World-sheet Duality for Superspace σ-Models

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Based on hep-th/0610070, arXiv:0712.3549, arXiv:0809.1046 and work in progress

(with C. Candu, T. Creutzig, G. Götz, V. Mitev, H. Saleur and V. Schomerus in various combinations)

SIXTH FRAMEWORK PROGRAMME

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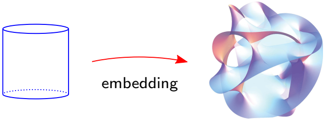
Motivation

World-sheet

2D Riemann surface
(w/wo boundaries)



Target space

Riemannian manifold
(plus extra structure)

Appearance of superspace σ-models

- String theory
  - Quantization of strings in flux backgrounds
  - AdS/CFT correspondence
  - Moduli stabilization in string phenomenology
- Disordered systems
  - Quantum Hall systems
  - Self avoiding random walks, polymer physics, ...
  - Efetov's supersymmetry trick

String quantization: The pure spinor formalism

Ingredients

- Superspace σ-model encoding geometry and fluxes
- Pure spinors: Curved ghost system
- BRST procedure

[Berkovits at al] [Grassi et al] [Hogeveen,Skenderis] [...]

Features

- Manifest target space supersymmetry
- Manifest world-sheet conformal symmetry
- Geometry encodes physical properties
- Action quantizable, but quantization hard in practice

Step 1: Trading disorder for supersymmetry

$$\overline{\langle\mathcal{O}\rangle} = \overline{\left(\frac{\int \mathcal{D}F\,\mathcal{O}\,e^{-\mathcal{S}(F)}}{\int \mathcal{D}F\,e^{-\mathcal{S}(F)}}\right)} = \overline{\left(\int \mathcal{D}F\mathcal{D}B\,\mathcal{O}\,e^{-\left[\mathcal{S}(F)+\mathcal{S}(B)\right]}\right)}$$

$$= \int \mathcal{D}F\mathcal{D}B\,\mathcal{O}\,e^{-\mathcal{S}_{\text{eff}}(F,B)} \leftarrow \text{supersymmetric}$$

Typically: $$\mathcal{S}_{\text{eff}}(F,B) = \mathcal{S}_{\text{free}} + g\int d^2x\big(B\bar{B}+F\bar{F}\big)^2$$

Step 2: Hubbard-Stratonovich transformation

- Remove BF interaction by introducing auxiliary field
- Integrate out B and F

$\Rightarrow$ Supersymmetric σ-model

The structure of this talk

Outline

1. Supercoset σ-models
   - Occurrence in string theory and condensed matter theory
   - Ricci flatness and conformal invariance
2. Some particular examples
   - Three-dimensional Anti-de Sitter space
   - Superspheres
   - Projective superspaces
3. Quasi-abelian perturbation theory
   - Exact boundary spectra
   - World-sheet duality for supersphere σ-models

Appearance of supercosets

String backgrounds as supercosets...

Minkowski	$\mathrm{AdS}_5 \times S^5$	$\mathrm{AdS}_4 \times \mathbb{CP}^3$	$\mathrm{AdS}_2 \times S^2$
$\frac{\text{super-Poincaré}}{\text{Lorentz}}$	$\frac{\mathrm{PSU}(2,2\|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)}$	$\frac{\mathrm{OSP}(6\|2,2)}{\mathrm{U}(3)\times\mathrm{SO}(1,3)}$	$\frac{\mathrm{PSU}(1,1\|2)}{\mathrm{U}(1)\times\mathrm{U}(1)}$

[Metsaev,Tseytlin] [Berkovits,Bershadsky,Hauer,Zhukov,Zwiebach] [Arutyunov,Frolov]

Supercosets in statistical physics...

IQHE	Dilute polymers (SAW)	Dense polymers
	$S^{2S+1\|2S}$	$\mathbb{CP}^{S-1\|S}$
$\frac{\mathrm{U}(1,1\|2)}{\mathrm{U}(1\|1)\times\mathrm{U}(1\|1)}$	$\frac{\mathrm{OSP}(2S+2\|2S)}{\mathrm{OSP}(2S+1\|2S)}$	$\frac{\mathrm{U}(S\|S)}{\mathrm{U}(1)\times\mathrm{U}(S-1\|S)}$

[Weidenmüller] [Read,Saleur]

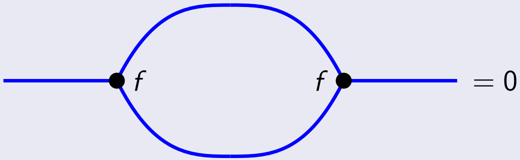
Symmetric and generalized symmetric spaces

Definition of the cosets

$$G/H \; = \; \{g \in G \big| gh \sim g, h \in H\}$$

Some additional requirements

- $H \subset G$ is invariant subgroup under an automorphism
- Ricci flatness $\Leftrightarrow$ super Calabi-Yau $\Leftrightarrow$ vanishing Killing form



Examples: Cosets of $\mathrm{PSU}(N|N)$, $\mathrm{OSP}(2S+2|2S)$, $\mathrm{D}(2,1;\alpha)$.

Remark: To describe supergroups choose $K = \frac{K\times K}{K_{\mathrm{diag}}}$

Properties of supercoset models

Properties in a nutshell

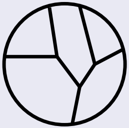
- Conformal invariance [Kagan,Young] [Babichenko]
- Family of CFTs with continuously varying exponents
- Geometric realization of supersymmetry: $g \mapsto hg$
- Completely new type of 2D conformal field theory

  Standard methods do not apply!
- Integrability [Pohlmeyer] [Lüscher] ... [Bena,Polchinski,Roiban] [Young]

Sketch of conformal invariance

The β-function vanishes identically...

$$\beta = \sum_{\substack{\text{certain}\\ G\text{-invariants}}}$$



$$= 0$$

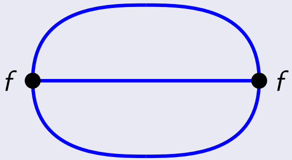
Ingredients:

Invariant form: $\kappa^{\mu\nu}$

Structure constants: $f^{\mu\nu\lambda}$

Sketch of conformal invariance

The β-function vanishes identically...



$= 0$

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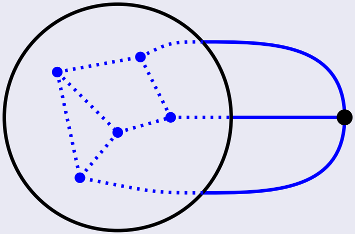
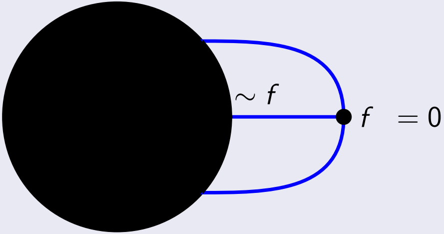


There is a unique invariant rank 3 tensor!

[Bershadsky,Zhukov,Vaintrob'99] [Babichenko'06]

Sketch of conformal invariance

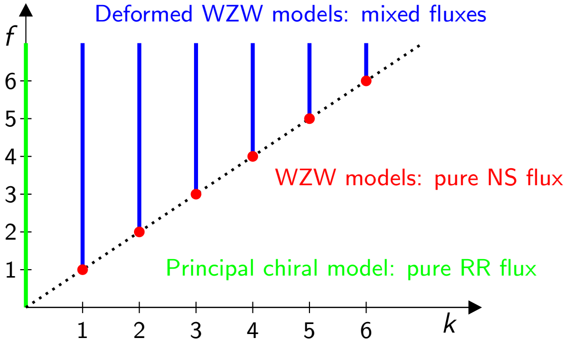
The β-function vanishes identically...



There is a unique invariant rank 3 tensor!

[Bershadsky,Zhukov,Vaintrob'99] [Babichenko'06]

$\mathrm{AdS}_3 \times \mathrm{S}^3$ alias $\mathrm{PSU}(1,1|2)$



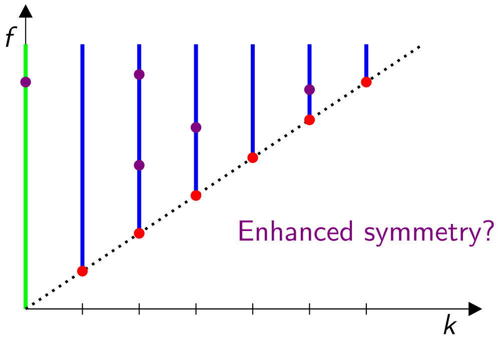
String theory
[Berkovits,Vafa,Witten]
[Bershadsky,Vaintrob,Zhukov]

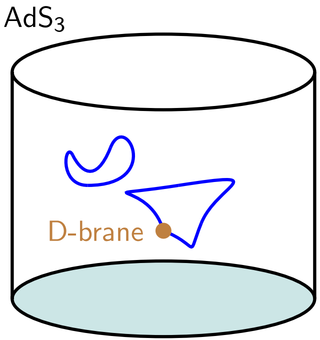
Condensed matter theory (IHQE)
[Zirnbauer]
[Bhaseen,Kog.,Sol.,Tan.,Tsvelik]
[Tsvelik]
[Obuse,Sub.,Fur.,Gruz.,Ludwig]

Action & physical interpretation of the PSU(1, 1|2) σ-model

$$\mathcal{S} = f\mathcal{S}_{\text{kin}} + k\mathcal{S}_{\text{top}} \qquad \begin{cases} k = Q_5^{\text{NS}} \\ f = \sqrt{(Q_5^{\text{NS}})^2 + g_s^2(Q_5^{\text{RR}})^2} \end{cases}$$

Recent progress





1. Solve the PSU(1, 1|2) WZW model → LCFT
2. Study marginal deformations by $\mathcal{S}_{\text{def}} = \int \text{str}\big(\text{J} \cdot \text{Ad}_g(\bar{\text{J}})\big)$
3. Quasi-abelian deformation theory → anomalous dimensions

$$Z(f, k) \longrightarrow \text{Degeneracies for certain values of } f \text{ and } k?$$

OSP(4|2) Gross-Neveu model

The OSP(4|2) Gross-Neveu model

Field content

- Fundamental OSP(4|2)-multiplet $(\psi_1, \psi_2, \psi_3, \psi_4, \beta, \gamma)$
- All these fields have conformal weight $h = 1/2$

Formulation as a Gross-Neveu model

$$\mathcal{S}_{\text{GN}} = \mathcal{S}_{\text{free}} + g^2\,\mathcal{S}_{\text{int}} \qquad \begin{cases} \mathcal{S}_{\text{free}} = \int \left[\psi\bar{\partial}\psi + 2\beta\bar{\partial}\gamma + h.c.\right] \\ \mathcal{S}_{\text{int}} = \int \left[\psi\bar{\psi} + \beta\bar{\gamma} - \gamma\bar{\beta}\right]^2 \end{cases}$$

Formulation as a deformed OSP(4|2) WZW model

$$\mathcal{S}_{\text{GN}} = \mathcal{S}_{\text{WZW}} + g^2\,\mathcal{S}_{\text{def}} \qquad \text{with} \qquad \mathcal{S}_{\text{def}} = \int \operatorname{str}(J\bar{J})$$

A brane / conformal boundary condition

More on the reformulation

- At $g = 0$ there is an affine $\widehat{\mathrm{OSP}}(4|2)_{-1/2}$ symmetry
- It has a "bosonic" realization as an orbifold

$$\widehat{\mathrm{OSP}}(4|2)_{-1/2} \cong \Big[\underbrace{\widehat{SU}(2)_{-1/2}}_{\beta\gamma} \times \underbrace{\widehat{SU}(2)_1}_{\psi_1,\psi_2} \times \underbrace{\widehat{SU}(2)_1}_{\psi_3,\psi_4}\Big]/\mathbb{Z}_2$$

Towards a boundary spectrum for $g = 0$

- Employ twisted gluing conditions
- The spectrum can be calculated using standard techniques

A boundary spectrum

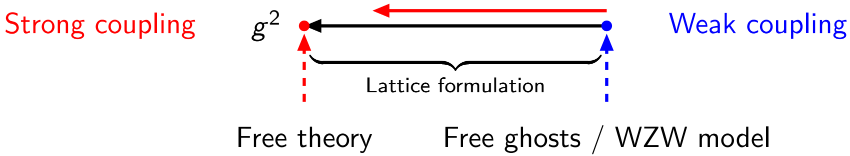
The main result: The full partition function

$$Z_{\mathrm{GN}}(g^2=0) \;=\; \sum \underbrace{\psi^{\mathrm{WZW}}_{[j_1,j_2,j_3]}(q)}_{\text{energy levels}} \; \underbrace{\chi_{[j_1,j_2,j_3]}(z)}_{\mathrm{OSP}(4|2)\text{ content}}$$

$$\psi^{\mathrm{WZW}}_{[j_1,j_2,j_3]}(q) \;=\; \frac{1}{\eta(q)^4}\sum_{n,m=0}^{\infty}(-1)^{n+m}q^{\frac{m}{2}(m+4j_1+2n+1)+j_1+\frac{n}{2}-\frac{1}{8}}$$

$$\times\,\bigl(q^{(j_2-\frac{n}{2})^2}-q^{(j_2+\frac{n}{2}+1)^2}\bigr)\bigl(q^{(j_3-\frac{n}{2})^2}-q^{(j_3+\frac{n}{2}+1)^2}\bigr)$$

A boundary spectrum



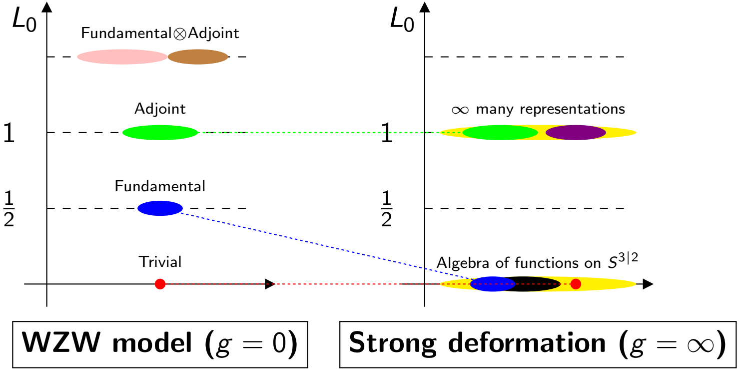
The main result: The full partition function

$$Z_{\mathrm{GN}}(g^2) = \sum \underbrace{q^{-\frac{1}{2}\frac{g^2}{1+g^2}C_\Lambda}}_{\text{anomalous dimension}} \underbrace{\psi^{\mathrm{WZW}}_{[j_1,j_2,j_3]}(q)}_{\text{energy levels}} \underbrace{\chi_{[j_1,j_2,j_3]}(z)}_{\mathrm{OSP}(4|2)\text{ content}}$$

$$\psi^{\mathrm{WZW}}_{[j_1,j_2,j_3]}(q) = \frac{1}{\eta(q)^4}\sum_{n,m=0}^{\infty}(-1)^{n+m}q^{\frac{m}{2}(m+4j_1+2n+1)+j_1+\frac{n}{2}-\frac{1}{8}}$$

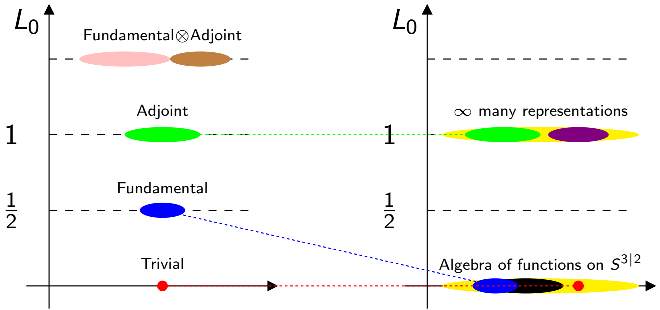
$$\times\left(q^{(j_2-\frac{n}{2})^2}-q^{(j_2+\frac{n}{2}+1)^2}\right)\left(q^{(j_3-\frac{n}{2})^2}-q^{(j_3+\frac{n}{2}+1)^2}\right)$$

At the two extremal values of g^2, the spectrum has the form...



Interpolation of the spectrum

At the two extremal values of g^2, the spectrum has the form...



WZW model ($g = 0$)

Strong deformation ($g = \infty$)

Supersphere σ-model at $R \to \infty$

A world-sheet duality for superspheres

The supersphere $S^{3|2}$

Realization of $S^{3|2}$ as a submanifold of flat superspace $\mathbb{R}^{4|2}$

$$\vec{X} = \begin{pmatrix} \vec{x} \\ \eta_1 \\ \eta_2 \end{pmatrix} \qquad \text{with} \qquad \vec{X}^2 = \vec{x}^2 + 2\eta_1\eta_2 = R^2$$

Realization as a symmetric space

$$S^{3|2} = \frac{\mathrm{OSP}(4|2)}{\mathrm{OSP}(3|2)}$$

The supersphere σ-model

Action functional

$$\mathcal{S}_\sigma = \int \partial\vec{X}\cdot\bar{\partial}\vec{X} \qquad \text{with} \qquad \vec{X}^2 = R^2$$

Properties of this σ-model

- There is no topological term
- Conformal invariance for each value of R
- Central charge: $c = 1$
- Non-unitarity [Read,Saleur] [Polchinski,Mann] [Candu,Saleur]^2 [Mitev,TQ,Schomerus]

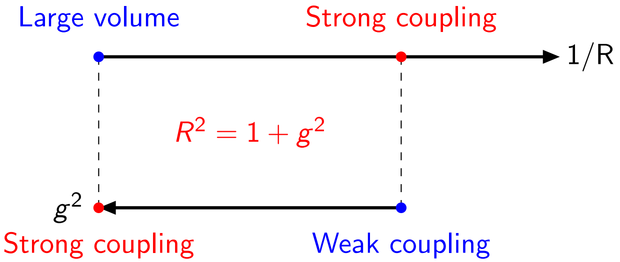
The space of states on a space-filling brane

$$\prod X^{a_i} \prod \partial X^{b_j} \prod \partial^2 X^{c_k} \cdots \qquad \text{and} \qquad \vec{X}^2 = R^2$$

$\Rightarrow$ Products of coordinate fields and their derivatives

A world-sheet duality for superspheres?

Supersphere σ-model geometric



$$Z_\sigma(q, z, R) \;\|\; Z_{\mathrm{GN}}(q, z, g^2)$$

OSP$(4|2)$ Gross-Neveu model non-geometric

[Candu,Saleur]^2 [Mitev,TQ,Schomerus]

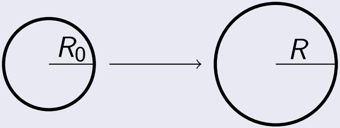
Quasi-abelian deformations

Radius deformation of the free boson

A Neumann brane on a circle of radius R...

$$Z_N(R) = \frac{1}{\eta(q)} \sum_{w\in\mathbb{Z}} q^{\frac{w^2}{2R^2}} = \frac{1}{\eta(q)} \sum_{w\in\mathbb{Z}} q^{\frac{w^2}{2R_0^2(1+\gamma)}}$$

Interpret this as a deformation...



$$R = R_0\sqrt{1+\gamma}$$

Anomalous dimensions

$$\delta_\gamma h_w = \frac{w^2}{2R_0^2}\left[\frac{1}{1+\gamma} - 1\right] = -\frac{\gamma}{1+\gamma}\,\frac{w^2}{2R_0^2} = -\frac{\gamma}{1+\gamma}\,C_2(w)$$

The effective deformation for conformal dimensions

- Vanishing Killing form ⇒ the perturbation is quasi-abelian (for the purposes of calculating anomalous dimensions)

  [Bershadsky,Zhukov,Vaintrob] [TQ,Schomerus,Creutzig]

- The currents behave as if they were abelian

$$J^\mu(z)\,J^\nu(w) = \frac{k\kappa^{\mu\nu}}{(z-w)^2} + \frac{if^{\mu\nu}{}_\lambda J^\lambda(w)}{z-w} \sim \frac{k\kappa^{\mu\nu}}{(z-w)^2}$$

- For $\widehat{\mathrm{OSP}}(4|2)_{-\frac{1}{2}}$ a representation Λ is shifted according to

$$\delta h_\Lambda(g^2) \;=\; -\frac{1}{2}\frac{g^2 C_\Lambda}{1+g^2} \;=\; -\frac{1}{2}\left(1-\frac{1}{R^2}\right)C_\Lambda$$

Projective superspaces

New features

- Non-trivial topology

$\Rightarrow$ Monopoles and θ-angle

- Symplectic fermions as a subsector [Candu,Creutzig,Mitev,Schomerus]

θ-angle $\Rightarrow$ twists

- σ-model brane spectrum can be argued to be

$$Z_{R,\theta}(q,z) = \underbrace{q^{-\frac{1}{2}\lambda(R,\theta)\left[1-\lambda(R,\theta)\right]}}_{\text{twist}} \sum_{\Lambda} \underbrace{q^{f(R,\theta)C_\Lambda}}_{\text{Casimir}} \underbrace{\psi_\Lambda^\infty(q)\,\chi_\Lambda(z)}_{\text{result for } R\to\infty}$$

- Currently no free field theory point is known...

Remark: The family contains the supertwistor space $\mathbb{CP}^{3|4}$ →[Witten]

Conclusions

Conclusions

- Using supersymmetry we determined the full spectrum of anomalous dimensions for certain boundary spectra in various models as a function of the radius
- The result provided strong evidence for a duality between supersphere σ-models and Gross-Neveu models

World-sheet methods appear to be more powerful than expected!

Open issues and outlook

Several open issues remain...

- More points with enhanced symmetry?
- Deformation of the bulk spectrum
- Correlation functions
- Interplay with integrability ("S-matrix approach")
- Path integral derivation?

Outlook

- Other spaces: AdS-spaces, conifold, nil-manifolds, ...
- Applications to condensed matter physics, ...