Crossover from weak to strong coupling, non-Abelian duality and confinement in $\mathcal{N} = 2$ supersymmetric QCD

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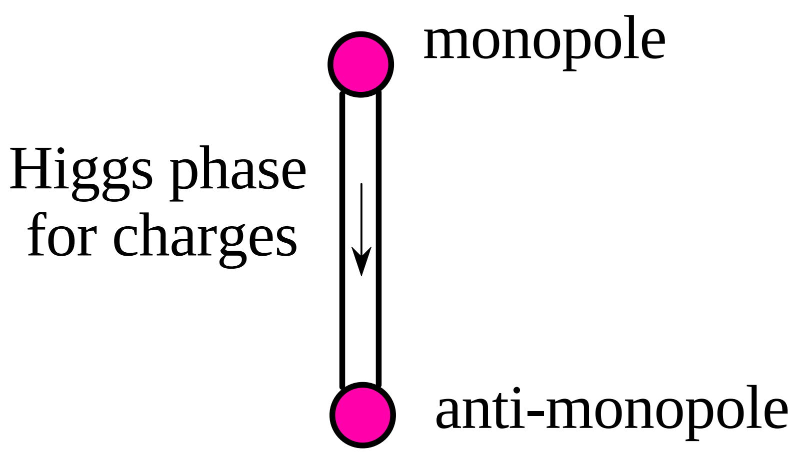
1 Introduction

Mandelstam and 't Hooft 1970's:

Confinement is a dual Meissner effect upon condensation of monopoles.

Electric charges condense → magnetic Abrikosov-Nielsen-Olesen flux tubes (strings) are formed → monopoles are confined

Monopoles condense → electric Abrikosov-Nielsen-Olesen flux tubes are formed → electric charges are confined



Seiberg and Witten 1994 : Abelian confinement in $\mathcal{N}=2$ QCD

Cascade gauge symmetry breaking:

- SU(N)$\longrightarrow$ U(1)$^{N-1}$ VEV's of adjoint scalars
- U(1)$^{N-1}$ $\longrightarrow$ 0 (or discrete subgroup) VEV's of quarks/monopoles

At the last stage Abelian Abrikosov-Nielsen-Olesen flux tubes are formed.

$$\pi_1(U(1)^{N-1}) = \mathcal{Z}^{N-1}$$

$(N-1)$ infinite towers of strings. In particular $(N-1)$ elementary strings

$\longrightarrow$ Too many degenerative hadron states

In search for non-Abelian confinement non-Abelian strings were suggested in $\mathcal{N} = 2$ U(N) QCD

Hanany, Tong 2003

Auzzi, Bolognesi, Evslin, Konishi, Yung 2003

Shifman Yung 2004

Hanany Tong 2004

Z_N Abelian string: Flux directed in the Cartan subalgebra, say for $SO(3) = SU(2)/Z_2$

$$flux \sim \tau_3$$

Non-Abelian string : Orientational zero modes

Rotation of color flux inside SU(N).

Non-Abelian strings were first found in $\mathcal{N} = 2$ QCD with U(N) gauge group and $N_f = N$ fundamental flavors (quarks).

Fayet Iliopoulos parameter ξ triggers quark condensation.

The theory is in the Higgs phase

To ensure weak coupling regime

$$\xi \gg \Lambda^2, \qquad g^2(\sqrt{\xi}) \ll 1$$

QUESTION:

What happens to the bulk theory and to non-Abelian strings in the strong coupling at

$$\xi \ll \Lambda^2?$$

What do we know about duality?

- Seiberg-Witten electromagnetic duality

  Example: SU(2) gauge theory. Near monopole vacuum it is described in terms of dual U(1) Abelian gauge theory of light monopoles.

  monopoles condense $\Rightarrow$ quarks are confined

- $\mathcal{N} = 1$ Seiberg Nonabelian duality

  $\mathrm{SU}(N) \Rightarrow \mathrm{SU}(N_f - N)$

Common belif: As we deform $\mathcal{N} = 2$ QCD with mass term for the adjoint matter $\mu(\Phi^a)^2$ Seiberg-Witten Abelian duality smoothly goes into Seiberg non-Abelian duality.

Non-Abelian monopoles condense $\Rightarrow$ quarks are confined

Two implicit assumptions:

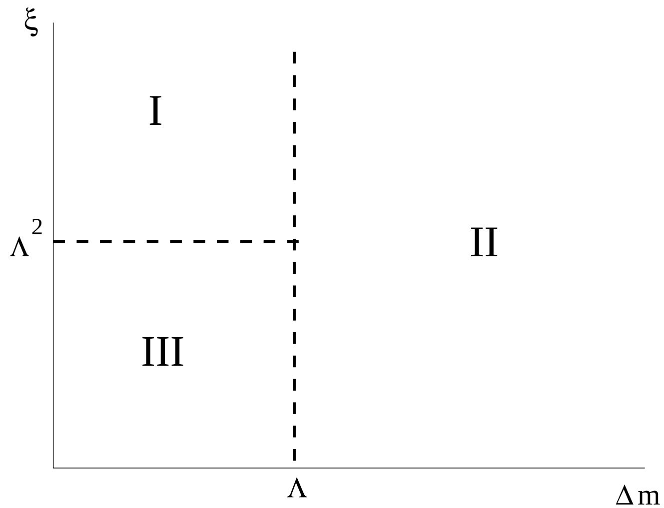
- Duality exchanges quarks and monopoles
- Smooth transition with respect to μ

We show:

- Non-Abelian duality is not the electro-magnetic duality.

  Monopoles are confined in both original and dual theories
- Crossover with respect to ξ

Fayet-Iliopoulos term $\xi \sim \mu\Lambda$ or $\xi \sim \mu m$



Three different regimes separated by crossovers

For $N_f = N$

I:

- N^2 non-Abelian degrees of freedom
- non-Abelian strings with N orientational moduli and $\langle(\mathrm{non-Abelian\ \ flux})\rangle = 0$

III:

- N Abelian degrees of freedom
- Abelian Z_N strings with fixed flux

For $N_f > N$

III:

- Gauge theory with dual gauge group

$$U(\tilde{N}) \times U(1)^{(N-\tilde{N})},$$

  where $\tilde{N} = N_f - N$.

- non-Abelian strings with $\tilde{N}$ orientational moduli

Why do we think it is a crossover rather then a phase transition?

Arguments which support this assumption:

- In the equal quark mass limit regions I and III have Higgs branches of the same dimensions

$$\dim \mathcal{H} = 4N\tilde{N}$$

  and the same pattern of global symmetry breaking

- At generic masses all three regimes has the same number of isolated vacua at non-zero ξ,

$$C^{N}_{N_f} = C^{\tilde{N}}_{N_f}$$

- Each of these vacua has the same number $(= N)$ of different elementary strings in all three domains. Moreover, BPS spectra of excitations on the non-Abelian string are the same in regions I and III,

Still both the perturbative spectrum and confining strings are dramatically different in the regimes I, II and III.

2 Bulk theory with $N_f = N$

$\mathcal{N} = 2$ QCD with gauge group $U(N) = SU(N) \times U(1)$ and
$N_f = N$ flavors of fundamental matter – quarks

+

Fayet-Iliopoulos term of U(1) factor

The bosonic part of the action

$$S = \int d^4x \left[\frac{1}{4g_2^2} \left(F^a_{\mu\nu}\right)^2 + \frac{1}{4g_1^2} \left(F_{\mu\nu}\right)^2 + \frac{1}{g_2^2} \left|D_\mu a^a\right|^2 + \frac{1}{g_1^2} \left|\partial_\mu a\right|^2 \right.$$
$$\left. + \left|\nabla_\mu q^A\right|^2 + \left|\nabla_\mu \bar{\tilde{q}}^A\right|^2 + V(q^A, \tilde{q}_A, a^a, a) \right] .$$

Here

$$\nabla_\mu = \partial_\mu - \frac{i}{2}\, A_\mu - i A^a_\mu\, T^a \, .$$

The potential is

$$
\begin{aligned}
V(q^A, \tilde{q}_A, a^a, a) &= \frac{g_2^2}{2}\left(\frac{i}{g_2^2}\, f^{abc}\bar{a}^b a^c + \bar{q}_A\, T^a q^A - \tilde{q}_A T^a\, \bar{\tilde{q}}^A\right)^2 \\
&+ \frac{g_1^2}{8}\left(\bar{q}_A q^A - \tilde{q}_A \bar{\tilde{q}}^A - N\xi\right)^2 \\
&+ 2g_2^2 \left|\tilde{q}_A T^a q^A\right|^2 + \frac{g_1^2}{2}\left|\tilde{q}_A q^A\right|^2 \\
&+ \frac{1}{2}\sum_{A=1}^{N}\left\{\left|(a + \sqrt{2}m_A + 2T^a a^a)q^A\right|^2\right. \\
&+ \left.\left|(a + \sqrt{2}m_A + 2T^a a^a)\bar{\tilde{q}}^A\right|^2\right\} .
\end{aligned}
$$

Vacuum

$$\langle \frac{1}{2}\, a + T^a\, a^a \rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & m_N \end{pmatrix},$$

For special choice

$$m_1 = m_2 = ... = m_N$$

U(N) gauge group is classically unbroken.

$$\langle q^{kA} \rangle = \sqrt{\xi} \begin{pmatrix} 1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 1 \end{pmatrix}, \qquad \langle \bar{\tilde{q}}^{kA} \rangle = 0,$$

$$k = 1, ..., N \qquad A = 1, ..., N\,,$$

Note

- Color-flavor locking

  Both gauge $U(N)$ and flavor $SU(N)$ are broken, however diagonal $SU(N)_{C+F}$ is unbroken

$$\langle q \rangle \to U \langle q \rangle U^{-1}$$

$$\langle a \rangle \to U \langle a \rangle U^{-1}$$

  All perturbative states come as singlets or adjoints of $SU(N)_{C+F}$

$$m_{singlet} = g_1 \sqrt{\xi}, \qquad m_{adjoint} = g_2 \sqrt{\xi}$$

- The way to stay at weak coupling:

$$\sqrt{\xi} \gg \Lambda$$

$$\frac{8\pi^2}{g_2^2(\xi)} = N \log \frac{\sqrt{\xi}}{\Lambda} \gg 1$$

Non-Abelian string

$$\frac{1}{N}\left\{U\begin{pmatrix} 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -(N-1) \end{pmatrix}U^{-1}\right\}^A_B = -n^A n^*_B + \frac{1}{N}\delta^A_B \ ,$$

with

$$n^*_A n^A = 1$$

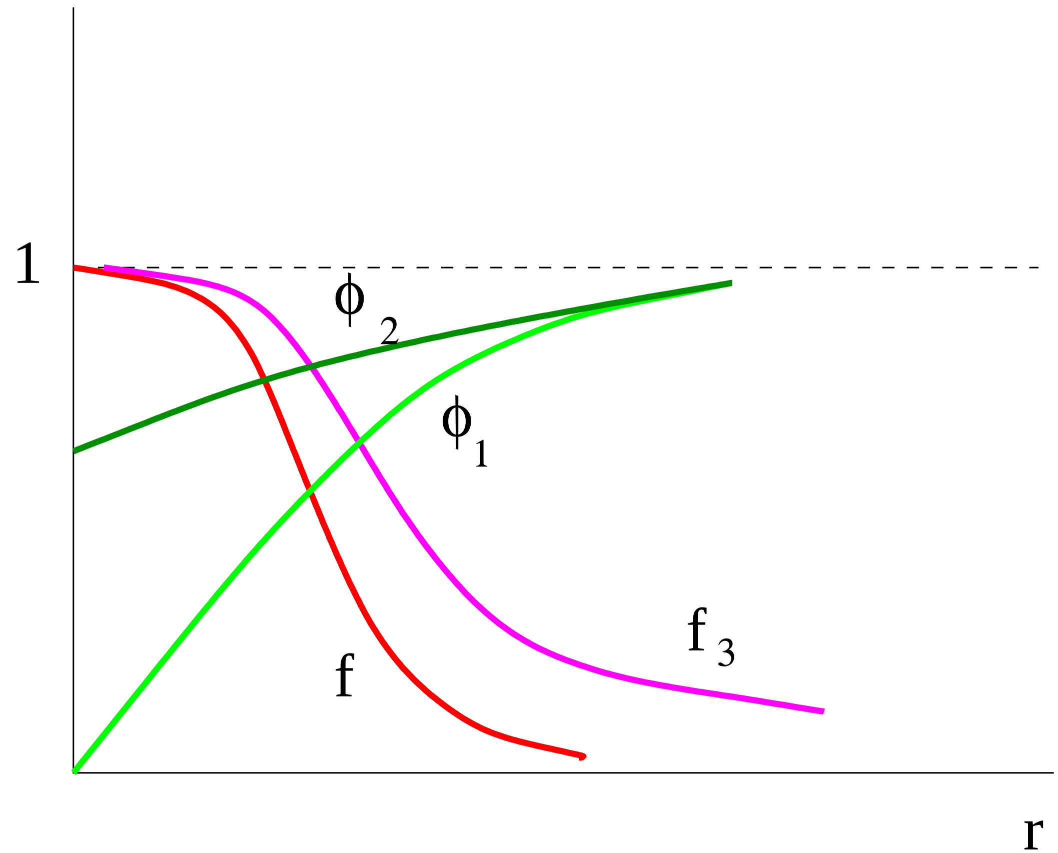
Then

$$q \ = \ \frac{1}{N}[(N-1)\phi_2 + \phi_1] + (\phi_1 - \phi_2)\left(n \ \cdot n^* - \frac{1}{N}\right),$$

$$A_i^{\mathrm{SU}(N)} \ = \ \left(n \ \cdot n^* - \frac{1}{N}\right)\varepsilon_{ij}\,\frac{x_j}{r^2}\,f_3(r)\,,$$

$$A_i^{\mathrm{U}(1)} \ = \ \frac{1}{N}\varepsilon_{ij}\,\frac{x_j}{r^2}\,f(r)\,,$$

Profile functions of the string (for $N = 2$)



3 $CP(N)$ model on the string

String moduli: x_{0i}, $i = 1, 2$ and n^A, $A = 1, ..., N$

Make them t, z-dependent

Z_N solution breaks $SU(N)_{C+F}$ down to $SU(N-1) \times U(1)$ Thus the orientational moduli space is

$$\frac{\mathrm{SU}(N)}{\mathrm{SU}(N-1) \times \mathrm{U}(1)} \sim \mathrm{CP}(N-1)$$

Gauge theory formulation of $\mathcal{N} = (2, 2)$ supersymmetric $CP(N-1)$ model, $e^2 \to \infty$

$$\begin{aligned} S_{\mathrm{CP}(N-1)} &= \int d^2x \left\{ |\nabla_\alpha n^A|^2 + \frac{1}{4e^2} F^2_{\alpha\beta} + \frac{1}{e^2} |\partial_\alpha \sigma|^2 \right. \\ &+ \left. 2\,|\sigma + \frac{m_A}{\sqrt{2}}|^2 |n^A|^2 + \frac{e^2}{2} \left(|n^A|^2 - 2\beta \right)^2 \right\}, \end{aligned}$$

where

$$\nabla_\alpha = \partial_\alpha - iA_\alpha, \qquad \alpha = 1, 2,$$

while σ is a complex scalar field, superpartner of A_α. The condition

$$n_A^* n^A = 2\beta$$

is implemented in the limit $e^2 \to \infty$.

The coupling constant β is given by

$$\beta = \frac{2\pi}{g_2^2}(\xi)$$

Gauge field can be eliminated:

$$A_k = -\frac{i}{4\beta}\,\bar{n}_A\,\overleftrightarrow{\partial}_\alpha\,n^A \qquad \sigma = 0$$

Number of degrees of freedom $= 2N - 1 - 1 = 2(N - 1)$

Non-Abelian flux of the string

$$\int d^2x \, (F_3^*)_{\text{SU}(N)} = 2\pi \left(\frac{n \cdot n^*}{2\beta} - \frac{1}{N} \right) ,$$

where $F_i^* = \frac{1}{2} \varepsilon_{ijk} F_{jk}$, $i, j, k = 1, 2, 3$

We can introduce a gauge invariant flux

$$\Phi = \int d^2x \, a^a F_3^{*a} .$$

As was shown by Witten (1979) in the CP$(N-1)$ model at zero (or small) m_A

$$\langle n^A \rangle = 0$$

Therefore, in the region I

$$\Phi \approx 0,$$

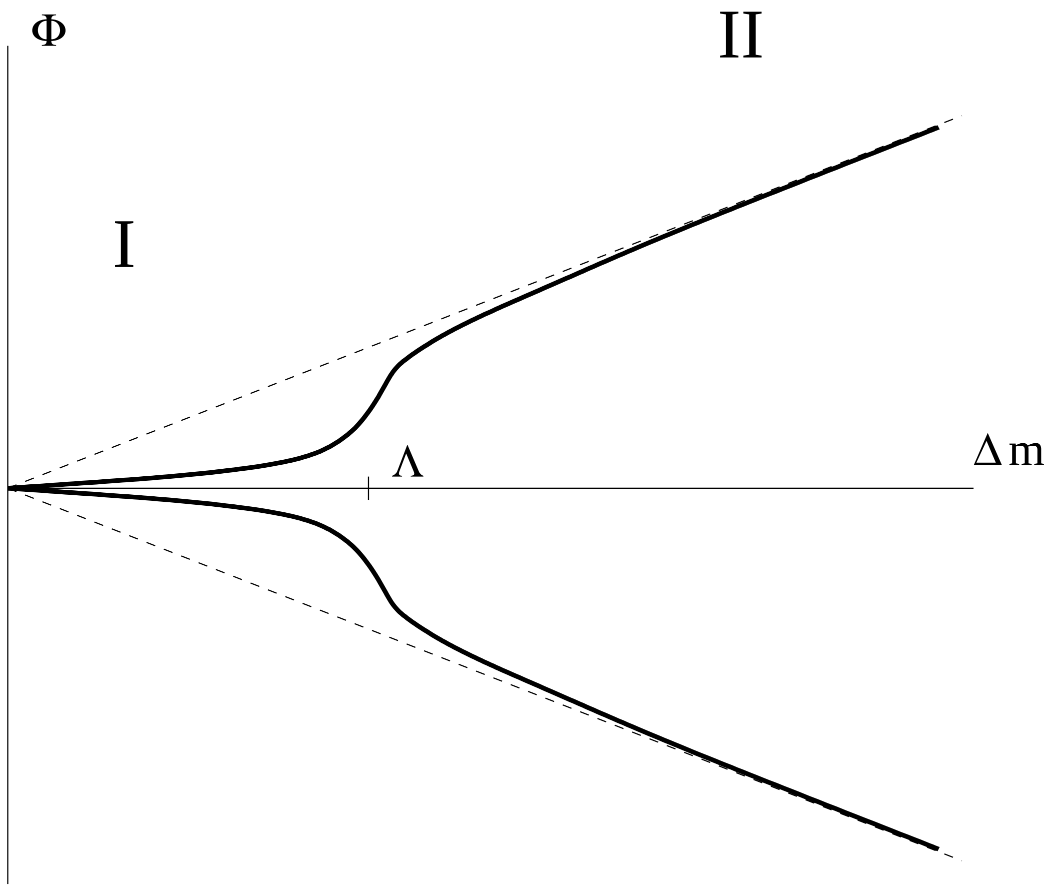
while in the region II

$$\langle n^A \rangle = \sqrt{2\beta} \, \delta^{AA_0} \qquad \langle \sigma \rangle = -\frac{m_{A_0}}{\sqrt{2}}$$

Therefore, in the region II

$$\Phi = -2\pi\sqrt{2}\, m_{A_0}$$

For $N = 2$.



Crossover

Note

- At $N \to \infty$: Crossover $\Rightarrow$ Phase transition
- In non-SUSY $CP(N-1)$ model with Z_N symmetry we have

phase transition

Region I: Z_N unbroken; only one vacuum (string)

Region II: Z_N broken; N vacua (strings)

4 Region III: small ξ, small Δm

- Go to the Coulomb branch at $\xi = 0$ and large Δm_{AB} to the region II. Abelian $U(1)^N$ gauge theory
- Then sitting at the $r = 2$ quark vacuum reduce Δm_{AB} using exact Seiberg-Witten solution

Simplest case $N = 2$ (U(2) theory with $N_f = 2$ flavors)

$$y^2 = (x - \phi_1)^2 (x - \phi_2)^2 - 2\Lambda^2 \left(x + \frac{m_1}{\sqrt{2}} \right) \left(x + \frac{m_2}{\sqrt{2}} \right),$$

ϕ_1, ϕ_2 – coordinates on the Coulomb branch

Argyres-Douglas point at

$$\Delta m^2 = 4\Lambda^2,$$

Here besides two massless quarks a SU(2) monopole becomes massless.

Monodromies

Quarks transform into dyons

$$q^{11} \longrightarrow D_1, \qquad q^{22} \longrightarrow D_2$$

$$(n_e, n_m; n_e^3, n_m^3) = \left(\frac{1}{2}, 0;\ \frac{1}{2}, 0\right)_{q^{11}} \quad \Rightarrow \quad \left(\frac{1}{2}, 0;\ \frac{1}{2}, 1\right)_{D_1},$$

$$(n_e, n_m; n_e^3, n_m^3) = \left(\frac{1}{2}, 0;\ -\frac{1}{2}, 0\right)_{q^{22}} \quad \Rightarrow \quad \left(\frac{1}{2}, 0;\ -\frac{1}{2}, -1\right)_{D_2}$$

while the monopole charge is

$$(n_e, n_m; n_e^3, n_m^3) = (0, 0;\ 0, 1)$$

According to this charges light dyons interact with two U(1) gauge fields

$$A_\mu$$

and

$$B_\mu = \frac{1}{\sqrt{5}} \left(A_\mu^3 + 2A_\mu^{3D}\right)$$

Effective action

$$S_{III} = \int d^4x \left[\frac{1}{4\tilde{g}_2^2} \left(F_{\mu\nu}^B \right)^2 + \frac{1}{4g_1^2} \left(F_{\mu\nu} \right)^2 + \frac{1}{\tilde{g}_2^2} \left| \partial_\mu b \right|^2 + \frac{1}{g_1^2} \left| \partial_\mu a \right|^2 \right.$$
$$+ \left| \nabla_\mu^1 D_1 \right|^2 + \left| \nabla_\mu^1 \tilde{D}_1 \right|^2 + \left| \nabla_\mu^2 D_2 \right|^2 + \left| \nabla_\mu^2 \tilde{D}_2 \right|^2$$
$$\left. + V(D, \tilde{D}, b, a) \right] ,$$

where

$$b = \frac{1}{\sqrt{5}} \left(a^3 + 2a_D^3 \right)$$

is the scalar $\mathcal{N} = 2$ superpartner of the photon.

Covariant derivatives

$$\nabla_\mu^1 = \partial_\mu - i \left(\frac{1}{2} A_\mu + \frac{1}{2} A_\mu^3 + A_\mu^{3D} \right) = \partial_\mu - \frac{i}{2} \left(A_\mu + \sqrt{5} B_\mu \right) ,$$
$$\nabla_\mu^2 = \partial_\mu - i \left(\frac{1}{2} A_\mu - \frac{1}{2} A_\mu^3 - A_\mu^{3D} \right) = \partial_\mu - \frac{i}{2} \left(A_\mu - \sqrt{5} B_\mu \right) .$$

The potential is

$$
\begin{aligned}
V(D, \tilde{D}, b, a) &= \frac{5\tilde{g}_2^2}{8}\left(|D_1|^2 - |\tilde{D}_1|^2 - |D_2|^2 + |\tilde{D}_2|^2\right)^2 \\
&+ \frac{g_1^2}{8}\left(|D_1|^2 - |\tilde{D}_1|^2 + |D_2|^2 - |\tilde{D}_2|^2 - 2\xi\right)^2 \\
&+ \frac{5\tilde{g}_2^2}{2}\left|\tilde{D}_1 D_1 - \tilde{D}_2 D_2\right|^2 + \frac{g_1^2}{2}\left|\tilde{D}_1 D_1 + \tilde{D}_2 D_2\right|^2 \\
&+ \frac{1}{2}\left\{\left|a + \sqrt{5}b + \sqrt{2}m_1\right|^2 \left(|D_1|^2 + |\tilde{D}_1|^2\right)\right. \\
&+ \left.\left|a - \sqrt{5}b + \sqrt{2}m_2\right|^2 \left(|D_2|^2 + |\tilde{D}_2|^2\right)\right\} .
\end{aligned}
$$

Vacuum

$$
a = -\sqrt{2}\, m, \qquad \sqrt{5}\, b = -\frac{\Delta m}{\sqrt{2}},
$$

$$
D_1 = \sqrt{\xi}, \qquad D_2 = \sqrt{\xi}, \qquad \tilde{D}_1 = \tilde{D}_2 = 0
$$

I: N^2 quarks and gauge bosons– non-Abelian regime

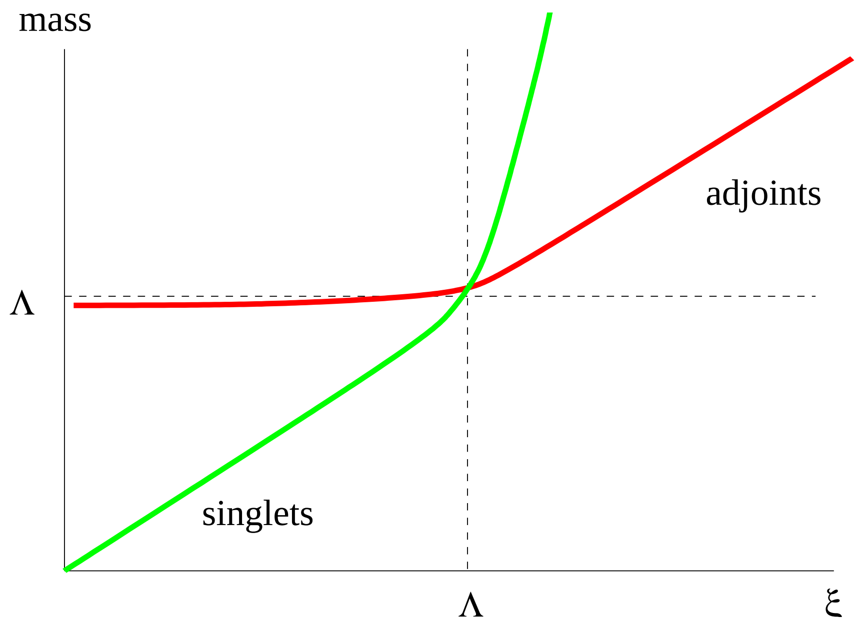
III: N dyons and dual gauge bosons–Seiberg-Witten Abelian regime

unbroken global $SU(N)_{C+F}$

Region I: quarks and gauge bosons are in adjoint

Region III: dyons and dual gauge bosons are singlets

Conclusion: These are different states



What is the physical nature of $(N^2 - 1)$ adjoints in the region III?

Consider W-boson and move from the region II to the region III along Coulomb branch at $\xi = 0$. Consider $N = 2$. $SU(2)_{C+F} \to U(1)_{\tau^3}$

W-boson is charged with respect to $U(1)_{\tau^3}$

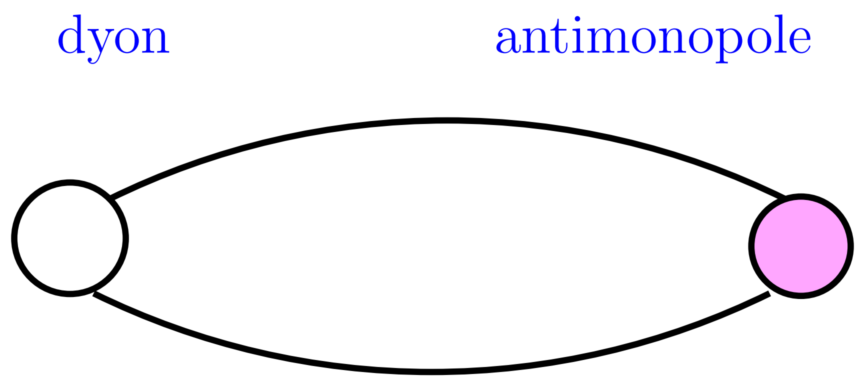
$$(n_e, n_m; n_e^3, n_m^3) = (0, 0;\ 1, 0)$$

decay to (anti)monopole and dyon

$$(0, 0;\ 0, -1) + (0, 0;\ 1, 1)$$

Now switch on small ξ

At $\xi \neq 0$ monopoles/dyons are confined and cannot move apart



5 $N_f > N$. Non-Abelian bulk duality

Region I. Large ξ Adjoint fields:

$$\langle \frac{1}{2}\, a + T^a\, a^a \rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & m_N \end{pmatrix},$$

Quarks

$$\langle q^{kA} \rangle \;=\; \sqrt{\xi} \begin{pmatrix} 1 & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & 0 & \ldots & 0 \end{pmatrix}, \qquad \langle \bar{\tilde{q}}^{kA} \rangle = 0,$$

$$k \;=\; 1, ..., N \qquad A = 1, ..., N_f\,,$$

In the equal mass limit $\mathrm{U}(N)_{\mathrm{gauge}} \times \mathrm{SU}(N_f)_{\mathrm{flavor}}$ is broken down to

$$\mathrm{SU}(N)_{C+F} \times \mathrm{SU}(\tilde{N})_F \times \mathrm{U}(1)\,,$$

where $\tilde{N} = N_f - N$.

Quarks and gauge fields fill following representations of the global group:

$$(1,1) \quad (N^2-1,1) \quad (\bar{N},\tilde{N}) \quad (N,\bar{\tilde{N}})$$

Region III. Small ξ and small Δm_{AB} ($\ll \Lambda$)

- First go to the Coulomb branch at $\xi = 0$ in the region II at weak coupling
- Then use Seiberg-Witten curve to go to small Δm_{AB}

We get theory of non-Abelian dyons and dual gauge fields with

$$U(\tilde{N}) \times U(1)^{(N-\tilde{N})}$$

gauge group and N_f non-Abelian dyons and $(N - \tilde{N})$ Abelian dyons.

The non-Abelian gauge factor $U(\tilde{N})$ is not broken by adjoint VEV's in the equal mass limit because this theory is not asymptotically free and stays at weak coupling

Argyres Plesser Seiberg:
$SU(\tilde{N}) \times U(1)^{(N-\tilde{N})}$ was identified at the root of the baryonic branch in SU(N) theory with massless quarks

Vacuum

$$\langle U(\tilde{N})\ adjoints \rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m_{N+1} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & m_{N_f} \end{pmatrix}$$

Dyons

$$\langle D^{lA} \rangle = \sqrt{\xi} \begin{pmatrix} 0 & \dots & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix}, \qquad \langle \bar{\tilde{D}} \rangle = 0,$$

$$\langle D^{ii} \rangle = \sqrt{\xi}, \qquad l = 1, ..., \tilde{N} \qquad i = \tilde{N}+1, ..., N \qquad A = 1, ..., N_f\ ,$$

$$(1, ..., N)|_{I,II} \ \Rightarrow \ (N+1, ..., N_f, \tilde{N}+1, ..., N)|_{III}$$

In the equal mass limit the global group is broken to

$$\mathrm{SU}(N)_F \times \mathrm{SU}(\tilde{N})_{C+F} \times \mathrm{U}(1)$$

Now dyons and dual gauge fields fill following representations of the global group:

$$III: \qquad (1,1) \quad (1, \tilde{N}^2 - 1) \quad (\bar{N}, \tilde{N}) \quad (N, \bar{\tilde{N}})$$

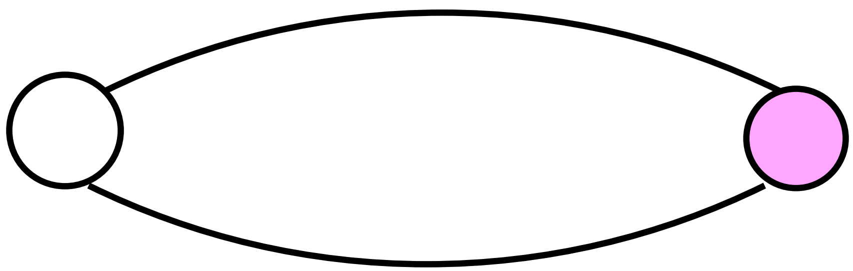
Recall that quarks and gauge bosons of the original theory are in

$$I: \qquad (1,1) \quad (N^2 - 1, 1) \quad (\bar{N}, \tilde{N}) \quad (N, \bar{\tilde{N}})$$

$(N^2 - 1)$ of $\mathrm{SU}(N)$ and $(\tilde{N}^2 - 1)$ of $\mathrm{SU}(\tilde{N})$

are different states

In the region III $(N^2 - 1)$ of SU(N) are stringy mesons formed by by pairs of (anti)monopoles and dyons connected by two strings



In the region I $(\tilde{N}^2 - 1)$ of SU$(\tilde{N})$ are stringy mesons formed by by pairs of (anti)monopoles and dyons connected by two strings

Conjecture:

These stringy mesons are Seiberg's neutral M-fields.

6 Conclusions

Sharp crossover

For $N_f = N$

- Region I: Non-Abelian bulk spectrum and Non-Abelian strings
- Region III: Abelian bulk spectrum and Abelian Z_N strings
  $+ (N^2 - 1)$ stringy mesons

For $N_f > N$

- Region III: Gauge theory with dual gauge group $U(\tilde{N}) \times U(1)^{(N-\tilde{N})}$
- Elementary adjoints of SU(N) of the region I become stringy mesons in region III, while elementary adjoints of SU($\tilde{N}$) of the region III become stringy mesons in region I.
- Bulk duality translates into world sheet duality

MAIN CONCLUSIONS:

- In both original and dual theories confined states are monopoles
- Non-Abelian confinement = Higgs screening + Decay on CMS + String formation