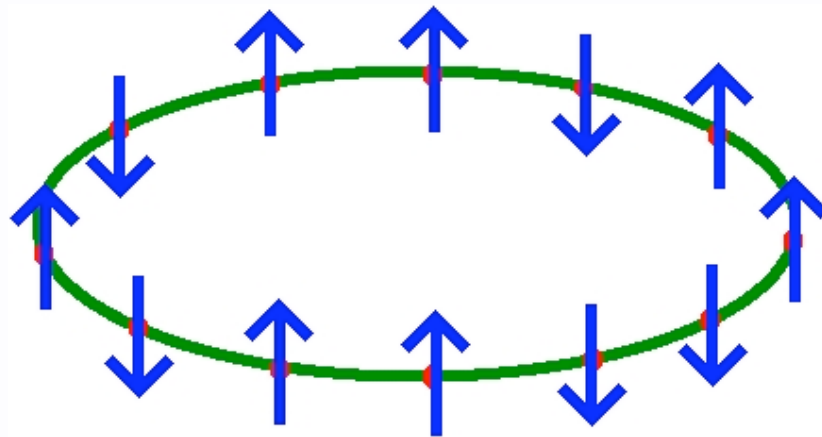
*Extends to every
spin group/supergroup,
spin representations,
inhomogeneity,
anisotropy....*

S.Shatashvili's
lecture

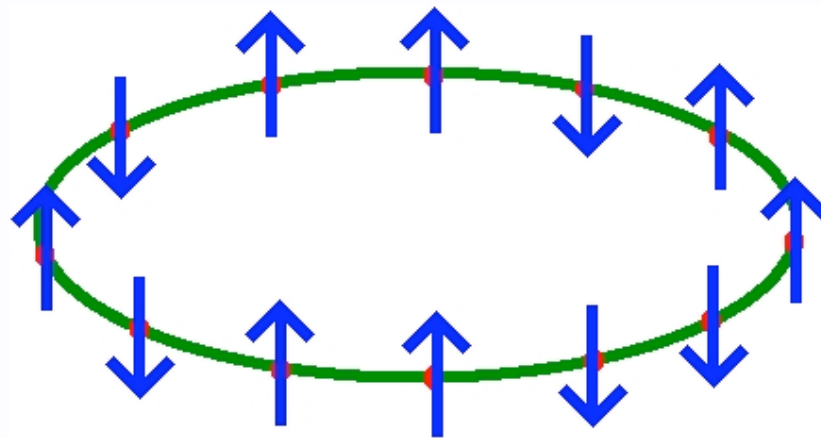
So far we were dealing
with «very» quantum
systems:
the Planck constant is
quantized



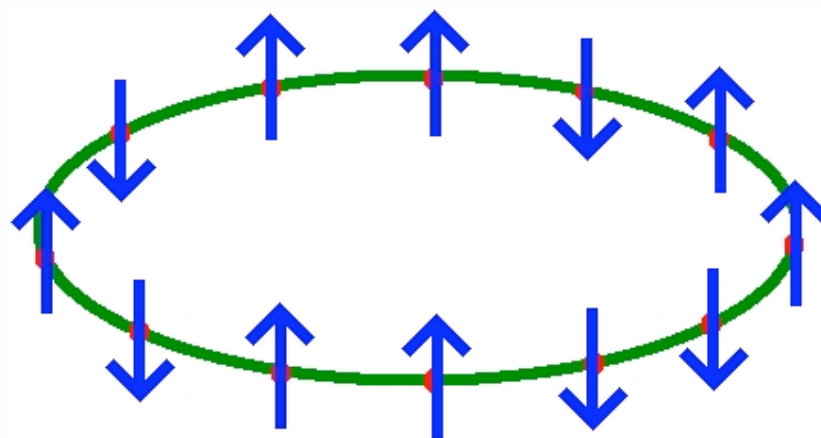
Spins were half-integers



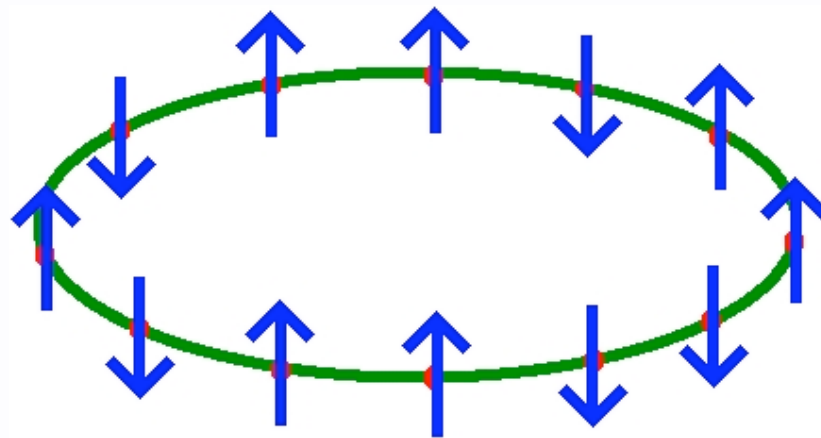
*The classical limit
unnatural ?*



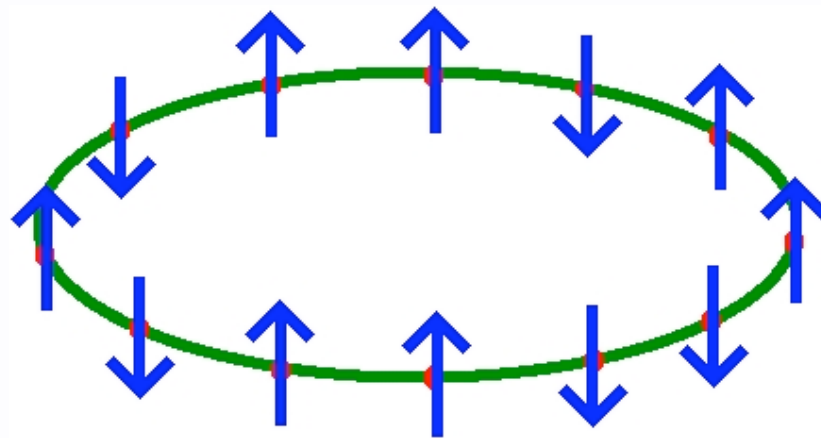
More formally,
we studied the finite-
dimensional representations
of
spin algebras
(compact spin groups)



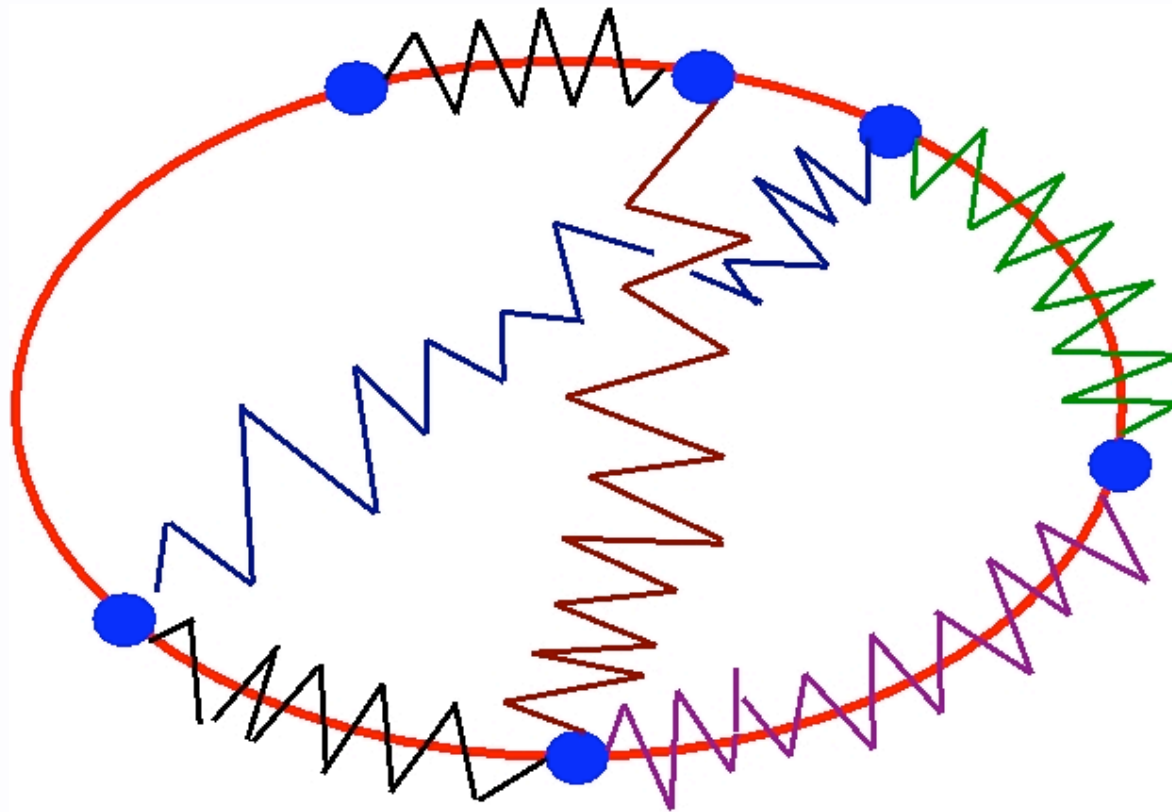
*Can we cross over to
the infinite-dimensional
representation theory?*



Non-compact spin groups?



*Many-body systems?
continuous degrees
of freedom?*



Many-body systems

The mother: Calogero system

$$H_{\text{eCM}} = \frac{1}{2} \sum_{i=1}^N p_i^2 + g^2 \sum_{i < j} U(x_i - x_j; \mathbf{q})$$

$$p_k = -i\hbar \frac{\partial}{\partial x_k}$$

Many-body systems

Calogero-Moser-Sutherland system

$$H_{\text{eCM}} = \frac{1}{2} \sum_{i=1}^N p_i^2 + g^2 \sum_{i < j} U(x_i - x_j; \mathbf{q})$$

$$p_k = -i\hbar \frac{\partial}{\partial x_k}$$

Many-body systems

Elliptic Calogero-Moser system

$$H_{\text{eCM}} = \frac{1}{2} \sum_{i=1}^N p_i^2 + g^2 \sum_{i < j} U(x_i - x_j; q)$$

$$U(x; q) = U(-x; q) = \sum_{n \in \mathbf{Z}} \frac{1}{\sinh^2(x + 2\pi n\beta)}$$

F.Calogero, '69-70

Many-body systems

The elliptic Calogero-Moser system
describes a system of identical particles
on a circle of radius

β

Elliptic Calogero-Moser

A system of identical particles on a circle of radius

β

subject to the pair-wise interaction potential, given by the elliptic (double-periodic) function

$$U(x; q) = U(-x; q) = \sum_{n \in \mathbf{Z}} \frac{1}{\sinh^2(x + 2\pi n\beta)}$$

Elliptic Calogero-Moser

One is interested in the β -periodic
symmetric, L^2 -normalizable wavefunctions

$$\Psi(x_1, \dots, x_N)$$

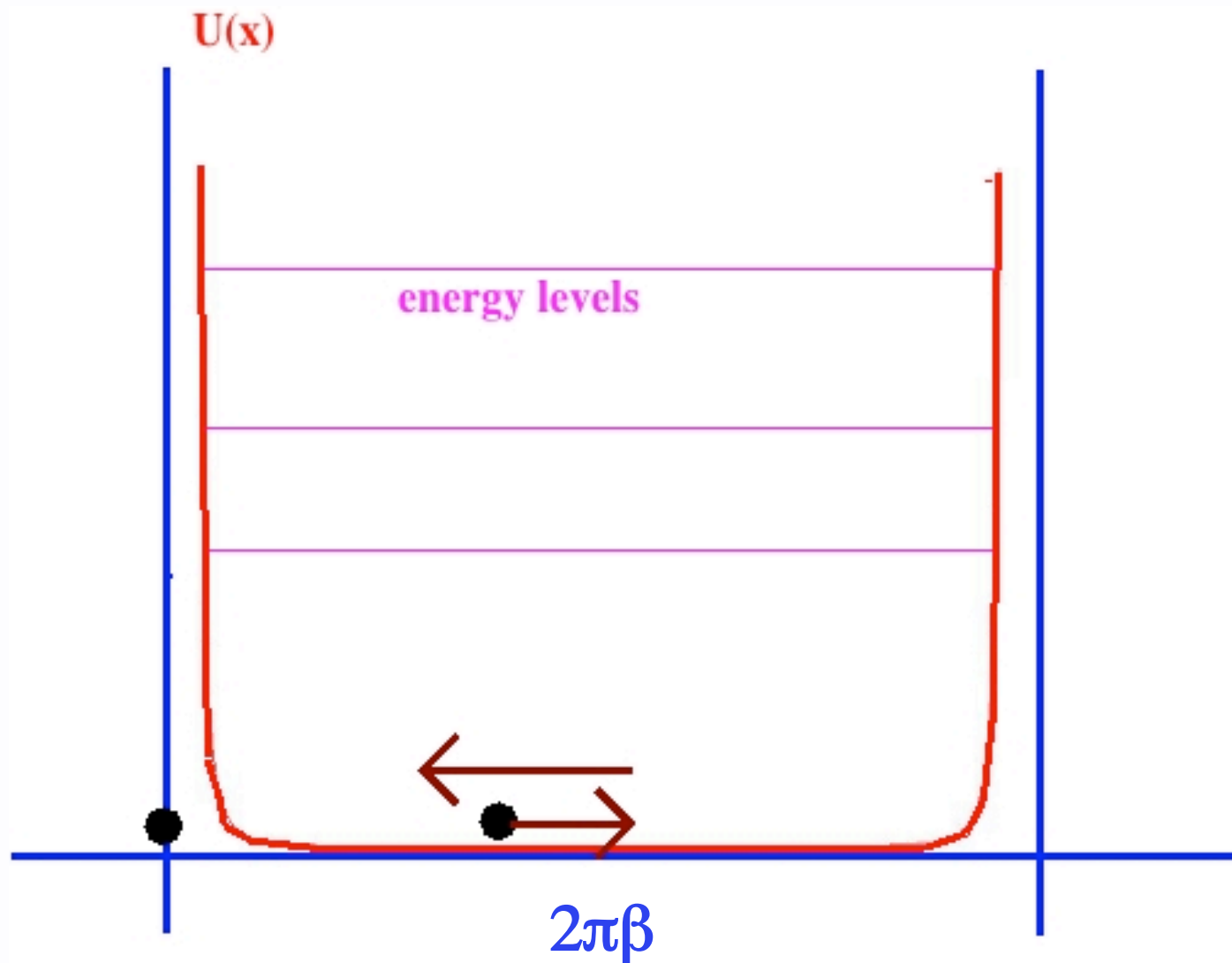
Elliptic Calogero-Moser

β -periodic, symmetric, L^2 -normalizable
wavefunctions

$$\Psi(x_1, \dots, x_N)$$

*The spectrum is
infinite discrete*

*One should get an
infinite discrete spectrum*



Many-body systems

Our main claim:
the
infinite discrete spectrum
of
the integrable many-body system
=
The vacua of the $N=2$ $d=2$ theory

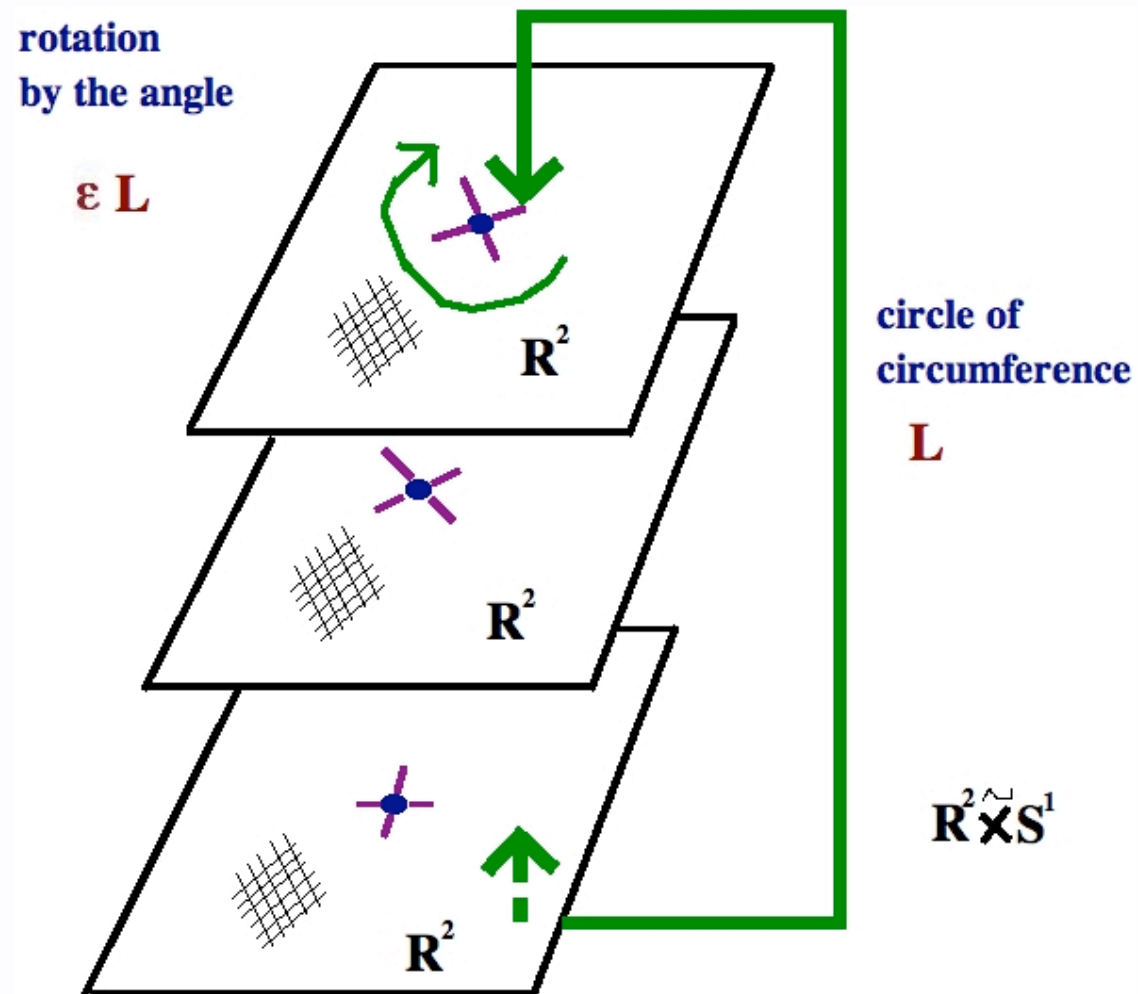
Many-body systems

The vacua of the $N=2$ $d=2$ theory,
Obtained by subjecting the $N=2$ $d=4$
Theory to an Ω -background in \mathbf{R}^2

The Ω -background

*The $N=2$ $d=4$
theory in the Ω -background in \mathbf{R}^2*

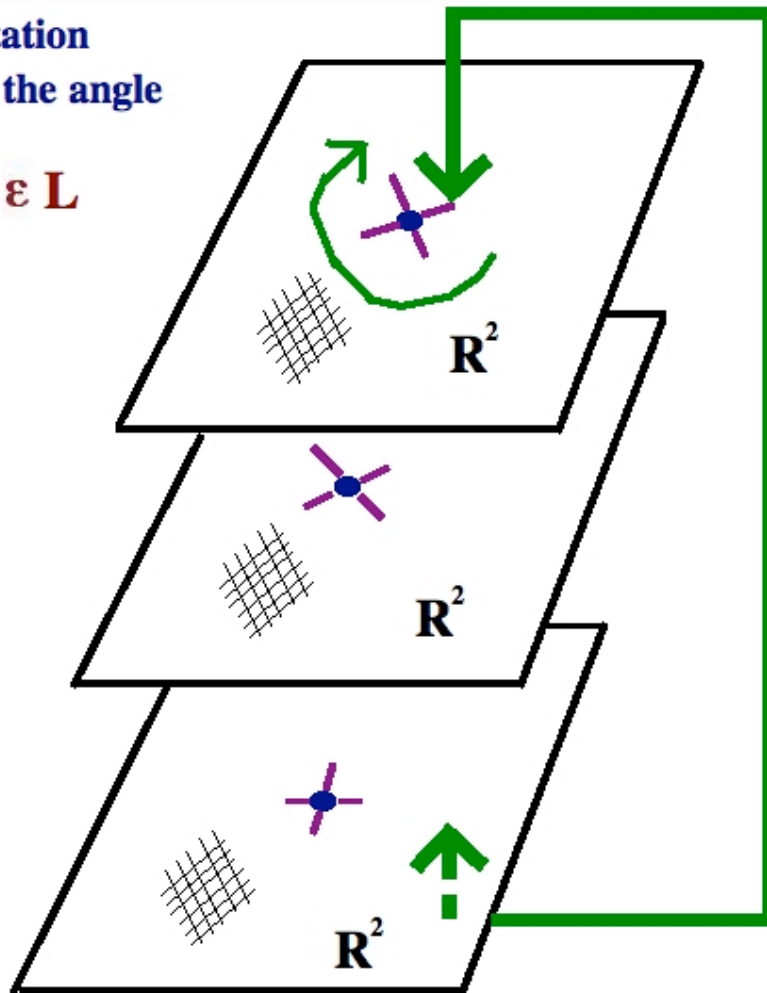
The Ω -background



The Ω -background

rotation
by the angle

ϵL



circle of
circumference

L

$R^2 \hat{\times} S^1$

Rotation
in space
is accompanied
by an **R-symmetry**
rotation

The Ω -background: Lagrangian of the theory

L \rightarrow 0 limit gives back the 4d theory

$$\begin{aligned} & \text{tr} \|F_A\|^2 + \text{tr} (D_A \phi - \iota_V F_A) \star (D_A \bar{\phi} - \iota_{\bar{V}} F_A) + \\ & + \text{tr} \|[\phi, \bar{\phi}] + \iota_V D_A \bar{\phi} - \iota_{\bar{V}} D_A \phi + \iota_V \iota_{\bar{V}} F_A\|^2 \end{aligned}$$

For pure $N=2$

The Ω -background: Lagrangian of the theory

$$\begin{aligned} & \text{tr} \|F_A\|^2 + \text{tr} (D_A \phi - \iota_V F_A) \star (D_A \bar{\phi} - \iota_{\bar{V}} F_A) + \\ & + \text{tr} \|[\phi, \bar{\phi}] + \iota_V D_A \bar{\phi} - \iota_{\bar{V}} D_A \phi + \iota_V \iota_{\bar{V}} F_A\|^2 \end{aligned}$$

$$V = \varepsilon [x^2 \partial_{x^3} - x^3 \partial_{x^2}]$$

$$\bar{V} = \bar{\varepsilon} [x^2 \partial_{x^3} - x^3 \partial_{x^2}]$$

The Ω -background: Lagrangian of the theory

$$V = \varepsilon \left[x^2 \partial_{x^3} - x^3 \partial_{x^2} \right]$$

$$\overline{V} = \overline{\varepsilon} \left[x^2 \partial_{x^3} - x^3 \partial_{x^2} \right]$$

Rotation in two dimensions

The theory in the Ω -background

*Looks two dimensional
in the infrared:
localization at
the «cosmic string»*

$$x^2 = x^3 = 0$$

The theory in the Ω -background

The two dimensional theory

Has $N=2$ $d=2$ susy

It has an

effective twisted

superpotential

The theory in the Ω -background

*The effective twisted
superpotential*

$$\widetilde{W}^{\text{eff}}(a_1, \dots, a_N; \varepsilon; \tau, m)$$

The theory in the Ω -background

The effective twisted superpotential

$$\widetilde{W}^{\text{eff}}(a_1, \dots, a_N; \varepsilon; \tau, m)$$

$\sigma_i = a_i$ **The special coordinates
on the moduli space of vacua of
the original four dimensional
 $N=2$ theory**

The effective twisted superpotential

$$\widetilde{W}^{\text{eff}}(a_1, \dots, a_N; \varepsilon; \tau, m)$$

**Computed by the instanton
Partition function**

$$Z(a, \varepsilon_1, \varepsilon_2; m, \tau)$$

The effective twisted superpotential

$$Z(a, \varepsilon_1, \varepsilon_2; m, \tau) \sim e^{\frac{1}{\varepsilon_2} \tilde{W}^{\text{eff}}(a_1, \dots, a_N; \varepsilon_1; \tau, m)} + \dots$$

$$\text{as } \varepsilon_2 \rightarrow 0$$

The effective twisted superpotential
leads to the vacuum equations

$$\exp \frac{\partial \widetilde{W}(a)}{\partial a_i} = \mathbf{1}$$

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leads to the vacuum equations

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*Lift the 4d vacuum
Degeneracy!*

The effective twisted superpotential
leads to the vacuum equations

$$\exp \frac{\partial \widetilde{W}(a)}{\partial a_i} = \mathbf{1}$$

*The solutions are isolated:
Discrete spectrum*

The effective twisted superpotential
leads to the vacuum equations

$$\exp \frac{\partial \widetilde{W}(a)}{\partial a_i} = \mathbf{1}$$

«Attractor SW curves»

**An interesting
possibility
for the brane world
scenarios**

The effective twisted superpotential
has one-loop perturbative
and all-order instanton corrections

$$\widetilde{W}^{\text{eff}}(a; \tau) = \widetilde{W}^{\text{pert}}(a) + \sum_{n=1}^{\infty} q^n \widetilde{W}_{n-\text{inst}}(a)$$

For example, for the $N=2^*$ theory
(adjoint hypermultiplet with mass m)

$$\exp \frac{\partial \widetilde{W}^{\text{pert}}(a)}{\partial a_i} =$$

$$e^{\frac{\pi i \tau a_i}{\varepsilon}} \prod_{j \neq i} S(a_i - a_j)$$

$$S(x) = \frac{\Gamma\left(\frac{-m+x}{\varepsilon}\right) \Gamma\left(1 - \frac{x}{\varepsilon}\right)}{\Gamma\left(\frac{-m-x}{\varepsilon}\right) \Gamma\left(1 + \frac{x}{\varepsilon}\right)}$$

Bethe equations

Factorized S-matrix

$$S(x) = \frac{\Gamma\left(\frac{-m+x}{\varepsilon}\right) \Gamma\left(1 - \frac{x}{\varepsilon}\right)}{\Gamma\left(\frac{-m-x}{\varepsilon}\right) \Gamma\left(1 + \frac{x}{\varepsilon}\right)}$$

*This is the two-body scattering
In hyperbolic Calogero-Sutherland*

Bethe equations

Factorized S-matrix

$$S(x) = \frac{\Gamma\left(\frac{-m+x}{\varepsilon}\right) \Gamma\left(1 - \frac{x}{\varepsilon}\right)}{\Gamma\left(\frac{-m-x}{\varepsilon}\right) \Gamma\left(1 + \frac{x}{\varepsilon}\right)}$$

$$U_0(x) = \frac{1}{\sinh^2(x)}$$

Two-body potential

Bethe equations

Factorized S-matrix

$$S(x) = \frac{\Gamma\left(\frac{-m+x}{\varepsilon}\right) \Gamma\left(1 - \frac{x}{\varepsilon}\right)}{\Gamma\left(\frac{-m-x}{\varepsilon}\right) \Gamma\left(1 + \frac{x}{\varepsilon}\right)}$$

Harish-Chandra, Gindikin-Karpelevich,
Olshanetsky-Perelomov, Heckmann,
final result: Opdam

The full superpotential of $N=2^*$ theory leads to the vacuum equations

Momentum phase shift

$$e^{\frac{\pi i \tau a_i}{\varepsilon}} \prod_{j \neq i} S(a_i - a_j) \times \left[1 + q \sum_{k \neq i} \prod_{l \neq k} \text{rational}(a_i, a_l, a_k, m(m + \varepsilon), \varepsilon) + \dots \right]$$

Two-body scattering

The finite size corrections

$$q = \exp (- N\beta)$$

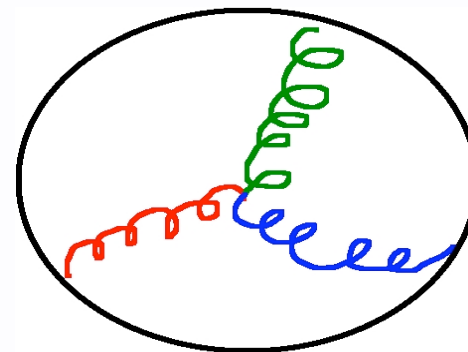
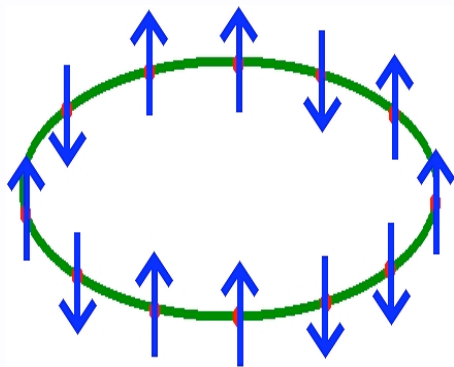
The main slogan this year

*Four dimensional
gauge theories
give rise to the
Instanton corrected
Bethe Ansatz
equations*

The main slogan this year

*aka the
Bethe Ansatz
with
the finite size effects*

Dictionary



Dictionary

Elliptic
CM
system



$N=2^*$ theory

Dictionary

classical
Elliptic
CM
system



4d $N=2^*$
theory

Dictionary

classical
Elliptic
CM
system



4d $N=2^*$
theory

Donagi-Witten, Martinec-Warner,
Gorsky-Nekrasov

Dictionary

quantum
Elliptic
CM
system



4d $N=2^*$
Theory
In the
 Ω -background

NN-Shatashvili

Dictionary

The
(complexified)
system
Size β



The gauge
coupling τ

Dictionary

The Planck constant \longleftrightarrow The Ω -background parameter

ε

Dictionary

The correspondence
Extends to other
integrable systems:

Today, relativistic Systems,
Perhaps all $1+1$ iQFTs

Classical Limit: back to 4d

In the limit $\varepsilon \longrightarrow 0$

$$\widetilde{W}(a; \varepsilon) \sim \frac{\mathcal{F}(a)}{\varepsilon} + \dots$$

*The prepotential $\mathcal{F}(a)$ of
the low-energy effective
theory in 4d*

The Classical Limit: back to 4d

*The prepotential $\mathcal{F}(a)$ of
the low-energy effective
theory
is governed by a
classical (holomorphic)
integrable system*

Gorsky-Krichever-Marshakov-Mironov-Morozov
Donagi-Witten'95

Classical Limit: back to 4d

*Liouville tori =
Jacobians (Prims) of the
Seiberg-Witten curves*

Classical Limit: back to 4d

$$a^i = \frac{1}{2\pi} \oint_{A_i} \mathbf{p} d\mathbf{q}, \quad a_{D,i} = \frac{1}{2\pi} \oint_{B_i} \mathbf{p} d\mathbf{q}$$

The Classical Limit: back to 4d

$$a^i = \frac{1}{2\pi} \oint_{A_i} \mathbf{p} d\mathbf{q}, \quad a_{D,i} = \frac{1}{2\pi} \oint_{B_i} \mathbf{p} d\mathbf{q}$$

*The redundant set of
the action variables*

The Classical Limit: back to 4d

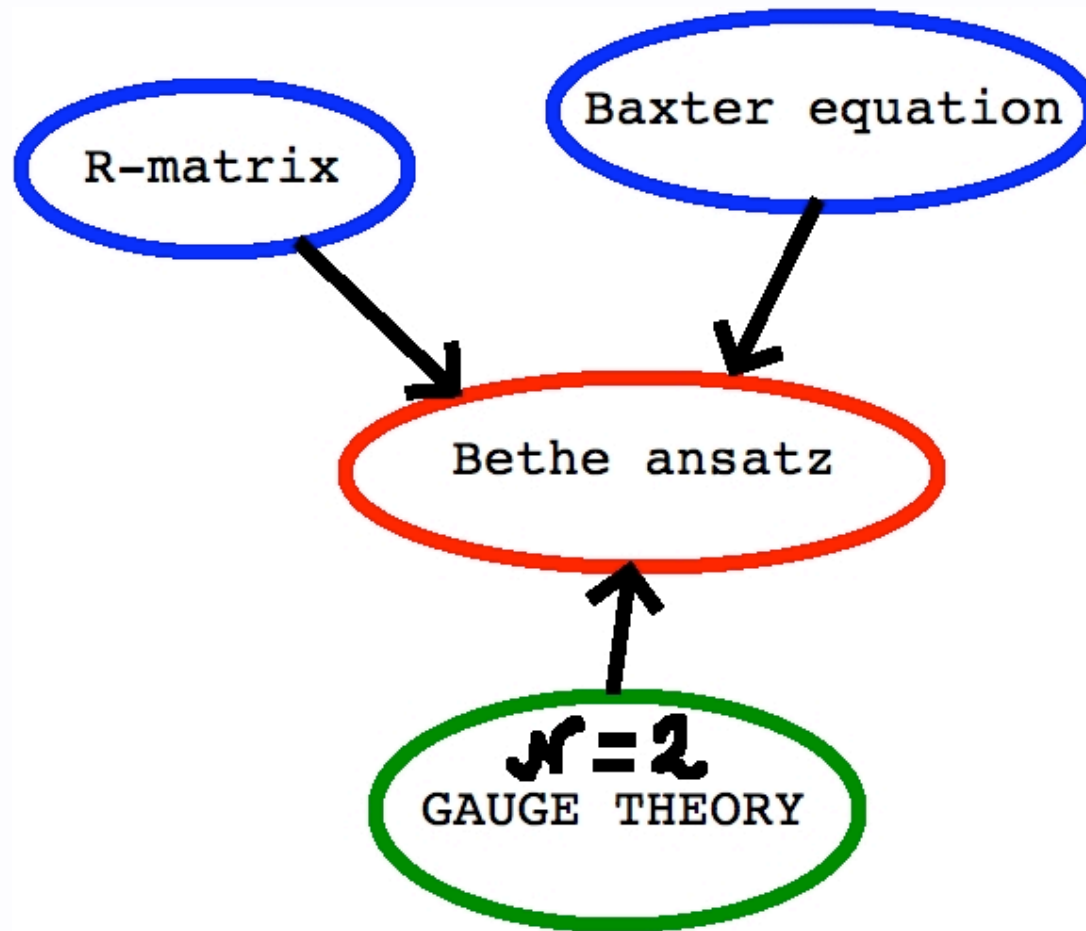
$$a_{D,i} = \frac{\partial \mathcal{F}}{\partial a^i}$$

Enters prepotential $\mathcal{F}(a)$

Classical integrable system vs Quantum integrable system

That system is quantized
when
the gauge theory is subject
to
the Ω -background

CONCLUSIONS



CONCLUSIONS

We have found a striking connection between the vacua of the $N=2$ theories and the eigenstates of the quantum integrable systems, both with finite and infinite number of degrees of freedom, Finite size effects etc.

CONCLUSIONS

*The connection is fruitful for both
gauge theorists and Bethe people*

WISHFUL CONCLUSIONS

*It would be nice to lift
this connection
further to string/M theory
constructions,
e.g. M5 branes wrapping curves
in K3*

WISHFUL CONCLUSIONS

This would shed light on the «real»
quantization of Hitchin systems
(cf. «complex» quantization of
Beilinson-Drinfeld)

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*This would shed light on the «real»
quantization of Hitchin systems
(cf. «complex» quantization of
Beilinson-Drinfeld)*

Kapustin-Witten
($N=4$ with boundaries)

VERY WISHFUL CONCLUSIONS

*And perhaps push the geometric
Langlands program
beyond geometry
back to the number theory....*

VERY WISHFUL CONCLUSIONS

Another possibility is that the
Planar $N=4$ super-Yang-Mills
Is the integrable system....

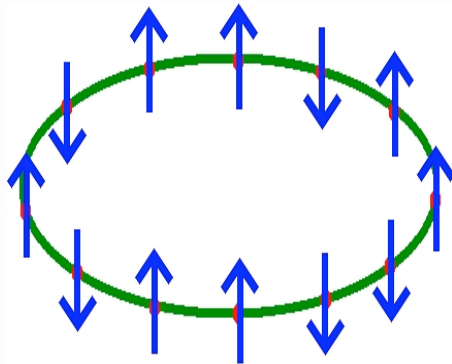
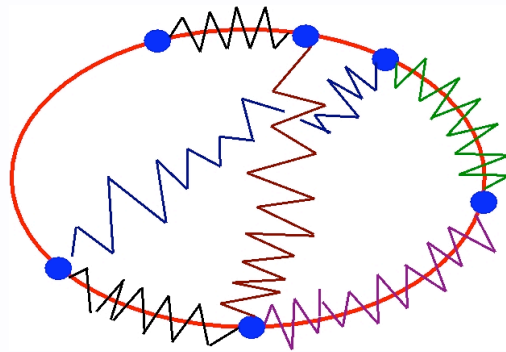
VERY WISHFUL CONCLUSIONS

Dual to the TOTALITY of
the $N=2$ $d=4$ theories
of a certain quiver type
In the VACUUM sector

CONCLUSIONS

These margins are
Too thin and the time
Is too short
To list them all....

The story is
beautiful,
complex,
and rich, yet
has real
applications



THANK
YOU

