Quantum strings in $AdS_5 \times S^5$ and gauge-string duality

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- Review of recent work on gauge-string duality
- Quantum corrections to energies of “short” strings

[R. Roiban, AT, arXiv:0906.4294]

General aims:

•understand quantum gauge theories at any coupling

[applications to both perturbative and non-perturbative issues]

•understand string theories in non-trivial backgrounds

[e.g. RR ones for flux compactifications]

AdS/CFT duality:

•relates the two questions suggesting solving them together

rather than separately is best strategy

•relates simplest most symmetric theories

use of symmetries on both sides to make progress

Integrability:

Existence of powerful hidden symmetries

allowing to solve problem "in principle"

Strategy:

solve simplest most symmetric ("harmonic oscillator") case

then hope to treat other cases "in perturbation theory"

"Harmonic oscillator" (or "Ising", or "WZW"):

planar $\mathcal{N} = 4$ SYM theory = free superstring in $AdS_5 \times S^5$

most symmetric 4-d gauge th. = most symmetric 10-d string th.

$\mathcal{N} = 4$ SYM:

•maximal supersymmetry; conformal invariance;

•integrability? its precise meaning? in which observables?

could be expected in anomalous dimensions

[1-loop gluonic sector – known emergence of XXX spin chain:

Lipatov; Faddeev-Korchemsky, ...]

•in fact, ∞ of hidden symmetries should play broader role:

"inherited" via AdS/CFT from 2-d integrable QFT –

string σ-model: use 2-d int. QFT to solve 4-d CFT

Superstring in $AdS_5 \times S^5$:

•integrable in "canonical" sense: sigma-model on symmetric space

classical equations admit infinite number of conserved charges

closely related (via Pohlmeyer reduction) to

(super) sine-Gordon and non-abelian Toda eqs

e.g. special motions of strings are described by

the integrable 1-d mechanical systems (Neumann, etc.)

•integrability extends to quantum level:

evidence directly on string-theory side to 2 loops

and also indirectly via AdS/CFT "bootstrap" reasoning

Quantum integrability: should control

•spectrum of string energies on $R \times S^1$

[anom. dim's of 2-d primary operators = vertex operators on $R^{1,1}$]

•correlation functions of vertex operators (to which extent?)*

[closed-string scattering amplitudes]

*not clear even in flat space; string field theory is not "integrable"

Integrability = hidden infinite dimensional symmetry

– if valid in quantum string theory –

i.e. at any value of string tension $\frac{\sqrt{\lambda}}{2\pi}$ – any $\lambda = g^2_{\rm YM} N_c$

should be "visible" also – via AdS/CFT – in

perturbative SYM theory

Integrability should then control:

•spectrum of dimensions of gauge-inv. single ${\rm tr}$ primary operators

[or spectrum of gauge-theory energies on $R \times S^3$]

•correlation functions of these operators (to which extent?)

What about scattering amplitudes and Wilson loops?

Amplitudes – IR divergent; Cusped Wilson loops – UV divergent

Hidden (Yangian) symmetries broken at loop level in a “useful” way?

Are there “better” observables? (from integrability point of view)

Cross-sections? Effective actions?

Relation to correlation functions of gauge-inv. ops.?

Hints from string theory ?

Recent remarkable progress:

Spectrum of states

I. Spectrum of "long" operators = "semiclassical" string states

determined by Asymptotic Bethe Ansatz (2002-2007)

•its final (BES) form found after intricate superposition
of information from perturbative gauge theory (spin chain, BA,...)
and perturbative string theory (classical and 1-loop phase,...),
use of symmetries (S-matrix), and assumption of exact integrability

•consequences checked against all available gauge and string data

Key example:

cusp anomalous dimension $\mathrm{Tr}(\Phi D^S \Phi)$

$$f(\lambda \ll 1) = \frac{\lambda}{2\pi^2}\Big[1 - \frac{\lambda}{48} + \frac{11\lambda^2}{2^8 \cdot 45} - (\frac{73}{630} + \frac{4\zeta^2(3)}{\pi^6})\frac{\lambda^3}{2^7} + ...\Big]$$

$$f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi}\Big[1 - \frac{3\ln 2}{\sqrt{\lambda}} - \frac{K}{(\sqrt{\lambda})^2} - ...\Big]$$

Extensions to subleading terms in large S expansion

II. Spectrum of “short” operators = all quantum string states

Thermodynamic Bethe Ansatz (2005-2009)

●reconstructed from ABA using solely
methods/intuition of 2-d integrable QFT, i.e. string-theory side
(how to incorporate wrapping terms directly on gauge-theory side?)

●highly non-trivial construction – lack of 2-d Lorentz invariance
in the standard “BMN-vacuum-adapted” l.c. gauge

●in few cases ABA “improved” by Luscher corrections is enough:
4- and 5-loop Konishi dimension, 4-loop minimal twist op. dimension

●crucial to check predictions against perturbative gauge and string data

Key example:

anomalous dimension of Konishi operator $\mathrm{Tr}(\bar\Phi_i \Phi_i)$

$$\gamma(\lambda \ll 1) = \frac{12\lambda}{(4\pi)^2}\Big[1 - \frac{4\lambda}{(4\pi)^2} + \frac{28\lambda^2}{(4\pi)^4} - [208 - 48\zeta(3) + 120\zeta(5)]\frac{\lambda^3}{(4\pi)^6} + 8[158 + 72\zeta(3) - 54\zeta^2(3) - 90\zeta(3) + 315\zeta(7)]\frac{\lambda^4}{(4\pi)^8} + ...\Big]$$

$$\gamma(\lambda \gg 1) = 2\sqrt[4]{\lambda} + \mathrm{b}_0 + \frac{b_1}{\sqrt[4]{\lambda}} + \frac{b_2}{(\sqrt[4]{\lambda})^2} + \frac{b_3}{(\sqrt[4]{\lambda})^3} + ...$$

Suppose sum up weak-coupling expansion and re-expand at large λ –

values of $\mathrm{b}_0, b_1, b_2, ...$?

directly from string theory ?

from TBA/Y-system that should be describing string spectrum ?

talks by Kazakov and Gromov

Many open questions:

Analytic form of strong-coupling expansion from TBA/Y-system?

Matching onto string spectrum in near-flat-space expansion?

No level crossing?

Strong-coupling expansion is Borel (non)summable?

Exponential corrections $e^{-a\sqrt{\lambda}}$ like in cusp anomaly case?

...

Deeper issues:

Solve string theory from first principles –

•fundamental variables? preserve 2-d Lorentz invariance?

•prove quantum integrability?

lattice version of “supercoset” sigma model?

Another remarkable recent progress:

Amplitudes, Wilson loops and their symmetries

Weak coupling:

various connections to hidden symmetries and integrability

conformal symmetry, dual conformal symmetry,

their “unification” in the Yangian (at classical level)

Other suggestions about role of integrability in the amplitudes

[talk by Lipatov]

Strong coupling:

use of integrability of string theory to determine (via relation to WL’s)

leading contributions to certain gluon scattering amplitudes

[Alday and Maldacena,...]

“tree-level” $AdS_5 \times S^5$ superstring = planar $\mathcal{N} = 4$ SYM

Recent remarkable progress in quantitative understanding

interpolation from weak to strong ‘t Hooft coupling

based on/checked by perturbative gauge theory (4-loop in λ)

and perturbative string theory (2-loop in $\frac{1}{\sqrt{\lambda}}$) “data”

and (strong evidence of) exact integrability

string energies = dimensions of local $\mathrm{Tr}(...)$ operators

$$E(\sqrt{\lambda}, C, m, ...) = \Delta(\lambda, C, m, ...)$$

C - “charges” of $SO(2,4) \times SO(6)$: $S_1, S_2;\ J_1, J_2, J_3$

m - windings, folds, cusps, oscillation numbers, ...

Operators: $\mathrm{Tr}(\Phi_1^{J_1} \Phi_2^{J_2} \Phi_3^{J_3} D_+^{S_1} D_\perp^{S_2} ... F_{mn} ... \Psi ...)$

Solve supersymmetric 4-d CFT

= Solve string in curved R-R background (2-d CFT):

compute $E = \Delta$ for any λ (and any C,m)

Problem: perturbative expansions are opposite

$\lambda \gg 1$ in perturbative string theory

$\lambda \ll 1$ in perturbative gauge theory

weak-coupling expansion convergent – defines $\Delta(\lambda)$

need to go beyond perturbation theory: integrability

Last 7 years – remarkable progress for subclass of states:

"semiclassical" string states with large quantum numbers

dual to "long" SYM operators (canonical dim. $\Delta_0 \gg 1$)

[BMN 02, GKP 02, FT 03,...]

$E = \Delta$ – same (in some cases !) dependence on $C, m, \ldots$

coefficients = "interpolating" functions of λ

Current status:

1. "Long" operators = strings with large quantum numbers:
Asymptotic Bethe Ansatz (ABA) [Beisert, Eden, Staudacher 06]
firmly established (including non-trivial phase factor)
2. "Short" operators = general quantum string states:
partial progress based on improving ABA by
"Luscher corrections" [Janik et al 08]
generalize ABA to TBA [Abjorn et al; Arutyunov, Frolov 07-09]
underlying Y-system, TBA eqs. for excited states
[Gromov, Kazakov, Vieira 09]

To justify from first principles need better understanding
of quantum $AdS_5 \times S^5$ superstring theory
1. Solve string theory on a plane $R^{1,1}$ $\rightarrow$
relativistic 2d S-matrix $\rightarrow$ asymptotic BA for the spectrum
2. Generalize to finite-energy closed strings – the theory on $R \times S^1$
$\rightarrow$ TBA (cf. integrable sigma models)

Superstring theory in $AdS_5 \times S^5$

bosonic coset $\frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

generalized to supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ [Metsaev, AT 98]

$$S = T \int d^2\sigma \Big[G_{mn}(x)\partial x^m \partial x^n + \bar{\theta}(D + F_5)\theta\partial x + \bar{\theta}\theta\bar{\theta}\theta\partial x\partial x + ... \Big]$$

tension $T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$

Conformal invariance: $\beta_{mn} = R_{mn} - (F_5)^2_{mn} = 0$

Classical (Luscher-Pohlmeyer 76) integrability of coset σ-model true for $AdS_5 \times S^5$ superstring [Bena, Polchinski, Roiban 02]

Progress in understanding of implications of (semi)classical integrability [Kazakov, Marshakov, Minahan, Zarembo 04,...]

Reformulation in terms of currents with Virasoro conditions solved: "Pohlmeyer reduction"

[Grigoriev, AT 07; Mikhailov, Schafer-Nameki 07, Roiban, AT 09]

1-loop quantum superstring corrections
[Frolov, AT; Park, Tirziu, AT, 02-04, ...]
used as an input data to fix 1-loop term
in strong-coupling expansion of the phase $\theta(\lambda)$ in ABA
[Beisert, AT 05; Hernandez, Lopez 06]

2-loop quantum superstring corrections
[Roiban, Tirziu, AT; Roiban, AT 07]
– check of finiteness of the GS superstring
– implicit check of integrability of quantum string theory
– non-trivial confirmation of BES phase in ABA
[Benna, Benvenuti, Klebanov, Scardicchio 07;
Basso, Korchemsky, Kotansky 07]

$AdS_5 \times S^5 = \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

Killing vectors and Killing spinors of $AdS_5 \times S^5$:

$PSU(2,2|4)$ symmetry

replace G/H=SuperPoincare/Lorentz in flat GS case by

$$\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$$

$PSU(2,2|4)$ invariant action:

$\int \mathrm{Tr}(g^{-1}dg)^2_{G/H}$ + WZ-term

$J = g^{-1}dg = J^m \mathcal{P}_m + J^I_\alpha \mathrm{Q}^\alpha_I + J^{mn}\mathcal{M}_{mn}$

$$I = \frac{\sqrt{\lambda}}{2\pi}\left[\int d^2\sigma (J^m J^m + a\bar{J}^I J^I) + b\int \; J^m \wedge \bar{J}^I \Gamma_m J^J s_{IJ}\right]$$

as in flat space $a = 0, \;\; b = \pm 1$ required by κ-symmetry

unique action with right symmetry and right flat-space limit

Formal argument for UV finiteness (2d conformal invariance):

1. global symmetry – only overall coefficient
   of J^2 term (radius) can run
2. non-renormalization of WZ term (homogeneous 3-form)
3. preservation of κ-symmetry at the quantum level
   – relating coefficients of J^2 and WZ terms

Equivalent form of the GS action:

$AdS_5 \times S^5 = \frac{SU(2,2)}{Sp(2,2)} \times \frac{SU(4)}{Sp(4)}$

generalized to

$\frac{\widehat{F}}{G} = \frac{PSU(2,2|4)}{Sp(2,2)\times Sp(4)}$

basic superalgebra $\widehat{\mathfrak{f}} = psu(2,2|4)$

bosonic part $\mathfrak{f} = su(2,2) \oplus su(4) \cong so(2,4) \oplus so(6)$

admits Z_4-grading:

(Berkovits, Bershadsky, Hauer, Zhukov, Zwiebach 89)

$$\widehat{\mathfrak{f}} = \mathfrak{f}_0 \oplus \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3 \,, \qquad [\mathfrak{f}_i, \mathfrak{f}_j] \subset \mathfrak{f}_{i+j \,\mathrm{mod}\, 4}$$

$\mathfrak{f}_0 = \mathfrak{g} = sp(2,2) \oplus sp(4)$

$\mathfrak{f}_2 = AdS_5 \times S^5$

current $J = f^{-1}\partial_a f, \ f \in \widehat{F}$

$$J_a = f^{-1}\partial_a f = \mathcal{A}_a + Q_{1a} + P_a + Q_{2a}$$

$$\mathcal{A} \in \mathfrak{f}_0, \quad Q_1 \in \mathfrak{f}_1, \quad P \in \mathfrak{f}_2, \quad Q_2 \in \mathfrak{f}_3 \,.$$

$$I = \int \mathrm{STr}\Big(P \wedge *P + Q_1 \wedge Q_2\Big)$$

How to solve quantum string theory in $AdS_5 \times S^5$?

GS string on supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$
not of known solvable type (cf. free oscillators; WZW)
analogy with exact solution of $O(n)$ model (Zamolodchikovs) or
principal chiral model (Polyakov-Wiegmann ...) ?
but 2d CFT – no mass generation

By analogy with flat space –
light-cone gauge: analog of $x^+ = p^+\tau,\ p^+ = \text{const},\ \Gamma^+\theta = 0$
Two natural options:
(i) null geodesic parallel to the boundary in Poincare patch –
action/Hamiltonian quartic in fermions (Metsaev, Thorn, AT, 01)
(ii) null geodesic wrapping S^5:
complicated action

Common problem:

lack of manifest 2d Lorentz symmetry

hard to apply known 2d integrable field theory methods –

S-matrix depends on two rapidities, not on their difference

constraints on it are a priori unclear...

An alternative approach: "Pohlmeyer reduction"

conformal gauge, solve Virasoro conditions

find "reduced" action in terms of currents

use it as a starting point for quantization

Aim: PR version for $AdS_5 \times S^5$ superstring

(i) introduce new fields locally related to supercoset currents

(ii) solve Virasoro condition explicitly

(iii) find local 2d Lorentz-invariant

action for independent (8B+8F) d.o.f

$\longrightarrow$ fermionic generalization of non-abelian Toda theory

PR: a nonlocal map that preserves integrable structure

1. gauge-equivalent Lax pairs; map between soliton solutions

gives integrable massive local field theory

2. quantum equivalence to original GS model ?

may expect for full $AdS_5 \times S^5$ string model = CFT

3. integrable theory: semiclassical solitonic spectrum

may essentially determine quantum spectrum

the two solitonic S-matrices should be closely related:

Lorentz-invariant S-matrix of PR-model should lead to

effective magnon S-matrix

Reduced action for $AdS_5 \times S^5$ superstring

(Grigoriev, AT 07; Mikhailov, Schafer-Nameki 07)

classical gauge-fixed 1-st order equations in terms of currents

follow from an action!

fermionic generalization of "gWZW+ potential" theory for

$\frac{G}{H} = \frac{Sp(2,2)}{SU(2)\times SU(2)} \times \frac{Sp(4)}{SU(2)\times SU(2)}$

$$
\begin{aligned}
L &= L_{\rm gWZW}(g, A_+, A_-) + \mu^2\, {\rm STr}(g^{-1}TgT) \\
&+ {\rm STr}\,(\Psi_L T D_+ \Psi_L + \Psi_R T D_- \Psi_R) \\
&+ \mu\, {\rm STr}\left(g^{-1}\Psi_L g \Psi_R\right)
\end{aligned}
$$

sum of PR theories for AdS_5 and S^5 "glued" by fermions

$$
\begin{aligned}
L &= \widetilde{L}_{AdS_5}(g_a, A_{\pm,a}) + \widetilde{L}_{S^5}(g_s, A_{\pm,s}) \\
&+ \psi_L D_+ \psi_L + \psi_R D_+ \psi_R + O(\mu)
\end{aligned}
$$

similar but not same as susy gWZW:

fermions are in "mixed" representation

standard 2d kin. terms

$$L_F = \mathrm{STr}(\Psi_L T \partial_+ \Psi_L + \Psi_R T \partial_- \Psi_R) + ...$$
$$= -2i \; \mathrm{Tr}(\xi_L^t \partial_+ \xi_L + \eta_L^t \partial_+ \eta_L + \xi_R^t \partial_- \xi_R + \eta_R^t \partial_- \eta_R) + ...$$

classically integrable model:

fermionic generalization of non-abelian Toda model

Lax pair encoding equations of motion

$$\mathcal{L}_- = \partial_- + A_- + \ell^{-1}\sqrt{\mu} g^{-1} \Psi_L g + \ell^{-2} \mu g^{-1} T g \,,$$
$$\mathcal{L}_+ = \partial_+ + g^{-1}\partial_+ g + g^{-1} A_+ g + \ell\sqrt{\mu}\Psi_R + \ell^2 \mu T$$

Quantum properties:

• UV finite theory [Roiban, AT 09]

• semiclassical (1-loop) partition function

same as in GS theory [Hoare, Iwashita, AT 09]

Comments:

- gWZW model coupled to the fermions interacting minimally and through the "Yukawa term"
- 2d Lorentz invariant with $\Psi_{_R}, \Psi_{_L}$ as 2d Majorana spinors
- 8 real bosonic and 16 real fermionic independent variables
- 2d supersymmetry? yes, in $AdS_2 \times S^2$ case: $n = 2$ super sine-Gordon
- μ-dependent interactions are equal to GS Lagrangian; gWZW produces MC eq.: path integral derivation via change from fields to currents?
- quadratic in fermions (like susy version of gWZW); integrating out $A_\pm$ gives quartic fermionic terms (reflecting curvature)
- linearisation of EOM in the gauge $A_\pm = 0$ around $g = \mathbf{1}$ describes 8+8 massive bosonic and fermionic d.o.f. with mass μ: same as in BMN limit
- symmetry of resulting relativistic S-matrix: $H = [SU(2)]^4$ – as bosonic part of magnon S-matrix symmetry $[PSU(2|2)]^2$

Open questions

- Quantum equivalence of reduced theory and GS theory?

  Path integral argument of equivalence?

- Indication of equivalence: semiclassical expansion

  near analog of (S, J) rigid string in $AdS_5 \times S^5$ leads to same characteristic frequencies

  – same 1-loop partition function

  (Roiban, AT 08; Hoare, Iwashita, Tseytlin 09)

- Tree-level S-matrix for elementary excitations?

  Manifest $SU(2) \times SU(2) \times SU(2) \times SU(2)$ symmetry?

  Relation to magnon S-matrix in BA?

Gauge states vs string states: principles of comparison

1. compare states with same global $SO(2,4) \times SO(6)$ charges

e.g., (S, J) – "sl(2) sector" – $\mathrm{Tr}(D_+^S \Phi^J)$

2. assume no "level crossing" while changing λ

min/max energy (S, J) states should be in correspondence

Gauge theory:

$\Delta \equiv E = S + J + \gamma(S, J, m, \lambda)$,

$\gamma = \sum_{k=1}^{\infty} \lambda^k \gamma_k(S, J, m)$

fix S, J, ... and expand in λ;

then may expand in large/small S, J, ...

Semiclassical string theory:

$E = S + J + \gamma(\mathcal{S}, \mathcal{J}, m, \sqrt{\lambda})$,

$\gamma = \sum_{k=-1}^{\infty} \frac{1}{(\sqrt{\lambda})^k} \widetilde{\gamma}_k(\mathcal{S}, \mathcal{J}, m)$

fix semiclassical parameters $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$, $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$, m

To match in general will need to resum – beyond ABA

Dimensions of short operators
= energies of quantum string states:

progress in understanding spectrum of conformal dimensions of planar $N = 4$ SYM or spectrum of strings in $AdS_5 \times S^5$ based on quantum integrability

Spectrum of states with large quantum numbers – solution of ABA equations

Recent proposal of how to extend this to "short" states with any quantum numbers – TBA / Y-system approach

compare to direct quantum string results

Aim: compute leading $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ correction to dimension of "lightest" massive string state dual to Konishi operator in SYM theory

– check against (numerical) prediction of Y-system approach

Konishi operator:

operators (long multiplet) related to singlet by susy

$[J_2 - J_3, J_1 - J_2, J_2 + J_3]^{\Delta_0}_{(s_L, s_R)} = [0,0,0]^2_{(0,0)}$

$\Delta = \Delta_0 + \gamma(\lambda), \qquad \Delta_0 = 2, \frac{5}{2}, 3, ..., 10$

– same anomalous dimension γ

singlet eigen-state of anom. dim. matrix with lowest eigenvalue

examples:

$\mathrm{Tr}(\bar{\Phi}_i \Phi_i), \quad i = 1,2,3, \quad \Delta_0 = 2$

$\mathrm{Tr}([\Phi_1, \Phi_2]^2)$ in $su(2)$ sector $\Delta_0 = 4$

$\mathrm{Tr}(\Phi_1 D_+^2 \Phi_1)$ in $sl(2)$ sector $\Delta_0 = 4$

Weak-coupling expansion of $\gamma(\lambda)$: $\qquad \lambda = g^2_{\mathrm{YM}} N_c$

$$\gamma(\lambda) = 12\Big[\frac{\lambda}{(4\pi)^2} - 4\frac{\lambda^2}{(4\pi)^4} + 28\frac{\lambda^3}{(4\pi)^6} + [-208 + 48\zeta(3) - 120\zeta(5)]\frac{\lambda^4}{(4\pi)^8} + ...\Big] + 5 - \mathrm{loop}$$

[Fiamberti et al; Bajnok,Janik; Velizhanin 08; Banjok et al 09]

Long Konishi multiplet

$\Delta_{0\ min} = 2, \;\; [m,n,k]_{(s,s')} = [0,0,0]_{(0,0)}$

$SO(6)$ and $SO(4)$ labels

[Andreanopoli,Ferrara 98; Bianchi,Morales,Samtleben 03]

see table

Finite radius of convergence ($N_c = \infty$) – if we could sum up and then re-expand at large λ – what to expect? (cf. $f(\lambda)$)

AdS/CFT duality: Konishi operator dual to
"lightest" among massive $AdS_5 \times S^5$ string states
large $\sqrt{\lambda} = \frac{\mathrm{R}^2}{\alpha'}$:
– "small" string at center of AdS_5 – in nearly flat space

Flat space case:

$m^2 = \frac{4(n-1)}{\alpha'}$, $\quad n = \frac{1}{2}(N + \bar{N}) = 1, 2, ...,\ \ N = \bar{N}$

$n = 1$: massless IIB supergravity (BPS) level

l.c. vacuum $|0>$: $(8+8)^2 = 256$ states

$n = 2$: first massive level (many states, highly degenerate)

$[(a^i_{-1} + S^a_{-1})|0>]^2 = [(8+8) \times (8+8)]^2$

in $SO(9)$ reps:

$([2,0,0,0] + [0,0,1,0] + [1,0,0,1])^2 = (44 + 84 + 128)^2$

e.g. $44 \times 44 = 1 + 36 + 44 + 450 + 495 + 910$

$84 \times 84 = 1 + 36 + 44 + 84 + 126 + 495 + 594 + 924 + 1980 + 2772$

switching on $AdS_5 \times S^5$ background fields lifts degeneracy

states with "lightest mass" at first excited string level

should correspond to Konishi multiplet

string spectrum in $AdS_5 \times S^5$:

long multiplets $\mathcal{A}^{\Delta}_{[k,p,q](j,j')}$ of $PSU(2,2|4)$

highest weight states: $[k,p,q]_{(s,s')}$ labels of $SO(6) \times SO(4)$

Remarkably, flat-space string spectrum can be re-organized

in multiplets of $SO(2,4) \times SO(6) \subset PSU(2,2|4)$

[Bianchi, Morales, Samtleben 03; Beisert et al 03]

$SO(4) \times SO(5) \subset SO(9)$ rep.

lifted to $SO(4) \times SO(6)$ rep. of $SO(2,4) \times SO(6)$

Konishi long multiplet

$\widehat{T}_1 = (1 + Q + Q \wedge Q + \ldots)[0,0,0]_{(0,0)}$

determines the KK "floor" of 1-st excited string level

$H_1 = \sum_{J=0}^{\infty} [0,J,0]_{(0,0)} \times \widehat{T}_1$

One expects for scalar massive state in AdS_5

$(-\nabla^2 + m^2)\Phi + ... = 0$

$\Delta(\Delta - 4) = (m\mathrm{R})^2 + O(\alpha') = 4(n-1)\frac{\mathrm{R}^2}{\alpha'} + O(\alpha')$

$\Delta = 2 + \sqrt{(m\mathrm{R})^2 + 4 + O(\alpha')}$

$$\Delta(\lambda \gg 1) = \sqrt{4(n-1)\sqrt{\lambda}} + ...$$

[Gubser, Klebanov, Polyakov 98]

e.g., for first massive level:

$n = 2: \qquad \Delta = 2\sqrt{\sqrt{\lambda}} + ...$

Subleading corrections?

Comparison between gauge and string theory states non-trivial:

GT ($\lambda \ll 1$): operators built out of free fields,
canonical dimension Δ_0 determines states that can mix
ST ($\lambda \gg 1$): near-flat-space string states built out of
free oscillators, level n determines states that can mix

meaning of Δ_0 at strong coupling?
meaning of n at weak coupling?

1. relate states with same global charges;
2. assume "non-intersection principle" [Polyakov 01]:
no level crossing for states with same quantum numbers
as λ changes from strong to weak coupling

Approaches to computation of corrections to string energies:

(i) vertex operator approach:
use $AdS_5 \times S^5$ string sigma model perturbation theory to find leading terms in anomalous dimension of corresponding vertex operator
[Polyakov 01; AT 03]

(ii) space-time effective action approach:
use near-flat-space expansion and NSR vertex operators to reconstruct $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections to corresponding massive string state equation of motion
(iii) "light-cone" quantization approach:
start with light-cone gauge $AdS_5 \times S^5$ string action and compute corrections to energy of corresponding flat-space oscillator string state
[Metsaev, Thorn, AT 00]

(iv) semiclassical approach:

identify short string state as small-spin limit of semiclassical string state

– reproduce the structure of strong-coupling corrections to short operators

[Tirziu, AT 08; Roiban, AT 09]

Spectrum of quantum string states

from target space anomalous dimension operator

Flat space: $k^2 = m^2 = \frac{4(n-1)}{\alpha'}$

e.g. leading Regge trajectory $(\partial x \bar\partial x)^{S/2} e^{ikx}, \ \ n = S/2$

spectrum in (weakly) curved background:

solve marginality (1,1) conditions on vertex operators

e.g. scalar anomalous dimension operator $\widehat{\gamma}(G)$

on $T(x) = \sum c_{n...m} x^n ... x^m$ or on coefficients $c_{n...m}$

differential operator in target space

found from β-function for the corresponding perturbation

$$I = \frac{1}{4\pi\alpha'} \int d^2 z [G_{mn}(x) \partial x^m \bar\partial x^n + T(x)]$$

$$\beta_T = -2T - \tfrac{\alpha'}{2} \, \widehat{\gamma} \, T + O(T^2)$$

$$\widehat{\gamma} = \Omega^{mn} D_m D_n + ... + \Omega^{m...k} D_m ... D_k + ...$$

$$\Omega^{mn} = G^{mn} + O(\alpha'^3), \qquad \qquad \Omega^{\cdots} \sim \alpha'^n R^p_{\cdots}$$

Solve $-\widehat{\gamma} \, T + m^2 T = 0$: diagonalize $\widehat{\gamma}$

similarly for massless (graviton, ...) and massive states

e.g. $\beta^G_{mn} = \alpha' R_{mn} + O(\alpha'^3)$

gives Lichnerowitz operator as anomalous dimension operator

$$(\widehat{\gamma} h)_{mn} = -D^2 h_{mn} + 2R_{mknl}h^{kl} - 2R_{k(m}h^k_{n)} + O(\alpha'^3)$$

Massive string states in curved background:

$$\int d^D x \sqrt{g} \left[\Phi_{...}(-D^2 + m^2 + X)\Phi_{...} + ... \right]$$
$$m^2 = \tfrac{4}{\alpha'}(n-1) \,, \qquad X = R_{....} + O(\alpha')$$

case of $AdS_5 \times S^5$ background

$$R_{mn} - \tfrac{1}{96}(F_5 F_5)_{mn} = 0, \quad R = 0 \,, \quad F_5^2 = 0$$

Find leading-order term in X ?

leading α' correction to scalar string state mass =0 (?!)

$$[-D^2 + m^2 + O(\tfrac{1}{\sqrt{\lambda}})]\Phi \ = 0$$
$$\Delta = 2 + \sqrt{4(n-1) + 4 + O(\tfrac{1}{\sqrt{\lambda}})}$$

$$\Delta_{(n=2)} = 2 + 2\sqrt{\sqrt{\lambda}}\,\Big[1 + \tfrac{1}{2\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2})\Big]$$

prediction (?) for leading term in strong-coupling expansion
of singlet Konishi state dimension
... but possible subtleties... 10d scalar vs singlet state...

What about non-singlet Konishi descendant states ?
– they should have the same dimension
$\text{Tr}[\Phi_1, \Phi_2]^2$ corresponds to $SO(6)$ (2,2,0) state $J_1 = J_2 = 2$
tensor wave function $\Phi_{mn;kl}$
or vertex operator like (see below)
$\sim N_+^{-\Delta}\partial n_x \bar{\partial} n_x \partial n_y \bar{\partial} n_y$
S^5: $\quad n_a n_a = 1, n_x = n_1 + i n_2, \quad n_y = n_3 + i n_4$
AdS_5: $\quad N_+ = N_0 + i N_5, \quad N_+ N_- - N_k N_k = 1$
$\text{Tr}(\Phi_1 D_+^2 \Phi_1)$ should correspond to state with spins $S = J = 2$

How to find $\widehat{\gamma}$: Effective action approach

derive equation of motion for a massive string field
in curved background from quadratic effective action S
reconstructed from flat-space NSR S-matrix
Example: totally symmetric NS-NS 10-d tensor
– state on leading Regge trajectory in flat space

symmetric tensor $\Phi_{\mu_1 \ldots \mu_{2n}}$ $\quad (m^2 = \frac{4(n-1)}{\alpha'})$
in metric+RR background

$$L = R - \tfrac{1}{2\cdot 5!} F_5^2 + O(\alpha'^3)$$
$$- \tfrac{1}{2}(D_\mu \Phi D^\mu \Phi + m^2 \Phi^2) + \sum_{k\geq 1} (\alpha')^{k-1} \Phi X_k(R, F_5, D)\Phi + \ldots$$

assumption: $\alpha' n R \ll 1, \quad i.e. \quad n \ll \sqrt{\lambda}$:
small massive string in the middle of AdS_5:
near-flat-space expansion should be applicable

$AdS_5 \times S^5$ background: $R_{ab} = -\frac{4}{\mathrm{R}^2} g_{ab}, \ R_{mn} = \frac{4}{\mathrm{R}^2} g_{mn}$

$\mu, \nu, \ldots = 0, 1, ...9; \quad a, b, \ldots$ in AdS_5 and $m, n, \ldots$ in S^5

$$L = \tfrac{1}{2}\Phi_{\mu_1\cdots\mu_{2n}}(-D^2 + m^2)\Phi^{\mu_1\cdots\mu_{2n}}$$
$$+\frac{n^2}{\mathrm{R}^2}\left(\Phi_{a_1 a_2 \mu_3\cdots\mu_{2n}}\Phi^{a_1 a_2 \mu_3\cdots\mu_{2n}} - \Phi_{m_1 m_2 \mu_3\cdots\mu_{2n}}\Phi^{m_1 m_2 \mu_3\cdots\mu_{2n}}\right) + ...$$

background is direct product – can consider particular tensor

with S indices in AdS_5 and K indices in S^5:

end up with anomalous dimension operator

$$[-D^2 + (m^2 + \tfrac{K^2 - S^2}{2\mathrm{R}^2})]\Phi = 0 \ , \qquad D^2 = D^2_{AdS_5} + D^2_{S_5}$$
$$m^2 = \tfrac{4}{\alpha'}(n-1) = \tfrac{2}{\alpha'}(S + K - 2), \qquad 2n = S + K$$

symmetric transverse traceless tensor – highest-weight state –

Young table labels $(\Delta, S, 0; J, K, 0)$,

extract AdS_5 radius R and set $\sqrt{\lambda} = \frac{\mathrm{R}^2}{\alpha'}$

$$(-D^2_{AdS_5} + M^2)\Phi = 0$$
$$M^2 = 2\sqrt{\lambda}(S + K - 2) + \frac{1}{2}(K^2 - S^2) + J(J+4) - K$$

For symmetric traceless rank S tensor in AdS_5:

$$\Delta - 2 = \sqrt{M^2 + S + 4}$$
$$= \sqrt{2\sqrt{\lambda}(S+K-2) + \tfrac{1}{2}(S+K-2)(K-S) + J(J+4) + 4 + O(\tfrac{1}{\sqrt{\lambda}})}$$

[Burrington, Liu 05]

condition of marginality of (1,1) vertex operator

for $(\Delta, S_1, S_2; J_1, J_2, J_3) = (\Delta, S, 0; K, J, 0)$ state

$$0 = -\sqrt{\lambda}(S+K-2)$$
$$+\frac{1}{2}[\Delta(\Delta-4) + \frac{1}{2}S(S-2) - \frac{1}{2}K(K-2) - J(J+4)] + O(\tfrac{1}{\sqrt{\lambda}})$$

BPS level: $n = \frac{1}{2}(S+K) = 1$

First massive level: $n = \frac{1}{2}(S+K) = 2$

minimal dimension shift

$S=2,\ K=2,\ J=0$ case: $[1,0,1]_{(1,1)}$

state with $\Delta_0 = 4$ or $\Delta_0 = 6$

To summarize: string states in $\mathrm{AdS}_5 \times S^5$ labeled by

$SU(2,2|4) \subset SO(2,4)\times SO(6)$ quantum numbers $(\Delta, S_1, S_2; J_1, J_2, J_3)$

condition of marginality of (1,1) operator for $(\Delta, S, 0; K, J, 0)$

$$0 = -\sqrt{\lambda}(S + K - 2)$$
$$+\frac{1}{2}[\Delta(\Delta - 4) + \frac{1}{2}S(S-2) - \frac{1}{2}K(K-2) - J(J+4)] + O(\tfrac{1}{\sqrt{\lambda}})$$

symmetry: analytic continuation between AdS_5 and S^5

$\Delta \leftrightarrow -J, \ K \leftrightarrow S$

Implications for Konishi state dimension ?

state from first massive level on leading Regge trajectory

$S = K = 2, \ J = 0$:

$$\Delta = 2 + \sqrt{4\sqrt{\lambda} + 4 + O(\tfrac{1}{\sqrt{\lambda}})} = 2 + 2\sqrt[4]{\lambda}(1 + \frac{1}{\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2}))$$

constant term 2=$\Delta_0 - 4$ for $\Delta_0 = 6$ operator

Vertex operator approach [Polyakov 01; AT 03]

calculate 2d anomalous dimensions from "first principles"–
superstring theory in $AdS_5 \times S^5$:

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \Big[\partial N_p \bar{\partial} N^p + \partial n_k \bar{\partial} n_k + \text{fermions} \Big]$$

$$N_+ N_- - N_u N_u^* - N_v N_v^* = 1 \,, \qquad n_x n_x^* + n_y n_y^* + n_z n_z^* = 1$$

$$N_\pm = N_0 \pm i N_5, \quad N_u = N_1 + i N_2, \; ..., \;\; n_x = n_1 + i n_2, ...$$

construct marginal (1,1) operators in terms of N_p and n_k

e.g. vertex operator for dilaton sugra mode (HW state)

$$\mathrm{V}_J = \; (N_+)^{-\Delta} \, (n_x)^J \, \Big[- \partial N_p \bar{\partial} N^p + \partial n_k \bar{\partial} n_k + \text{fermions} \Big]$$

$N_+ \equiv N_0 + i N_5 = \frac{1}{z}(z^2 + x_m x_m) \sim e^{it}$

$n_x \equiv n_1 + i n_2 \sim e^{i\varphi}$

$$0 = 2 - 2 + \frac{1}{2\sqrt{\lambda}} [\Delta(\Delta - 4) - J(J+4)] + O(\tfrac{1}{(\sqrt{\lambda})^2})$$

i.e. $\Delta = 4 + J$ (BPS)

cf. vertex operator for bosonic string state

on leading Regge trajectory in flat space $\alpha' E^2 = 2(S-2)$

$$\mathrm{V}_S = e^{-iEt}\big(\partial x \bar\partial x\big)^{S/2}\,, \quad x = x_1 + i x_2$$

candidate operators for states on leading Regge trajectory:

$$\mathrm{V}_J = (N_+)^{-\Delta}\big(\partial n_x \bar\partial n_x\big)^{J/2}\,, \qquad n_x \equiv n_1 + i n_2$$

$$\mathrm{V}_S(\xi) = (N_+)^{-\Delta}\big(\partial N_u \bar\partial N_u\big)^{S/2}\,, \qquad N_u \equiv N_1 + i N_2$$

+ fermionic terms

+ $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ terms from diagonalization of anom. dim. op.

how they mix with ops with same charges and dimension?

in general $\big(\partial n_x \bar\partial n_x\big)^{J/2}$ mixes with singlets

$$(n_x)^{2p+2q}(\partial n_x)^{J/2-2p}(\bar\partial n_x)^{J/2-2q}(\partial n_m \partial n_m)^p(\bar\partial n_k \partial n_k)^q$$

ops. for states on leading Regge trajectory

$$O_{\ell,s} = f_{k_1 \ldots k_\ell m_1 \ldots m_{2s}} n_{k_1} \ldots n_{k_\ell} \partial n_{m_1} \bar\partial n_{m_2} \ldots \partial n_{m_{2s-1}} \bar\partial n_{m_{2s}}$$

their renormalization studied before [Wegner 90]

simplest case: $f_{k_1\ldots k_\ell} n_{k_1}...n_{k_\ell}$ with traceless $f_{k_1\ldots k_\ell}$
same anom. dim. $\widehat{\gamma}$ as its highest-weight rep $V_J = (n_x)^J$

$$\widehat{\gamma} = 2 - \frac{1}{2\sqrt{\lambda}} J(J+4) + ...$$

scalar spherical harmonic that solves Laplace eq. on S^5
similarly for AdS_5 or $SO(2,4)$ model:
replacing n_x^J and $\partial n_m \bar{\partial} n_m$ with $N_+^{-\Delta}$ and $\partial N^p \bar{\partial} N_p$, with
$J = -\Delta$ and $\frac{1}{\sqrt{\lambda}} \to -\frac{1}{\sqrt{\lambda}}$
e.g. dimension of $n_x^J \partial n_m \bar{\partial} n_m$:
$\widehat{\gamma} = -\frac{1}{2\sqrt{\lambda}} J(J+4) + O(\frac{1}{(\sqrt{\lambda})^2})$
dimension of $N_+^{-\Delta} \partial N^p \bar{\partial} N_p$:
$\widehat{\gamma} = \frac{1}{2\sqrt{\lambda}} \Delta(\Delta-4) + O(\frac{1}{(\sqrt{\lambda})^2})$

Example of scalar higher-level operator:

$$N_+^{-\Delta}[(\partial n_k \bar{\partial} n_k)^r + ...] \ , \qquad r = 1, 2, ...$$

[Kravtsov, Lerner, Yudson 89; Castilla, Chakravarty 96]

$$0 = -2(r-1) + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) + 2r(r-1)]$$
$$+ \frac{1}{(\sqrt{\lambda})^2}[\tfrac{2}{3}r(r-1)(r-\tfrac{7}{2}) + 4r] + ...$$

$r = 1$: ground level

fermionic contributions should make $r = 1$ exact zero of $\widehat{\gamma}$

$r = 2$: first excited level

candidate for singlet Konishi state $\Delta_0 = 2$

$$\Delta(\Delta - 4) = 4\sqrt{\lambda} - 4 + O(\tfrac{1}{\sqrt{\lambda}}) \ ,$$
$$\Delta - \Delta_0 = 2\sqrt[4]{\lambda}\Big[1 + 0 \times \tfrac{1}{\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2})\Big]$$

but fermionic contribution may change this

Operators with two spins $J_1 = J,\ J_2 = K$ in S^5:

$$V_{K,J} = N_+^{-\Delta} \sum_{u,v=0}^{K/2} c_{uv} M_{uv}$$
$$M_{uv} \equiv n_y^{J-u-v} n_x^{u+v} (\partial n_y)^u (\partial n_x)^{K/2-u} (\bar{\partial} n_y)^v (\bar{\partial} n_x)^{K/2-v}$$

highest and lowest eigen-values of 1-loop anom. dim. matrix

$$\widehat{\gamma}_{\min} = 2 - K + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta-4) - \frac{1}{2}K(K+10) - J(J+4) - 2JK] + O(\tfrac{1}{(\sqrt{\lambda})^2})$$
$$\widehat{\gamma}_{\max} = 2 - K + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta-4) - \frac{1}{2}K(K+6) - J(J+4)] + O(\tfrac{1}{(\sqrt{\lambda})^2})$$

fermions may alter terms linear in K

How to take fermionic contribution into account?

Semiclassical expansion: spinning strings

$$E = E(\frac{J}{\sqrt{\lambda}}, \sqrt{\lambda}) = \sqrt{\lambda}\mathcal{E}_0(\mathcal{J}) + \mathcal{E}_1(\mathcal{J}) + \frac{1}{\sqrt{\lambda}}\mathcal{E}_2(\mathcal{J}) + ...$$

in "short" string limit $\mathcal{J} \ll 1$

$$\mathcal{E}_n = \sqrt{\mathcal{J}}\ (a_{0n} + a_{1n}\mathcal{J} + a_{2n}\mathcal{J}^2 + ...)$$

expansion valid for $\sqrt{\lambda} \gg 1$ and $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$ fixed: $J \sim \sqrt{\lambda} \gg 1$
but if knew all terms in this expansion – could express $\mathcal{J}$
in terms of J, fix J to finite value and re-expand in $\sqrt{\lambda}$

$$E = \sqrt{\sqrt{\lambda} J}\Big[a_{00} + \frac{a_{10}J + a_{01}}{\sqrt{\lambda}} + \frac{a_{20}J^2 + a_{11}J + a_{02}}{(\sqrt{\lambda})^2} + ...\Big]$$

to trust the coeff of $\frac{1}{(\sqrt{\lambda})^n}$ need coeff of up to n-loop terms
e.g. classical a_{10} and 1-loop a_{01} sufficient to fix $\frac{1}{\sqrt{\lambda}}$ term
[cf. "fast string" expansion $\mathcal{J} \gg 1$ for fixed J
positive powers of $\sqrt{\lambda}$ – need to resum]

Logic: interested in short string probing flat-space limit–

(i) start with classical string solutions in flat space

representing states at 1-st excited string level

(ii) embed into $AdS_5 \times S^5$ and compute 1-loop correction to energy

Two basic examples:

(1) circular string with 2 spins in two orthogonal planes

(2) folded spinning string

Rigid circular string rotating in two planes of R^4

$$t = \kappa\tau \ , \qquad \mathrm{x}_x \equiv x_1 + i x_2 = \ a\, e^{i(\tau+\sigma)} \ , \qquad \mathrm{x}_y \equiv x_3 + i x_4 = \ a\, e^{i(\tau-\sigma)} \ ,$$
$$E_{\rm flat} = \frac{\kappa}{\alpha'} = \sqrt{\frac{4}{\alpha'} J} \ , \qquad J_1 = J_2 = J = \frac{a^2}{\alpha'} \ .$$

Identifying oscillator modes that are excited

associate it with the quantum string state created by

$$e^{-iEt} \left[(\partial n_x \bar\partial \mathrm{x}_x)^{\frac{J_1}{2}} (\partial n_y \bar\partial \mathrm{x}_y)^{\frac{J_2}{2}} + ... \right] , \qquad \alpha' E^2 = 2(J_1 + J_2 - 2) \ .$$

$J_1 = J_2$ case - quantum-state analog – shift $J \to J - 1$

$$E_{\rm flat} = \sqrt{\frac{4}{\alpha'} (J-1)} \ .$$

$J_1 = J_2 = 2$ corresponds to state on 1-st string level $n = 2$

Folded string rotating in a plane

$$t = \kappa\tau \ , \qquad \mathrm{x}_1 \equiv x_1 + ix_2 = \ a \sin\sigma \, e^{i\tau} \ ,$$
$$E_{\rm flat} = \sqrt{\tfrac{2}{\alpha'} S} \ , \qquad\qquad S = \tfrac{a^2}{2\alpha'} \ ,$$

semiclassical counterpart of quantum string state
on the leading Regge trajectory

$$e^{-iEt} \left[(\partial \mathrm{x}_x \bar\partial \mathrm{x}_x)^{\frac{S}{2}} + ... \right] , \qquad\qquad \alpha' E^2 = 2(S-2) \ .$$

3 obvious choices how to embed these solutions into $AdS_5 \times S^5$:
(i) the two 2-planes may belong to S^5: $J_1 = J_2$ "small string"
(ii) the two 2-planes may belong to AdS_5: $S_1 = S_2$ "small string"
(iii) one plane in AdS_5 and the other in S^5: $S = J$ "small string"

similar 3 choices for folded string:
study each case in $AdS_5 \times S^5$ and interpolate to small values of S, J
match to states in Konishi table
verify universality of strong-coupling expansion of 4-d anom. dim
of dual gauge theory operators in same supermultiplet

Final result:

$$E = 2\sqrt[4]{\lambda} + b_0 + \frac{b_1}{\sqrt[4]{\lambda}} + \frac{b_2}{\sqrt{\lambda}} + \frac{b_3}{(\sqrt[4]{\lambda})^3} \ldots$$

$$b_0 = \Delta_0 - 4\,, \qquad\qquad b_1 = 1$$

$\Delta_0 = 4$ for 3 circular string cases

$\Delta_0 = 6$ for 3 folded string cases

b_2 is sensitive to 2-loop string corrections not computed
[conjecture $b_2 = 0$ that it is zero due to supersymmetry:
cf. vertex op. approach]

Definite predictions:

b_0 is integer; b_1 is rational; b_3 is transcendental
(contains $\zeta(3)$)

Example: circular rotating string in S^5 with $J_1 = J_2 = J$:

cf. Konishi descendant with $J_1 = J_2 = 2$: $\mathrm{Tr}([\Phi_1, \Phi_2]^2)$

try represent it by "short" classical string with same charges

flat space $R_t \times R^4$: circular string solution

$$x_1 + ix_2 = \ a\, e^{i(\tau+\sigma)} \ , \qquad x_3 + ix_4 = \ a\, e^{i(\tau-\sigma)}$$

$$E = \sqrt{\frac{4}{\alpha'} J}, \qquad J = \frac{a^2}{\alpha'}$$

this solution can be directly embedded into

$R_t \times S^5$ in $AdS_5 \times S^5$ [Frolov, AT 03] :

string on *small* sphere inside S^5: $X_1^2 + ... + X_6^2 = 1$

$$\begin{aligned} &X_1 + iX_2 = a\, e^{i(\tau+\sigma)}, \quad X_3 + iX_4 = a\, e^{i(\tau-\sigma)} \ , \\ &X_5 + iX_6 = \sqrt{1 - 2a^2}, \qquad t = \kappa\tau \\ &\mathcal{J} = \mathcal{J}_1 = \mathcal{J}_2 = a^2, \qquad \mathcal{E}^2 = \kappa^2 = 4\mathcal{J} \end{aligned}$$

Remarkably, exact E_0 is just as in flat space

$$E_0 = \sqrt{\lambda}\mathcal{E} = \sqrt{4\sqrt{\lambda}J} \ , \qquad J = \sqrt{\lambda}\mathcal{J}$$

1-loop quantum string correction to the energy:

sum of bosonic and fermionic fluctuation frequencies ($n = 0, 1, 2, ...$)

Bosons (2 massless + massive):

$$
\begin{aligned}
AdS_5: \quad & 4\times \qquad \omega_n^2 = n^2 + 4\mathcal{J} \\
S^5: \quad & 2\times \qquad \omega_{n\pm}^2 = n^2 + 4(1-\mathcal{J}) \pm 2\sqrt{4(1-\mathcal{J})n^2 + 4\mathcal{J}^2}
\end{aligned}
$$

Fermions:

$$
4\times \qquad \omega_{n\pm}^{2\,f} = n^2 + 1 + \mathcal{J} \pm \sqrt{4(1-\mathcal{J})n^2 + 4\mathcal{J}}
$$

$$
E_1 = \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} \Big[4\omega_n + 2(\omega_{n+} + \omega_{n-}) - 4(\omega^f_{n+} + \omega^f_{n-}) \Big]
$$

expand in small $\mathcal{J}$ and do sums (UV divergences cancel)

$$
E_1 = \frac{1}{\sqrt{\mathcal{J}}} \Big[\mathcal{J} - [3 + \zeta(3)]\mathcal{J}^2 - \tfrac{1}{4}\big[5 + 6\zeta(3) + 30\zeta(5)\big]\mathcal{J}^3 + \ldots \Big]
$$

$$
E = E_0 + E_1 = 2\sqrt{\sqrt{\lambda} J}\Big[1 + \frac{1}{2\sqrt{\lambda}} - \tfrac{3}{4}[1 + 2\zeta(3)]\frac{J}{(\sqrt{\lambda})^2} + ...\Big]
$$

To interpolate to quantum short string in flat space shift $J \to J-1$

then for finite $J = J_1 = J_2 = 2$

that would suggest for Konishi state $[2,0,2]_{(0,0)}$

$$E = 2\sqrt[4]{\lambda}\Big[1 + \frac{1}{2\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2})\Big]$$

dual state in Konishi table has $\Delta_0 = 4$

thus: $\quad b_1 = 1$

$b_2 = 0$ (at least 1-loop contribution to it)

$b_3 = 0$ (contains $\zeta(3)$ – at least 1-loop contribution to it)

consistent results are found for 2+3 other solutions

representing 5 other states at the 1st massive string level

dual to 5 operators in Konishi table

Conclusion

Beginning of understanding
of quantum string spectrum in $AdS_5 \times S^5$
= spectrum of “short” SYM operators

need to understand the origin of partial disagreement
$b_1 = 1$ vs $b_1 = 2$
with numerical solution of TBA by Gromov, Kazakov, Vieira

need better understanding of quantum string theory in $AdS_5 \times S^5$
in particular, near flat space expansion

Δ_0	
2	$[0,0,0]_{(0,0)}$
$\frac{5}{2}$	$[0,0,1]_{(0,\frac{1}{2})}+[1,0,0]_{(\frac{1}{2},0)}$
3	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})}+[0,0,2]_{(0,0)}+[0,1,0]_{(0,1)+(1,0)}+[1,0,1]_{(\frac{1}{2},\frac{1}{2})}+[2,0,0]_{(0,0)}$
$\frac{7}{2}$	$[0,0,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)+(\frac{3}{2},0)}+[0,1,1]_{(0,\frac{1}{2})+(1,\frac{1}{2})}+[1,0,0]_{(0,\frac{1}{2})+(0,\frac{3}{2})+(1,\frac{1}{2})}+[1,0,2]_{(\frac{1}{2},0)}$ $+[1,1,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)}+[2,0,1]_{(0,\frac{1}{2})}$
4	$[0,0,0]_{(0,0)+(0,2)+(1,1)+(2,0)}+[0,0,2]_{(\frac{1}{2},\frac{1}{2})+(\frac{3}{2},\frac{1}{2})}+[0,1,0]_{2(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})}+[2,0,2]_{(0,0)}$ $+[0,1,2]_{(1,0)}+[0,2,0]_{2(0,0)+(1,1)}+[1,0,1]_{(0,0)+2(0,1)+2(1,0)+(1,1)}+[1,1,1]_{2(\frac{1}{2},\frac{1}{2})}+[2,0,0]_{(\frac{1}{2}}$
6	$[0,0,0]_{3(0,0)+3(1,1)+(2,2)}+[0,0,2]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})}+[0,1,0]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})+}$ $+[0,1,2]_{(0,0)+2(0,1)+2(1,0)+(1,1)}+[0,2,0]_{3(0,0)+(0,1)+(0,2)+(1,0)+3(1,1)+(2,0)}+[0,2,2]_{(\frac{1}{2},\frac{1}{2})}$ $+[0,3,0]_{2(\frac{1}{2},\frac{1}{2})}+[0,4,0]_{(0,0)}+[1,0,1]_{(0,0)+3(0,1)+3(1,0)+4(1,1)+(1,2)+(2,1)}+[1,0,3]_{(\frac{1}{2},\frac{1}{2})}+[0,$ $+[1,1,1]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})}+[1,2,1]_{(0,0)+(0,1)+(1,0)}+[2,0,0]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})}$ $+[2,0,2]_{(0,0)+(1,1)}+[2,1,0]_{(0,0)+2(0,1)+2(1,0)+(1,1)}+[2,2,0]_{(\frac{1}{2},\frac{1}{2})}+[3,0,1]_{(\frac{1}{2},\frac{1}{2})}+[4,0,0]_{(0,0}$
$\frac{17}{2}$	$[0,0,1]_{(0,\frac{1}{2})+(0,\frac{3}{2})+(1,\frac{1}{2})}+[0,1,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)}+[1,0,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)+(\frac{3}{2},0)}+[1,0,2]_{(0,\frac{1}{2})}$ $+[1,1,0]_{(0,\frac{1}{2})+(1,\frac{1}{2})}+[2,0,1]_{(\frac{1}{2},0)}$
9	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})}+[0,0,2]_{(0,0)}+[0,1,0]_{(0,1)+(1,0)}+[1,0,1]_{(\frac{1}{2},\frac{1}{2})}+[2,0,0]_{(0,0)}$
$\frac{19}{2}$	$[0,0,1]_{(\frac{1}{2},0)}+[1,0,0]_{(0,\frac{1}{2})}$
10	$[0,0,0]_{(0,0)}$