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12 – 16 July 2010

Dedicated to the memory of M. Sh. Birman (1928–2009)

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Program

Abstracts

St.Petersburg, 2010

Organizers:

Alexandre Fedotov, Nikolai Filonov, Alexander Pushnitski

Organizing committee:

Alexandre Fedotov, Nikolai Filonov, Alexander Pushnitski, Nadia Zalesskaya

International Conference in Spectral Theory.
Program and abstracts of the conference. Euler International Mathematical Institute, St.Petersburg, 2010.

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The conference website: http://www.pdmi.ras.ru/EIMI/2010/st/

Speakers

Michael Agranovich (Moscow Institute of Electronics and Mathematics)
Michail Belishev (Steklov Intstitute, St.Petersburg)
Vladimir Buslaev (St.Petersburg State University)
Andrea Cianchi (University of Florence)
Brian Davies (King's College London)
Maria Esteban (University of Paris-Dauphine)
Pavel Exner (Nuclear Physics Institute, Czech Academy of Sciences)
Alexander Fedotov (St.Petersburg State University)
Rupert Frank (Princeton University)
Francois Germinet (University of Cergy-Pontoise)
Yulia Karpeshina (University of Alabama at Birmingham)
Frederic Klopp (University of Paris Nord)
Shinichi Kotani (Kwansei Gakuin University)
Konstantin Makarov (University of Missouri at Columbia)
Leonid Pastur (Institute for Low Temperature Physics, Kharkov)
Vladimir Peller (Michigan State University at East Lansing)
Boris Plamenevsky (St.Petersburg State University)
Alexey Pozharsky (St.Petersburg State University)
Oleg Safronov (University of North Carolina at Charlotte)
Roman Shterenberg (University of Alabama at Birmingham)
Gunter Stolz (University of Alabama at Birmingham)
Boris Vainberg (University of North Carolina at Charlotte)
Timo Weidl (University of Stuttgart)

and young scientists:

Maxim Demchenko (St.Petersburg Dept of Steklov Institute)
Ilya Kachkovskiy (St.Petersburg State University)
Anna Kononova (Baltiiskii Technical State University),
Acia Metelkina (Fern Universitat im Hagen)
Fabian Portmann (KTH, Stockholm)
Sergei Simonov (St.Petersburg State University)
Valentin Strazdin (St.Petersburg State University)

Scientific programme

MONDAY 12 July:

9:30–10:00: REGISTRATION

10:00–11:00: Rupert Frank *Critical Lieb-Thirring bounds in gaps*

COFFEE BREAK

11:30–12:30: Yulia Karpeshina *Quasi-intersections of an isoenergetic surface and complex angle variable*

LUNCH

14:30–15:30: Oleg Safronov *Absolutely continuous spectrum of multi-dimensional Schrödinger operators*

15:40–16:40: Roman Shterenberg *Complete asymptotic expansion of the integrated density of states of multidimensional almost-periodic Schrödinger operators*

COFFEE BREAK

17:00–18:00: Andrea Cianchi *Eigenvalue problems for the Laplacian on noncompact Riemannian manifolds*

TUESDAY 13 July:

10:00–11:00: Vladimir Buslaev *Scattering for the system of three one-dimensional Coulomb particles*

COFFEE BREAK

11:30–12:30: Boris Plamenevsky *On a method for computing waveguide scattering matrices*

LUNCH

14:30–15:30: Brian Davies *Resonances of quantum graphs*

15:40–16:40: Pavel Exner *Loops and trees: spectral and resonance properties of quantum graphs*

COFFEE BREAK

17:00–18:00: Alexey Pozharskii *Averaging in scattering problems*

WEDNESDAY 14 July:

10:00–11:00: Konstantin Makarov *On non-unitary representations of the generalized Weyl commutation relations*

COFFEE BREAK

11:30–12:30: Vladimir Peller *Functions of normal operators under perturbations*

LUNCH

YOUNG SCIENTISTS SESSION:

14:30–14:50: Ilya Kachkovskiy *Positive polynomials and mapping of pseudospectra*

14:55–15:15: Anna Kononova *On spectral perturbations of bounded Jacobi operators*

15:20–15:40: Fabian Portmann *Spectral inequalities for a class of non-elliptic operators*

COFFEE BREAK

16:10–16:30: Sergei Simonov *Zeros of the spectral density of the discrete Schrdinger operator with Wigner-von Neumann potential*

16:35–16:55: Asya Metelkina *Lyapunov exponent and integrated density of states for the slowly oscillating perturbations of the periodic Schrdinger operators.*

17:00–17:20: Maxim Demchenko *The dynamical inverse problem for the Maxwell system*

17:30–17:50: Valentin Strazdin *Matrix Schrödinger operator on the half-line: the differential equation with respect to the spectral parameter and an analog of Freud's equations*

time and venue TBC: BOAT TRIP

THURSDAY 15 July:

9:30–10:30: Leonid Pastur *On Links Between the Random Matrix and Random Operator Theories*

10:40–11:40: Alexander Fedotov *Behavior at infinity of solutions of almost periodic equations*

COFFEE BREAK

12:10–13:10: Francois Germinet *About currents, magnetic perturbations, magnetic barriers and magnetic guides in quantum Hall systems*

LUNCH

14:30–15:30: Frederic Klopp *Decorrelation estimates for the eigenlevels of random operators in the localized regime*

15:40–16:40: Gunter Stolz *Localization properties of the random displacement model*

COFFEE BREAK

17:00–18:00: Boris Vainberg *Negative spectrum of a perturbed Anderson Hamiltonian*

18:15: CONFERENCE DINNER

FRIDAY 16 July:

10:00–11:00: Michael Agranovich *General and spectral boundary value problems for strongly elliptic second-order systems in bounded Lipschitz domains*

COFFEE BREAK

11:30–12:30: Michael Belishev *Wave spectrum of symmetric semi-bounded operator and its applications*

LUNCH

14:30–15:30: Timo Weidl *Trapped modes in elastic media for zero Poisson ratio*

15:40–16:40: Maria Esteban *Critical threshold for electronic stability under the action of an intense magnetic field*

COFFEE BREAK

17:00–18:00: Shinichi Kotani *KdV flow on the space of generalized reflectionless potentials*

Abstracts

General and spectral boundary value problems for strongly elliptic second-order systems in bounded Lipschitz domains

Michael Agranovich

Moscow Institute of Electronics and Mathematics

In the last three decades, the general and spectral theory for strongly elliptic equations and systems in Lipschitz domains was the subject of unfailing interest for many mathematicians and was carried forward considerably. A Lipschitz boundary can contain edges, conical points and other singularities. All main difficulties are closely connected with the non-smoothness of the boundary. In the talk, a short survey of this theory will be presented. We consider a strongly elliptic second-order system $Lu = f$ written in the divergent form under minimized smoothness assumptions concerning the coefficients. Examples: Beltrami–Laplace equation, systems of isotropic and anisotropic elasticity and their generalizations.

Here is the plan of the talk.

1. The variational approach to the Dirichlet and Neumann problems in bounded Lipschitz domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, in the simplest (Hilbert) spaces H^s, $|s| \leq 1$. Two-sided *a priori* estimates.

2. Properties of the spectral Dirichlet and Neumann problems: spectral asymptotics, basis property of eigenfunctions of formally selfadjoint problems, completeness of root functions of more general problems, Abel–Lidskii summability of Fourier series with respect to these functions.

3. Surface single layer potential $u = \mathcal{A}\psi$ and double layer potential $u = \mathcal{B}\psi$ on a Lipschitz surface Γ, the integral representation of solutions and relations between the main operators on Γ: the restriction $A\psi$ of the single layer potential to Γ, the direct value $B\varphi$ of the double layer potential on Γ, the hyper-singular operator H, and the Neumann-to-Dirichlet operator N. For simplicity, we consider the system on the standard torus $\mathbb{T} = \mathbb{T}^n$ (with periodic coordinates) divided by a Lipschitz surface Γ into two domains $\Omega^{\pm}$.

4. Spectral Poincaré–Steklov and transmission problems for the system $Lu = 0$ with spectral parameter in conditions at the boundary. Reduction of these problems to the spectral equations for the operators N, A, and H on Γ. Spectral properties of these operators in the spaces $H^s(\Gamma)$, $|s| \leq 1/2$.

5. Regularity of solutions: their belonging to more general space H^s_p of Bessel potentials and Besov spaces B^s_p (with deviation of s from ± 1 and p from 2). Two approaches to regularity problems: the more easy variational approach and the approach on the base of the deep investigation of potentials. The extrapolation of the invertibility of operators acting in interpolation scales of spaces.

6. The same spectral problems and spectral equations in (Banach) spaces H^s_p and B^s_p. Specific feature of the spectral theory in a Banach space. The independence of the spectrum of (s, t). Estimates of approximation numbers. Optimal resolvent estimates and the representation

of the resolvent as a ratio of entire analytic functions, operator-valued and numerical, with estimating of their growth. Generalizations of theorems on the completeness and the Abel–Lidskii summability.

7. Some generalizations, including the cases of an unbounded domain with compact boundary, higher-order systems, and the boundary conditions on a non-closed boundary.

As a result, several classical achievements in the spectral theory of "smooth" elliptic problems admit a generalization to "non-smooth" problems considered in short scales of function spaces. However, by far not all problems have been solved up to the desired end.

The references will be indicated in the talk.

Wave spectrum of symmetric semi-bounded operator and its applications

Michail Belishev

Steklov Intstitute, St.Petersburg

Motivation The paper introduces the notion of a *wave spectrum* of a symmetric semi-bounded operator in a Hilbert space. The impact comes from inverse problems of mathematical physics; the following is one of the motivating questions.

Let Ω be a smooth compact Riemannian manifold with the boundary Γ, $-\Delta$ the (scalar) Laplace operator, $L_0 = -\Delta|_{C_0^\infty(\Omega\backslash\Gamma)}$ the *minimal Laplacian* in $\mathcal{H} = L_2(\Omega)$. Assume that we are given with a unitary copy $\widetilde{L}_0 = UL_0U^*$ in $\widetilde{\mathcal{H}} = U\mathcal{H}$ (but U is unknown!). To what extent does $\widetilde{L}_0$ determine the manifold Ω?

So, we have no points, boundaries, tensors, etc, whereas the only thing given is an operator $\widetilde{L}_0$ in a Hilbert space $\widetilde{\mathcal{H}}$. Provided the operator is unitarily equivalent to L_0, is it possible to extract Ω from $\widetilde{L}_0$? Such a question is an "invariant" version of various setups of dynamical and spectral inverse problems on manifolds [1].

Wave spectrum Substantially, the answer is affirmative: for a generic class of manifolds, any unitary copy of the minimal Laplacian determines Ω up to isometry (Theorem 1). A wave spectrum is a construction that realizes the determination $\widetilde{L}_0 \Rightarrow \Omega$ and, thus, solves inverse problems. In more detail,

- With a closed densely defined symmetric semi-bounded operator L_0 of nonzero defect indexes in a separable Hilbert space $\mathcal{H}$ we associate a metric space Ω_{L_0} (its wave spectrum). The space consists of the so-called eikonal operators (*eikonals*), so that Ω_{L_0} is a subset of the bounded operators algebra $\mathfrak{B}(\mathcal{H})$, whereas the metric on Ω_{L_0} is $\|\tau - \tau'\|_{\mathfrak{B}(\mathcal{H})}$.

  The eikonals are constructed from the projections on the reachable sets of an abstract *dynamical system with boundary control* governed by the evolutionary equation $u_{tt}+L_0^*u = 0$. More precisely, they appear in the framework of a von Neumann algebra $\mathfrak{N}_{L_0}$ associated with the system, whereas $\Omega_{L_0} \subset \mathfrak{N}_{L_0}$ is a set of the so-called *maximal eikonals*. The peculiarity is that this algebra is endowed with an additional operation that we call a *space extension*.

  Since the definition of Ω_{L_0} is of invariant character, the spectra Ω_{L_0} and $\Omega_{\widetilde{L}_0}$ of the unitarily equivalent operators L_0 and $\widetilde{L}_0$ turn out to be isometric (as metric spaces). So, a wave spectrum is a (hopefully, new) unitary invariant of a symmetric semi-bounded operator.

- A wide generic class of the so-called *simple manifolds* is introduced. Roughly speaking, a simplicity means that the symmetry group of Ω is trivial. The central Theorem 1 establishes that for a simple Ω, the wave spectrum of its minimal Laplacian L_0 is isometric to Ω. Hence, any unitary copy $\widetilde{L}_0$ of L_0 determines the simple Ω up to isometry by the scheme $\widetilde{L}_0 \Rightarrow \Omega_{\widetilde{L}_0} \stackrel{\rm isom}{=} \Omega_{L_0} \stackrel{\rm isom}{=} \Omega$. In applications, it is the procedure, which recovers manifolds by the BC-method [1]: concrete inverse data determine a relevant $\widetilde{L}_0$, and it is the fact, which enables one to realize the scheme.

- We discuss one more option: once the wave spectrum of the copy $\widetilde{L}_0$ is found, the BC-procedure realizes elements of the space $\widetilde{\mathcal{H}}$ as functions on $\Omega_{\widetilde{L}_0}$. In the BC-method, such an option is interpreted as *visualization of waves* [1]. Thereafter, one can construct a *functional model* $L_0^{\rm mod}$ of the original Laplacian L_0, the model being an operator in $\mathcal{H}^{\rm mod} = L_{2,\,\mu}(\Omega_{\widetilde{L}_0})$ related with L_0 through a similarity (gauge transform). Hopefully, this observation can be driven to a functional model of a class of symmetric semi-bounded operators. Presumably, this model will be *local*, i.e., satisfying ${\rm supp}\, L_0^{\rm mod} y \subseteq {\rm supp}\, y$.

Comments The concept of wave spectrum summarizes rich "experimental material" accumulated in inverse problems in the framework of the BC-method, and elucidates operator background of the latter. Owing to its invariant nature, Ω_{L_0} promises to be useful for further applications to unsolved inverse problems of elasticity theory, electrodynamics of crystals, graphs, etc.

Actually, a wave spectrum is an attribute not of a single operator but a so-called *algebra with space extension* [2]. In the scalar problems on manifolds, this algebra is commutative, whereas its wave spectrum is identical to Gelfand's spectrum of the norm-closed subalgebra generated by eikonals. However, it is not clear whether this fact is of general character. The algebras that appear in the above mentioned unsolved problems, are *noncommutative* and the relation between their wave and Jacobson's spectra is not understood yet.

By the recent trend in the BC-method, to recover unknown manifolds via boundary inverse data is to find spectra of relevant algebras determined by the data [3]. We hope for further promotion of this approach: see [2].

References

[1] M.I.Belishev. Recent progress in the boundary control method. *Inverse Problems*, 23 (2007), no 5, R1–R67.

[2] M.I.Belishev. An unitary invariant of semi-bounded operator and its application to inverse problems. http://arxiv.org/abs/1004.1646.

[3] M.I.Belishev. Geometrization of Rings as a Method for Solving Inverse Problems. *Sobolev Spaces in Mathematics III. Applications in Mathematical Physics, Ed. V.Isakov.*, Springer, 2008, 5–24.

Asymptotic behavior of the eigenfunctions of three-particle Schrödinger operator. II. One-dimensional charged particles.

Vladimir Buslaev

St.Petersburg State University

The talk is based on a joint work with S.B. Levin: Asymptotic behavior of the eigenfunctions of three-particle Schrödinger operator II. One-dimensional charged particles. Algebra i Analiz, vol. 22 (2010), no. 3, 60-79 (in Russian).

We consider the three particle Schrödinger operator with slowly decreasing pair potentials. More precisely, we suppose that the pair potentials for large distances behave like the Coulomb potentials. We develop an heuristic approach that allows to describe the asymptotic behavior of the generalized eigen-functions of the continuous spectrum at infinity in the configuration space. We hope to give the rigorous justification of these asymptotic formulas later.

We consider the eigen-functions that can be treated as perturbations of the plane waves, and we call the corresponding eigen-functions the scattered plane waves. It is clear that after the separation of the motion of the centre of mass for one-dimensional particles the dimension of the configuration space is equal to 2. The centers of three two particle pair interactions defines on the configuration plane three straight lines. We call them screens. We suppose that the masses of the particles and the pair potentials are identical. It allows to simplify the presentation. With given wave vector of the falling plane wave we define the scattered plane wave by very rough assumptions on its asymptotic behavior. The corresponding scattering amplitude is a rather singular distribution. Our goal is to find smooth functions that correctly describe the asymptotic behavior and in a weak topology generate the mentioned distribution. Moreover, we are able to describe not only the general structure of the singular contributions, but their precise form.

The solution has natural singularities on screens, on the ray corresponding to the wave vector of the falling plane wave and on five other rays that can be obtained from the mentioned one by its reflections with respect to screens. The general structure of the asymptotic behavior can be described in terms of the reflections of the falling plane wave by the system of the screens. It is natural that near the screens the simple plane wave must be replaced by a solution that includes the effect of the pair potential centered around the screen. It turns out that this geometrical solution defines on the phase plane the smooth function everywhere except two rays obtained from the reflection of the wave vector. It is very similar to the singularity that we have in the wave field that is generated by the classical diffraction of the plane wave by a plane semi-screen. And analogously to the classical problem we can overcome the difficulty with the jump the scattering amplitude with the help of the Fresnel integral.

In fact, we described above the results that were obtained for the case of sufficiently fast decay of the pair potentials. These results were presented in the talk of the same author at the previous conference. The generalization to the case of slowly decreasing pair potentials demands the development a lot of new technique. The simplest is the replacement of the classical plane wave by the Coulomb plane wave that contains some more or less known logarithmic corrections at infinity. The crucial generalization is the description of the approximate separation of the variables near the screens. There is no analog of this procedure in the case of fast decaying pair potentials. Another essential generalization of the previous constructions is the replacement well known Fresnel integrals some other special functions containing the Cauchy singular integrals.

Finally, we can obtain for the case of the slowly decaying pair potentials the results that are more general, but in their flavor are similar to the results described earlier for the case of fast decaying at infinity pair potentials. By the moment these results were used for the computer description of the scattered plane waves. The approach turned out to be quite efficient.

Eigenvalue problems for the Laplacian on noncompact Riemannian manifolds

Andrea Cianchi

University of Florence

This talk is based on a joint work with V.Maz'ya

We begin by dealing with a class of eigenvalue problems for the Laplacian on n-dimensional Riemannian manifolds M whose weak formulation is:

$$\int_M \langle \nabla u \,, \nabla v \rangle \, d\mathcal{H}^n(x) = \gamma \int_M u \, v \, d\mathcal{H}^n(x) \tag{1}$$

for every test function v in the Sobolev space $W^{1,2}(M)$. Here, $u \in W^{1,2}(M)$ is an eigenfunction associated with the eigenvalue $\gamma \in \mathbb{R}$, ∇ is the gradient operator, $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure on M, i.e. the volume measure on M induced by its Riemannian metric, and $\langle \cdot \,, \cdot \rangle$ stands for the associated scalar product.

Note that various special instances are included in this framework. For example, if M is a complete Riemannian manifold, then (1) is equivalent to a weak form of the equation

$$\Delta u + \gamma u = 0 \qquad \text{on } M. \tag{2}$$

In the case when M is an open subset of a Riemannian manifold, and in particular of the Euclidean space $\mathbb{R}^n$, equation (1) is a weak form of the eigenvalue problem for the Laplacian with homogenous Neumann boundary condition

$$\begin{cases} \Delta u + \gamma u = 0 & \text{on } M, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial M. \end{cases} \tag{3}$$

Our discussion focuses the case when M need not be compact. We exhibit minimal assumptions on M ensuring $L^q(M)$ bounds for all $q < \infty$, or $L^\infty(M)$ bounds for eigenfunctions of the Laplacian on M. Our estimates are new even for problem (3) when M is an on open subset of $\mathbb{R}^n$ of finite volume.

The assumptions to be imposed on M are formulated in terms of either its isocapacitary function ν_M, or its isoperimetric function λ_M. They are the largest functions of the measure of subsets of M which can be estimated by the capacity, or by the perimeter of the relevant subsets.

Both the conditions in terms of ν_M, and those in terms of λ_M, for eigenfunction estimates in $L^q(M)$ or $L^\infty(M)$ that will be presented are sharp in the class of manifolds M with prescribed asymptotic behavior of ν_M and λ_M at 0.

In the second part of the talk we discuss the problem of the discreteness of the spectrum of the Laplacian Δ on M, regarded as the semi-definite self-adjoint Laplace operator in the Hilbert space $L^2(M)$ associated with the closed bilinear form defined for u and v in $W^{1,2}(M)$ by the left-hand side of (1). If the space $C_0^\infty(M)$ of smooth compactly supported functions on M is dense in $W^{1,2}(M)$, the operator Δ agrees with the Friedrichs extension of the classical Laplacian, regarded as an unbounded operator on $L^2(M)$ with domain $C_0^\infty(M)$. This is certainly the case when M is complete. A different situation occurs when M is an open subset of $\mathbb{R}^n$, or, more generally, of a Riemannian manifold; in this case, Δ corresponds to the so called Neumann Laplacian on M.
A necessary and sufficient condition in terms of ν_M is established for the spectrum of Δ to be discrete. As a corollary, we derive a sufficient condition for the discreteness of the spectrum of the Laplacian on M involving the isoperimetric function of M.

Each one of the two approaches (via ν_M or via λ_M) to these problems has its own advantages. The isoperimetric function λ_M has a transparent geometric character, and it is usually easier to investigate. The isocapacitary function can be less simple to compute; however its use is in a sense more appropriate in the present framework since it not only implies the results involving λ_M, but leads to finer conclusions in general. Typically, this is the case when manifolds with complicated geometric configurations are taken into account.

Resonances of Quantum Graphs

Brian Davies

King's College London

This work was carried out jointly with A. Pushnitski in [1]; it was further investigated with P. Exner and J. Lipovský in [2].

Consider a compact quantum graph $\mathcal{G}_0$ consisting of finitely many edges of finite length joined in some manner at certain vertices. Let $\mathcal{G}$ be obtained from $\mathcal{G}_0$ by attaching a finite number of semi-infinite leads to $\mathcal{G}_0$, possibly with more than one lead attached to some vertices. For later reference we say that a vertex v of $\mathcal{G}$ is *balanced* if the number of edges with finite length attached to v equals the number of leads attached to v.

Let $H_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ acting in $L^2(\mathcal{G}_0)$ subject to continuity and Kirchhoff boundary conditions at each vertex, and let H be defined in the same way in $L^2(\mathcal{G})$. It is well-known that H_0 has discrete spectrum, and an application of standard variational methods implies that the eigenvalue asymptotics is given by Weyl's law in one dimension, the 'volume' of $\mathcal{G}_0$ being defined as the sum of the lengths of the edges. The same cannot hold for H, because its spectrum is $[0, \infty)$. Unlike the normal case for Schrödinger operators H may possess many L^2 eigenvalues corresponding to eigenfunctions that have compact support. However some eigenvalues of H_0 turn into resonances of H, and when defining the resonance counting function

$$N(r) = \#\{\text{ resonances } \lambda = k^2 \text{ of } H \text{ such that } |k| < r\}$$

one should regard eigenvalues of H as special kinds of resonance. One might hope that $N(r)$ obeys the same leading order asymptotics as $r \to \infty$ as in the case of $\mathcal{G}_0$. The analogous result for one-dimensional Schrödinger operators is known to hold whenever the potential concerned has compact support, the volume in question being the length of the shortest interval containing the support of the potential.

The following is the main theorem in [1].

Theorem *The resonances of H obey the Weyl asymptotic law if and only if the graph $\mathcal{G}$ does not have any balanced vertex. If there is a balanced vertex then one still has a Weyl law, but the effective volume is smaller than the volume of $\mathcal{G}_0$.*

The main tool in the proof is a theorem of Langer [3] which describes the asymptotic behaviour of the zeros of an 'exponential polynomial' of the form

$$F(k) = \sum_{r=1}^{n} \alpha_r \mathrm{e}^{i\sigma_r k}$$

where $\sigma_1 < \sigma_2 < \ldots < \sigma_n \in \mathbf{R}$. Langer's theorem asserts that the zeros are all confined to a strip $\{k : |\mathrm{Im}(k)| < c\}$ and that the counting function has Weyl asymptotics, the relevant volume being $\sigma_n - \sigma_1$, provided α_1 and α_n are non-zero.

The application of this theorem involves several problems. The first is to identify the zeros k of F with the complex resonances of H. Then one must prove that the order of each zero of F equals the algebraic multiplicity of the relevant resonance. Finally, after finding the σ_r, one has to prove that one does indeed have $\alpha_1 \neq 0 \neq \alpha_n$. It turns out that this last step is not valid if $\mathcal{G}$ contains a balanced vertex. The proof of this statement depends on an induction in which one starts with $\mathcal{G}_0$ and adds the leads one vertex at a time until one reaches $\mathcal{G}$. This leads to a sequence of exponential polynomials F_r, whose relationship to each other has to be understood by examining the structures of certain large matrices.

References

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[2] E. B. Davies, P. Exner, J. Lipovský, Non-Weyl asymptotics for quantum graphs with general coupling conditions, preprint 2010, J. Phys. A to appear.

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Critical threshold for electronic stability under the action of an intense magnetic field

Maria Esteban

University of Paris-Dauphine

This work has been done in collaboration with Jean Dolbeault (CNRS and Universit Paris-Dauphine) and Michael Loss (Georgia Tech University)

My talk for the Conference in Spectral Theory will be about some results on the point spectrum of the Dirac operator with external magnetic field.

Our aim is to see what happens to the first eigenvalue of the operator $H_A+V=-i\,\alpha\cdot\nabla_A+V$ in the spectral gap $(-1,1)$, where the matrices α_i are the Pauli-Dirac matrices, V is an external electrostatic potential and A is a potential related to an external magnetic field B. If for instance V is a not too strong potential of Coulombic type and A corresponds to a not too large constant magnetic field, it is easy to prove that there is a sequence of eigenvalues of H_A in the spectral gap $(-1,1)$, sequence which has as unique accumulation point the point 1. The first eigevalue in that sequence, that we denote by $\lambda_1^{V,A}$ depends on V and B (or A). If we let V be fixed, and be equal for instance to the Coulombic potential $-\nu/|x|$, $\nu\in(0,1)$, the first eigenvalue will depend only on A. Our aim is to show that if B is constant, $\lambda_1^{V,A}$ belongs to $(-1,1)$ if B is not too strong, but tends to -1 when the intensity of B increases and tends to a critical value. Our second aim is to evaluate the intensity of that critical value as a function of ν.

Our main results are the following: let us consider the particular case $V=-\nu/|x|$, $\nu\in(0,1)$ and $A=-\frac{b}{2}(-x_2,x_1,0)$, that is, $B=(0,0,b)$. And let us then denote $\lambda_1^{V,A}$ by $\lambda(b)$ and ∇_A by ∇_b.

– For every $\nu\in(0,1)$, there exists a number $b(\nu)>0$ such that $\lambda(b)\in(-1,1)$ for all $0\le b<b(\nu)$. Moreover, $\lambda(b) to -1$ as $b\to b(\nu)$.

– For all $\nu\in(0,1)$, $b\ge 0$, $\lambda(b)<1$.

– $\lim_{\nu\to1} b(\nu)>0$.

– $\lim_{\nu\to0}\ \nu\log B(\nu)\ =\ \pi$.

– $b(\nu)=4/\mu(\nu)^2$, where

$$\mu(\nu):=\inf\left\{E_\nu[\varphi]\,;\quad \int_{R^3}|\varphi|^2\,dx=1\right\},$$

and the functional E_ν is defined by

$$E_\nu[\varphi]:=\int_{R^3}\frac{|x|}{\nu}\,|(\sigma\cdot\nabla_1)\,\varphi|^2\,dx-\int_{R^3}\frac{\nu}{|x|}\,|\varphi|^2\,dx\ ,$$

The basic ingredients that we use to prove the above results are: on one hand a variational characterization of the first eigenvalue of operators with gaps that was proved by Dolbeault, Esteban and Séré in [1]. Then, a very nice scaling property which allows us to "isolate" b and get an explicit expression for it (see above).

The above variational characterization is based on an implicit minimization argument. This characterization was devised for general operators with gaps in the essential spectrum. Applied to Dirac operators, it yields a very useful tool to have very good control on the variation of $\lambda(b)$ with respect to b. More generally, this tool is extremely easy to use in order to build algorithms which yield all the eigenvalues in the gap in a robust and efficient way.

The results presented in this talk are contained in [2] and [3].

References

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Loops and trees: spectral and resonance properties of quantum graphs

Pavel Exner

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In this talk three new results about Schrödinger operators on metric graphs will be discussed which have been obtained in collaboration with Jiří Lipovský and Brian Davies. The first one [4] is related to the well-known fact about invalidity of the uniform continuation principle for quantum-graph Hamiltonian. One manifestation of it are embedded eigenvalues which can appear in case when some graph edge lengths are rationally related. This is, of course, a non-generic situation and such eigenvalues disappear once a geometric perturbation turns them into resonances; we discuss this effect generally as well as on simple solvable examples.

The second problem is related to high-energy asymptotics of resonances on quantum graphs. In a recent intriguing observation, Davies and Pushnitski [2] demonstrated that graphs with Kirchhoff vertex coupling and balanced vertices can exhibit a behavior deviating from the usual Weyl formula. Following [3] we analyze this problem for graphs with more general vertex coupling and show that if the latter exhibits edge permutation symmetry, there are exactly two situations in which the unusual behavior occurs, while without the symmetry requirement it can happen in many cases. We present also an insight explaining the effect.

The third problem concerns spectra of radial trees which are sparse in the sense that there is a subsequence of edges with lengths growing to infinity. Breuer and Frank [1] proved recently that in case of Kirchhoff coupling the corresponding Hamiltonian has empty *ac* spectrum. We have shown in [4] that the result remains valid for a large class of vertex couplings, but on the other hand, there are nontrivial couplings for which transport on such a tree is possible.

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Behavior at infinity of solutions of almost periodic equations

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The talk is based on a joint work with Frederic Klopp

Discuss the behavior at infinity of the solutions of the one dimensional difference Schrödinger equation

$$\psi(k+1)+\psi(k-1)+\lambda v(\omega k+\theta)\psi(k)=E\psi(k), \quad k\in\mathbb{Z}, \tag{4}$$

where v a 1-periodic function of $\mathbb{R}$, and $\lambda >> 1$, $0<\omega<1$ and $0\le\theta<1$ are constant parameters. We are interessed in the case where $\omega\notin\mathbb{Q}$, i.e., the Schrödinger equation is almost periodic.

It is well known that the absolutely continuous spectrum and, thus, the singular spectrum of (4) in $l^2(\mathbb{Z})$ can be characterized in terms of the of the Lyapunov exponent defined by the limit

$$\gamma(E,\theta)=\lim_{k\to\infty}\frac{1}{k}\ln\|M(\theta+k\omega)\dots M(\theta+\omega)M(\theta)\|, \quad M(\theta)=\begin{pmatrix}E-\lambda v(\theta) & -1\\ 1 & 0\end{pmatrix}. \tag{5}$$

If v is good enough (say, bounded and measurable), for a.e. θ, the limit exists and is independent of θ. The independent of θ value is denoted by $\gamma(E)$. By Isii-Kotani-Pastur theorem, if on an interval $I\subset\mathbb{R}$ the $\gamma(E)>0$, then, for a.e. θ, the spectrum situated on I is singular. It is known (follows from theorems of Oseledets and of Ruelle) that if the limit $\gamma(E,\theta)$ exists and is positive, then all the solutions of (4) are either eponentially increasing or decaying at infinity. Such behavior of the solutions can lead only to the point spectrum with exponentially decaying eigenfunctions. But, there are many examples of the equations with singular continuous spectrum (first was construced by B.Simon by means of A.Gordon results). For a given θ, the singular continuous spectrum can essentially be situated only at the energies E where the limit defining $\gamma(E,\theta)$ does not exist.

In this talk we discuss the pointwise existence of the limit $\gamma(E,\theta)$ and the behavior at infinity of the solutions in the cases where the limit does not exists.
Our approach to the investigation of the solutions is based on the monodromization idea. The monodromization method is a renormalization method proposed in 90-ies by V.Buslaev and A.Fedotov to construct Bloch solutions of difference equations with periodic (as the potential v does) coefficients; first, it was aimed to reproduce the result on the cantorien geometry of the spectrum of the Almost Mathieu equation obtained by B.Helffer and J.Sjöstrand. It appeared that the ideas of the approach allow also to control (recurrently) the solutions at larger and larger distances.
The idea of the approach is suggested by the standard Bloch-Floquet theory for the differential periodic equations. The coefficients of the differential equation being 1-periodic, the space of its solutions is invariant with respect to the translation by 1. This translation defines a linear map in the space of the solutions. The matrix M_1 representing this map (for a fixed base in the space of the solutions) is called the monodromy matrix. It is independent of the variable of the eqiuation. The behavior of the solutions at infinity is determined by the behavior of $(M_1)^k$ as $k \to \infty$. Turn to the difference equation (4). Instead of considering this equation on the **lattice** $\mathbb{Z}$, we consider anlogous difference equation on the *real line* $\mathbb{R}$:

$$\psi(x+\omega) + \psi(x-\omega) + \lambda v(x)\psi(x) = E\psi(x), \quad x \in \mathbb{R}. \tag{6}$$

As the behavior at infinity of the solutions of (4), so the behavior of the solutions of (6) is determined by the behavior of the product

$$P_k(\theta) = M(k\omega+\theta)\dots M(\omega+\theta)M(\theta) \quad \text{as} \quad k \to \infty. \tag{7}$$

Note that $P_k(\theta)$ is the matrix fundamental solution of the equation $P_{k+1} = M(k\omega+\theta)\,P_k$ with the initial condition $P_0 = I$. The function v being 1-periodic, the space of the solutions of (6) is invariant with respect to the translation by 1. This again leads to the notion of the monodromy matrix. But, now, the monodromy matrices appear to be non-constant. They are ω-periodic in x (the variable in (6)). It appears that the behavior of the solutions of (4) at infinity can be equally characterized in terms of the product

$$P_k^{(1)}(\theta_1) = M_1(k\omega_1+\theta_1)\dots M(\omega_1+\theta_1)M(\theta_1), \tag{8}$$

where M_1 is the one periodic matrix obtained from a monodromy matrix just by the linear change of the variable $x \mapsto x_1 = x/\omega$, and

$$\omega_1 = \left\{\frac{1}{\omega}\right\}, \quad \theta_1 = \left\{\frac{\theta}{\omega}\right\}, \tag{9}$$

$\{x\}$ denoting the fractional part of $x \in \mathbb{R}$. More precisely, one has

$$P_k(\theta) = \Psi(\{k\omega+\theta\})\sigma P_{k_1}^{(1)}(\theta_1)\sigma\Psi^{-1}(\theta), \quad k_1 = -[k\omega+\theta], \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{10}$$

where $\Psi(x)$ is a fundamental matrix solution of the equation $\Psi(x+\omega) = M(x)\Psi(x)$ on the real line $x \in \mathbb{R}$, and $P^{(1)}$ is constructed in terms of the monodromy matrix corresponding to Ψ.
We call the passage from the matri product P to $P^{(1)}$ the monodromization (or the renormalization).

The matrix product $P^{(1)}$ having the same structure as P (constructed of 1-periodic unimodular matrices), one can apply the renormalization to the product $P^{(1)}$ so on. This defines the sequences $\{M_l\}$, $\{\omega_l\}$, $\{\theta_l\}$ and $\{k_l\}$.
A very important feature of the relation (10) is that k_1, the new number of the matrices M_1 in $P^{(1)}$, the new matrix product, is typically smaller than k, and that $|k_l|$ exponentially quickly decreases to 1 as the number l increases. So, in course of the renormalizations, one quickly diminishes the number of the matrices in the product to study. That is this property that allows to study effectively the behavior of the input matrix product as $k \to \infty$ or, equivalently, the behavior of the solutions of (4) at infinity.
Of course, to carry out the renormalizations effectively, one has to control effectively the "sandwich" terms Ψ in (10) and can be able to compute effectivelly the monodromy matrices. This is to enable such control that we have assumed that the coupling constant λ in the input equation (4) is sufficiently large. The striking feature of the equation with a sufficiently large λ is that all the equations arizing in course of the renormalizations contain effective large parameters, and that these parameters increase hyperexponentially quickly with the subsequent renormalizations.
Though our approach is quite general, and we already have results for a wide class of analytic potentials v, in this talk, we concentrate on the model equation with

$$v(x) = 2e^{i\pi\omega/2} \sin \pi x, \quad \text{and} \quad E = 0. \tag{11}$$

We describe the set of θs for which the pointwise limit defining Lyapunov exponent exists and describe the behavior of the solutions at infinity for the complementary values of θ.
The advantage of the working with the model equation is that one can construct explicitely (by contour integration of combinations of non-trivial special functions) solutions of the corresponding continuous equation (6) and compute explicitely the corresponding monodromy matrices. Moreover, having the rather explicit description of the solutions of (6), we can get their asymptotics (for $\lambda >> 1$ and $0 \leq x \leq 1$) "almost for free". The solutions of (6) have the most simple asymptotic behavior (on the interval $0 \leq x \leq 1$) if ω is not too small. More precisely, one has to assume that $\omega\lambda >> 1$. Trying to remain in such a case when studing each equation arizing in the course of the renormalizations, we have to assume that for the given $\lambda > 1$, the frequency ω we consider has to satisfy the following condition: we assume that there is a non-decreasing function $M : \mathbb{N} \to \mathbb{N}$ such that $M(L) < L$ and

$$\omega_M \dots \omega_L \omega_{L-1} \to 0, \quad \left(\lambda^{\frac{1}{\omega\omega_1 \cdots \omega_{M-1}}} \omega_M\right)^{1/2} \omega_{M+1} \dots \omega_L \omega_{L+1} \to \infty,$$
$$\text{when} \quad M = M(L), \quad \text{and } L \to \infty.$$

This condition is satisfied for almost all ωs.
We describe the sufficient condition for the pointwise existence of the Lyapunov exponent in terms of the dynamical system defined by (9) on $(0,1) \times (0,1) \subset \mathbb{R}$. We prove that the limit defining the Lyapunov exponent exists and is equal to $\ln \lambda$ if, along the trajectory $\{\omega_l, \theta_l\}$, $\omega_0 = \omega$, $\theta_0 = \theta$, the conditions

$$\begin{aligned} |\theta_{L+1} - 1| &\leq \tfrac{1}{(\lambda_{M(L)}\omega_{M(L)})^{1/2}\omega_{M(L)+1}\cdots\omega_L} && \text{if } L+1 \text{ is even,} \\ |\theta_{L+1} - \omega_{L+1}| &\leq \tfrac{1}{(\lambda_{M(L)}\omega_{M(L)})^{1/2}\omega_{M(L)+1}\cdots\omega_L} && \text{if } L+1 \text{ is odd,} \end{aligned} \tag{12}$$

are satisfied at most for finitely many L. The set of "bad" θs for which the Lyapunov eponent does not exist appears to be a dence G_δ-set of zero Lebeague measure. It admits rather simple constructive description. Roughly, the set θs for which Θ_{L+1} appears too close to the forbidden values, see (12), consists of small neighborhoods of the points $0 < \theta = n\omega + m < 1, \ \ m, n \in \mathbb{N}$, such that $0 < m \leq \frac{1}{\omega_1 \omega_2 \ldots \omega_L}$.
For the "bad" values of θ (the Lyapunov exponent does not exist), we describe the behavior of $\Gamma_k(\theta) = \frac{1}{k} \ln \|P_k(\theta)\|$. We show that if θ_{L+1} appears too close to the forbiden value, then $\ln \|P_k(\theta)\|$ instead of increasing (roughly linearly) becomes to decrease (roughly linearly). More precisely, on the "interval" $0 \leq k \leq \frac{\text{Const}}{\omega_0 \omega_1 \ldots \omega_{L-1}}$, there is a subinterval, where $\Gamma_k(\theta)$ that normally has to tend to $\ln \lambda$ can fall by a number of order $\ln \lambda$.
There are two possible scenarios of behavior of $\Gamma_k(\theta)$ for the bad θs. One of them is possible when the sequence $\{\omega_l\}$ contains a decreasing subsequence, and the other is the only one possible in the case when $\{\omega_l\}$ is bounded from below. In the talk, we describe these scenarios and give the asymptotic description of $\Gamma_k(\theta)$.

Critical Lieb-Thirring bounds in gaps

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Lieb-Thirring inequalities [4] bound moments of eigenvalues of Schrödinger operators in terms of integrals of the potential. These inequalities play a crucial role in Mathematical Physics (in particular, in the problem of stability of matter), but recently, they have also found an application in purely spectral theoretic questions, namely, in the characterization of spectral measures. A particular case of a Lieb-Thirring inequality is the following endpoint inequality in dimension one [7], which states that

$$\mathrm{Tr}\left(-\frac{\mathrm{d}^2}{\mathrm{dx}^2} + \mathrm{V}\right)_-^{1/2} \leq \mathrm{const} \int_{\mathbb{R}} \mathrm{V(x)}_- \, \mathrm{dx}\,.$$

In this talk we will discuss generalizations of this inequality to Schrödinger operators $-\frac{d^2}{dx^2} + W + V$, where W is a background potential and V is a decaying perturbation. An initial result concerns eigenvalues below the bottom λ_W of the spectrum of $-\frac{d^2}{dx^2} + W$. Assuming that there is a positive function ω which is bounded from above and away from zero and which satisfies $-\omega'' + W\omega = \lambda_W \omega$, we showed in [2] that

$$\mathrm{Tr}\left(-\frac{\mathrm{d}^2}{\mathrm{dx}^2} + \mathrm{W} + \mathrm{V} - \lambda_\mathrm{W}\right)_-^{1/2} \leq \mathrm{const} \int_{\mathbb{R}} \mathrm{V(x)}_- \, \mathrm{dx}$$

with a constant depending only on the infimum and the supremum of ω. We note that the existence of an ω with the required properties is known, for instance, in the case of periodic W's.

Our second main result concerns an internal gap (a, b) in the spectrum of $-\frac{d^2}{dx^2} + W$. Under suitable assumptions on the spectral representation near the band edges of $-\frac{d^2}{dx^2} + W$, we prove that

$$\sum_{\lambda_j \in \mathrm{spec}(-\frac{d^2}{dx^2}+W+V)\cap(a,b)} \mathrm{dist}\left(\lambda_j, \mathrm{spec}(-\tfrac{d^2}{dx^2}+W)\right)^{1/2} \leq \mathrm{const} \int_{\mathbb{R}} |V(x)|\, dx$$

with a constant depending only on spectral characteristics of $-\frac{d^2}{dx^2}+W$. Again the assumptions are verified for periodic potentials W.

Both results have analogues for discrete Schrödinger operators. In that case, the assumptions are satisfied, for instance, if the spectrum of the unperturbed operator has finitely many gaps (which is weaker than requiring the background potential to be periodic) and we are able to prove a generalized Nevai conjecture about an ℓ_1-condition implying a Szegő condition for the spectral measure.

Our proofs make use of, and extend, operator theoretic techniques developed in the study of eigenvalue asymptotics in gaps (see, e.g., [1,5,6] and references therein).

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About currents, magnetic perturbations, magnetic barriers and magnetic guides in quantum Hall systems

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We report on two recent joint works with N. Dombrowski and G. Raikov [2,3].

Since a seminal paper of Halperin, the physics of the Quantum Hall effect can be studied from two points of view: bulk and edge. They both give rise to quantized currents measured through, respectively, the Hall conductance and the edge conductance. These two points of view coincide since these two conductances are simultaneously quantized.

BULK:

In quantum Hall systems, namely 2DEG submitted to a transverse constant magnetic field, localized states are responsible for the celebrated plateaux of the quantum Hall effect. Where the Hall conductance is discontinuous, non trivial transport has been proved to take place in [6] for electric disorder, using a characterization of the region of localization in terms of the dynamics proved in [5]. We provide a similar picture but with magnetic disorder in [3]. The random magnetic potential is shown to create both strongly localized states at the edges of the spectrum and dynamical delocalization near the center of the band in the sense that wave packets travel at least at a given minimum speed. We thus consider 2D-random magnetic perturbations of the Landau Hamiltonian and prove a transition between dynamical localization and dynamical delocalization inside an arbitrary number of bands.

The proof of localization exploits the Wegner estimate of Hislop and Klopp [9], revisited by Ghribi, Hislop and Klopp [7], together with a simple weak disorder argument to start the multiscale analysis, provided some information on the location of the spectrum that we address in a separate argument; then dynamical localization follows from [4]. Delocalization is proved along the lines of [6]; in particular the Hall conductance is quantized, constant in the region of localization and jumps by one as a Landau level is crossed.

We further exhibit an explicit family of small periodic magnetic perturbations for which the splitting gives rise to a full interval of spectrum. This is achieved by direct computation using translation invariance of our potential in one direction. Such examples are then good enough to be randomized and used as random magnetic fields.

ONE EDGE:

In [2], the wall is designed by an Iwatsuka magnetic field [10], a y-independent magnetic field with a decaying profile in the x-asis. As a matter of fact the particle is subjected to, say, a strong magnetic field on the left half plane, and to a weaker one on the right half plane, creating currents along such an interface.

Perturbations are also of magnetic nature. As a preliminary but essential result, we prove that magnetic perturbations carried by magnetic fields compactly supported in the x axis do not affect the edge conductance. Next, we consider non compactly supported perturbations that do not vanish at infinity, and provide a sum rule similar to that obtained in [1]. Namely, the edge conductance of the perturbed system is the sum of the edge conductance of the magnetic confining potential and of the edege conductance of the system without magnetic wall defined by a reference Landau Hamiltonian pertubed by the magnetic potential. This enables us to compute the edge conductance of the perturbed Hamiltonian when energies fall inside a gap of the Landau Hamiltonian of magnetic strength B_- and perturbed by the magnetic potential. To consider energies corresponding to localized states, one has to go one step further and regularize the trace that defines the edge conductance.

TWO EDGES:

If we now consider a magnetic strip created by two large positive magnetic fields and a (not too big) magnetic field inside, the net current flowing along these axes is zero, like in the electric case. An interesting phenomenon appears when the two walls are generated by magnetic fields of opposite signs. Existence of quantized current is proved, with quantization equal to two times the value provided by the quantum Hall effect in the particular case of opposite value of the magnetic strength [2]. Such currents are sometimes called "snake currents" in the physics literature.

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Quasi-intersections of an Isoenergetic Surface and Complex Angle Variable

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Joint work with Young-Ran Lee[2]

We study the operator

$$H = -\Delta + V(x) \tag{13}$$

in two dimensions, $V(x)$ being a limit-periodic potential:

$$V(x) = \sum_{r=1}^{\infty} V_r(x), \tag{14}$$

where $\{V_r\}_{r=1}^{\infty}$ is a family of periodic potentials with doubling periods and decreasing L_∞-norms, namely, V_r has orthogonal periods $2^{r-1}\vec{\beta}_1,\ 2^{r-1}\vec{\beta}_2$ and $\|V_r\|_\infty < \exp(-2^{\eta r})$ for some $\eta > 0$.

We concentrate here on properties of the spectrum and eigenfunctions of (13), (14) in the high energy region. We prove the following results for the case $d = 2$.

1. The spectrum of the operator (13), (14) contains a semiaxis. A proof of the analogous result by different means is to appear in in the paper [1]. In [1], more general case $H = (-\Delta)^l + V$, $8l > d + 3$, $d \neq 1(\mathrm{mod} 4)$, is considered, however, under additional restriction on the potential: all the lattices of periods Q_r of periodic potentials V_r need to contain a nonzero vector γ in common.

2. There are generalized eigenfunctions $\Psi_\infty(\vec{k}, \vec{x})$, corresponding to the semiaxis, which are close to plane waves: for every $\vec{k}$ in an extensive subset $\mathcal{G}_\infty$ of $\mathbb{R}^2$, there is a solution $\Psi_\infty(\vec{k}, \vec{x})$ of the equation $H\Psi_\infty = \lambda_\infty \Psi_\infty$ which can be described by the formula:

$$\Psi_\infty(\vec{k}, \vec{x}) = e^{i\langle \vec{k}, \vec{x}\rangle} \left(1 + u_\infty(\vec{k}, \vec{x})\right), \tag{15}$$

$$\|u_\infty\| =_{|\vec{k}|\to\infty} O(|\vec{k}|^{-\gamma_1}), \quad \gamma_1 > 0, \tag{16}$$

where $u_\infty(\vec{k}, \vec{x})$ is a limit-periodic function:

$$u_\infty(\vec{k}, \vec{x}) = \sum_{r=1}^{\infty} u_r(\vec{k}, \vec{x}), \tag{17}$$

$u_r(\vec{k}, \vec{x})$ being periodic with periods $2^{r-1}\vec{\beta}_1,\ 2^{r-1}\vec{\beta}_2$. The eigenvalue $\lambda_\infty(\vec{k})$ corresponding to $\Psi_\infty(\vec{k}, \vec{x})$ is close to $|\vec{k}|^2$:

$$\lambda_\infty(\vec{k}) =_{|\vec{k}|\to\infty} |\vec{k}|^2 + O(|\vec{k}|^{-\gamma_2}), \quad \gamma_2 > 0. \tag{18}$$

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The "non-resonant" set $\mathcal{G}_\infty$ of the vectors $\vec{k}$, for which (15) – (18) hold, is an extensive Cantor type set: $\mathcal{G}_\infty = \cap_{n=1}^{\infty}\mathcal{G}_n$, where $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets with larger and larger number of holes in each bounded region, holes added on each step being of smaller and smaller size. The set $\mathcal{G}_\infty$ satisfies the estimate:

$$\frac{|(\mathcal{G}_\infty \cap \mathbf{B_R})|}{|\mathbf{B_R}|} =_{R\to\infty} 1 + O(R^{-\gamma_3}), \quad \gamma_3 > 0, \tag{19}$$

where $\mathbf{B_R}$ is the disk of radius R centered at the origin, $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^2$.

3. The set $\mathcal{D}_\infty(\lambda)$, defined as a level (isoenergetic) set for $\lambda_\infty(\vec{k})$,

$$\mathcal{D}_\infty(\lambda) = \left\{ \vec{k} \in \mathcal{G}_\infty : \lambda_\infty(\vec{k}) = \lambda \right\},$$

is proven to be a slightly distorted circle with infinite number of holes. It can be described by the formula:

$$\mathcal{D}_\infty(\lambda) = \{\vec{k} : \vec{k} = \varkappa_\infty(\lambda, \vec{\nu})\vec{\nu},\ \vec{\nu} \in \mathcal{B}_\infty(\lambda)\}, \tag{20}$$

where $\mathcal{B}_\infty(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_\infty(\lambda)$ can be interpreted as the set of possible directions of propagation for the almost plane waves (15). The set $\mathcal{B}_\infty(\lambda)$ has a Cantor type structure and an asymptotically full measure on S_1 as $\lambda \to \infty$:

$$L\left(\mathcal{B}_n(\lambda)\right) =_{\lambda\to\infty} 2\pi + O\left(\lambda^{-\gamma_3/2l}\right), \tag{21}$$

here and below $L(\cdot)$ is the length of a curve. The value $\varkappa_\infty(\lambda, \vec{\nu}) - \lambda^{1/2l}$ in (20) gives the deviation of $\mathcal{D}_\infty(\lambda)$ from the perfect circle of the radius $\lambda^{1/2l}$ in the direction $\vec{\nu}$. It is proven that the deviation is asymptotically small

$$\varkappa_\infty(\lambda, \vec{\nu}) =_{\lambda\to\infty} \lambda^{1/2l} + O\left(\lambda^{-\gamma_4}\right), \quad \gamma_4 > 0. \tag{22}$$

4. Absolute continuity of the branch of the spectrum (the semiaxis) corresponding to $\Psi_\infty(\vec{k}, \vec{x})$ is proven.

To prove the results listed above we develop a modification of the Kolmogorov-Arnold-Moser (KAM) method. The method includes an iteration procedure. On each step of the procedure we describe a resonant set, which is a neighborhood of intersections and quasi-intersections of an isoenergetic surface in space of $\vec{k}$. Description of resonant sets is the main technical challenge of the proof. We introduce an angle variable φ: $\vec{k} = |\vec{k}|(\cos\varphi, \sin\varphi)$. All functions of $\vec{k}$ we consider as analytical functions of φ. We describe resonant sets in terms of φ. In the case of a polyharmonic operator $H = (-\Delta)^l + V(x)$, $l > 5$, the result is proven in [2].

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Decorrelation estimates for the eigenlevels of random operators in the localized regime

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We report on recent results obtained in [4]. On $\ell^2(\mathbb{Z}^d)$, consider the random Anderson model

$$H_\omega = -\Delta + V_\omega$$

where $-\Delta$ is the free discrete Laplace operator

$$(-\Delta u)_n = \sum_{|m-n|=1} u_m \quad \text{for } u = (u_n)_{n\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$$

and V_ω is the random potential

$$(V_\omega u)_n = \omega_n u_n \quad \text{for } u = (u_n)_{n\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d). \tag{23}$$

We assume that the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$ are independent identically distributed and that their distribution admits a compactly supported bounded density, say g.
It is then well known (see e.g. [3]) that

- let $\Sigma := [-2d, 2d]+\operatorname{supp} g$ and S_- and S_+ be the infimum and supremum of Σ; for almost every $\omega = (\omega_n)_{n\in\mathbb{Z}^d}$, the spectrum of H_ω is equal to Σ;
- for some $S_- < s_- \le s_+ < S_+$, the intervals $I_- = [S_-, s_-)$ and $I_+ = (s_+, S_+]$ are contained in the region of localization for H_ω, in particular, $I := I_- \cup I_+$ contains only pure point spectrum associated to exponentially decaying eigenfunctions. If the disorder is sufficiently large or if the dimension $d = 1$ then, one can pick $I = \Sigma$;
- there exists a bounded density of states, say $\lambda \mapsto \nu(E)$, such that, for any continuous function $\varphi :\ \mathbb{R} \to \mathbb{R}$, one has

$$\int_{\mathbb{R}} \varphi(E)\nu(E)dE = \mathbb{E}(\langle \delta_0, \varphi(H_\omega)\delta_0\rangle). \tag{24}$$

  Here, and in the sequel, $\mathbb{E}(\cdot)$ denotes the expectation with respect to the random parameters, and $\mathbb{P}(\cdot)$ the probability measure they induce.
  Let N be the integrated density of states of H_ω i.e. N is the distribution function of the measure $\nu(E)dE$. The function ν is only defined E-almost everywhere. In the sequel, when we speak of $\nu(E)$ for some E, we mean that the non decreasing function N is differentiable at E and that $\nu(E)$ is its derivative at E.

For $L \in \mathbb{N}$, let $\Lambda = \Lambda_L = [-L, L]^d$ be a large box and $\#\Lambda_L = (2L+1)^d$ be its cardinality. Let $H_\omega(\Lambda)$ be the operator H_ω restricted to Λ with periodic boundary conditions. The notation $|\Lambda| \to +\infty$ is a shorthand for considering $\Lambda = \Lambda_L$ in the limit $L \to +\infty$. Let us denote the eigenvalues of $H_\omega(\Lambda)$ ordered increasingly and repeated according to multiplicity by $E_1(\omega, \Lambda) \le E_2(\omega, \Lambda) \le \cdots \le E_N(\omega, \Lambda)$.

Let E be an energy in I such that $\nu(E) > 0$. The local level statistics near E is the point process defined by

$$\Xi(\xi, E, \omega, \Lambda) = \sum_{j=1}^{N} \delta_{\xi_j(E,\omega,\Lambda)}(\xi) \tag{25}$$

where

$$\xi_j(E, \omega, \Lambda) = |\Lambda| \, \nu(E) \, (E_j(\omega, \Lambda) - E), \quad 1 \le j \le N. \tag{26}$$

One of the most striking results describing the localization regime for the Anderson model is

Theorem [6] *Assume that $E \in I$ be such that $\nu(E) > 0$.*
When $|\Lambda| \to +\infty$, the point process $\Xi(\cdot, E, \omega, \Lambda)$ converges weakly to a Poisson process on $\mathbb{R}$ with intensity the Lebesgue measure i.e. for $(U_j)_{1\le j\le J}$, $U_j \subset \mathbb{R}$ bounded measurable and $U_{j'} \cap U_j = \emptyset$ if $j \neq j'$ and $(k_j)_{1\le j\le J} \in \mathbb{N}^J$, one has

$$\mathbb{P}\left(\left\{\omega; \left\{ \begin{array}{ccc} \#\{j; \xi_j(E,\omega,\Lambda) \in U_1\} = k_1 \\ \vdots & & \vdots \\ \#\{j; \xi_j(E,\omega,\Lambda) \in U_J\} = k_J \end{array} \right\}\right\}\right) \underset{\Lambda \to \mathbb{Z}^d}{\to} \prod_{j=1}^{J} e^{-|U_j|} \frac{|U_j|^{k_j}}{k_j!}.$$

A natural question that arises once this result is known:

- for $E \neq E'$, are the limits of $\Xi(\xi, E, \omega, \Lambda)$ and $\Xi(\xi, E', \omega, \Lambda)$ stochastically independent?

This question has arisen and has been answered for random matrices (see e.g. [5]); note that, in this case, the local statistics are not Poissonian.
For the Anderson model, this question has been open (see e.g. [7,8]) and to the best of our knowledge, the paper [4] is the first to bring an answer. The conjecture is also open for the continuous Anderson model and random CMV matrices where the local statistics have also been proved to be Poissonian (see e.g. [1,2,8,9]).
The main result is

Theorem 1 [4] *Assume that the dimension $d = 1$. Pick $E \in I$ and $E' \in I$ such that $E \neq E'$, $\nu(E) > 0$ and $\nu(E') > 0$.*
When $|\Lambda| \to +\infty$, the point processes $\Xi(E, \omega, \Lambda)$ and $\Xi(E', \omega, \Lambda)$, defined in (25), converge weakly respectively to two independent Poisson processes on $\mathbb{R}$ with intensity the Lebesgue measure. That is, for $(U_j)_{1\le j\le J}$, $U_j \subset \mathbb{R}$ bounded measurable and $U_{j'} \cap U_j = \emptyset$ if $j \neq j'$ and $(k_j)_{1\le j\le J} \in \mathbb{N}^J$ and $(U'_j)_{1\le j\le J'}$, $U'_j \subset \mathbb{R}$ bounded measurable and $U'_{j'} \cap U'_j = \emptyset$ if $j \neq j'$ and $(k'_j)_{1\le j\le J} \in \mathbb{N}^{J'}$ one has

$$\mathbb{P}\left(\left\{\omega; \begin{pmatrix} \#\{j; \xi_j(E,\omega,\Lambda) \in U_1\} = k_1 \\ \vdots \qquad\qquad \vdots \\ \#\{j; \xi_j(E,\omega,\Lambda) \in U_J\} = k_J \end{pmatrix} \text{ and } \begin{pmatrix} \#\{j; \xi_j(E',\omega,\Lambda) \in U'_1\} = k'_1 \\ \vdots \qquad\qquad \vdots \\ \#\{j; \xi_j(E',\omega,\Lambda) \in U_{J'}\} = k_{J'} \end{pmatrix}\right\}\right)$$
$$\underset{\Lambda \to \mathbb{Z}^d}{\to} \prod_{j=1}^{J} e^{-|U_j|} \frac{|U_j|^{k_j}}{k_j!} \cdot \prod_{j=1}^{J'} e^{-|U_{j'}|} \frac{|U_{j'}|^{k_{j'}}}{k_{j'}!}.$$

When $d \ge 2$, we also prove

Theorem 2 *Assume that d is arbitrary. Pick $E \in I$ and $E' \in I$ such that $|E - E'| > 2d$, $\nu(E) > 0$ and $\nu(E') > 0$.*
When $|\Lambda| \to +\infty$, the point processes $\Xi(E, \omega, \Lambda)$ and $\Xi(E', \omega, \Lambda)$, defined in (25), converge weakly respectively to two independent Poisson processes on $\mathbb{R}$ with intensity the Lebesgue measure.

In [2], the authors extensively study the distribution of the energy levels of random systems in the localized phase. Their results apply also to the discrete Anderson model. As a consequence of this work, Theorems 1 and 2 follow immediately from the decorrelation estimates that we present now. They are the main technical results of the paper [4].

Lemma 1 [4] *Assume $d = 1$ and pick $\beta \in (1/2, 1)$. For $\alpha \in (0, 1)$ and $\{E, E'\} \subset I$ s.t. $E \neq E'$, for any $c > 0$, there exists $C > 0$ such that, for $L \geq 3$ and $cL^\alpha \leq \ell \leq L^\alpha/c$, one has*

$$\mathbb{P}\left(\begin{Bmatrix}\sigma(H_\omega(\Lambda_\ell)) \cap (E + L^{-d}(-1,1)) \neq \emptyset, \\ \sigma(H_\omega(\Lambda_\ell)) \cap (E' + L^{-d}(-1,1)) \neq \emptyset\end{Bmatrix}\right) \leq C(\ell/L)^{2d} e^{(\log L)^\beta}. \tag{27}$$

This lemma shows that, up to sub-polynomial errors, the probability to obtain simultaneously an eigenvalue near E and another one near E' is bounded by the product of the estimates given for each of these events by Wegner's estimate. In this sense, (27) is similar to Minami's estimate for two distinct energies.
Lemma 1 proves a lemma conjectured in [7,8] in dimension 1.

In arbitrary dimension, we prove (27), actually a somewhat stronger estimate, only when the two energies E and E' are sufficiently far apart.

Lemma 2 [4] *Assume d is arbitrary. Pick $\beta \in (1/2, 1)$. For $\alpha \in (0, 1)$ and $\{E, E'\} \subset I$ s.t. $|E - E'| > 2d$, for any $c > 0$, there exists $C > 0$ such that, for $L \geq 3$ and $cL^\alpha \leq \ell \leq L^\alpha/c$, one has*

$$\mathbb{P}\left(\begin{Bmatrix}\sigma(H_\omega(\Lambda_\ell)) \cap (E + L^{-d}(-1,1)) \neq \emptyset, \\ \sigma(H_\omega(\Lambda_\ell)) \cap (E' + L^{-d}(-1,1)) \neq \emptyset\end{Bmatrix}\right) \leq C(\ell/L)^{2d} (\log L)^C. \tag{28}$$

This e.g. proves the independence of the processes for energies in opposite edges of the almost sure spectrum.
The estimate in Lemma 2 is somewhat stronger than (27); one can obtain an analogous estimate in dimension 1 if one restricts oneself to energies E and E' such that $E - E'$ does not belong to some set of measure 0.

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KdV flow on the space of generalized reflectionless potentials

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For a real valued bounded measurable function V on $\mathbb{R}$ we consider a 1D-Schrödinger operator L with potential V, namely

$$L = -\frac{d^2}{dx^2} + V,$$

which can be realized as a self-adjoint operator in $L^2(\mathbb{R})$.Then it is well known that for each $\lambda \in \mathbb{C}\backslash\mathbb{R}$ there exist unique solutions $f_\pm$ of

$$Lf_\pm = \lambda f_\pm, \quad f_\pm \in L^2(\mathbb{R}_\pm) \quad \text{and} \quad f_\pm(0) = 1.$$

Define the Weyl functions $m_\pm$ by

$$m_\pm(\lambda) = \pm f'_\pm(0).$$

They are holomorphic functions with positive imaginary parts on the upper half plane $\mathbb{C}_+$, which we call **Herglotz functions**. If the potential V is a sample path of a certain stationary random process $V^\omega(x)$, we have a family of Schrödinger operators L^ω with potentials V^ω. We assume the **ergodicity** of the stationary process $\{V^\omega(x)\}_{x\in\mathbb{R}}$. In this formulation, not only random processes but also periodic or almost periodic potentials are included. Due to the ergodicity of the process, every spectral property of the family of self-adjoint operators $\{L^\omega\}$

holds with probability one. Therefore the absolutely continuous spectrum of L^{ω} which we denote by Σ_{ac} is independent of a sample ω. If Σ_{ac} is non-empty, then it is known that

$$m_{+}^{\omega}(\xi + i0) = -\overline{m_{-}^{\omega}(\xi + i0)} \quad \text{for a.e. } \xi \in \Sigma_{ac}. \tag{29}$$

On the other hand, the property of (29) between two Weyl functions $m_{\pm}$ is known to be held on the half axis $[0, \infty)$ if the potential V decays sufficiently fast at $\pm\infty$ and the reflectionless coefficient vanishes on $[0, \infty)$, and such a potential is called reflectionless. It is also well known that the KdV equation is completely solvable if the initial function is a reflectionless potential.

These facts naturally lead us to define a class of potentials

$$\Omega = \left\{ \begin{array}{l} V;\ \text{real valued bounded measurable and} \\ m_{+}(\xi + i0) = -\overline{m_{-}(\xi + i0)} \quad \text{for a.e. } \xi \in [0, \infty) \end{array} \right\},$$

an element of which is called as a **generalized reflectionless potential**. This class was introduced by V.A.Marchenko and his collaborators and they proved several important properties of generalized reflectionless potentials. Especially such potentials are known to be holomorphic on a strip including the whole real line. The speaker has constructed a solution of the hierarchy of KdV equations with initial function from Ω, which we call **KdV flow**. The purpose of the talk is to give several properties of this flow and future research programme of the flow.

On non-unitary representations of the generalized Weyl commutation relations

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Joint work with E. Tsekanovskii

Let U and V be strongly continuous one-parameter (semi-)groups of unitary and contractive operators in a separable Hilbert space, respectively. Assume, in addition, that $\{g_t\}_{t\in\mathbb{R}}$ is a one-parameter group of affine transformations of the real line preserving the orientation. That is,

$$g_t(x) = a^t(x - \gamma) + \gamma, \quad t, x \in \mathbb{R},$$

for some $a > 0$ and $\gamma \in \mathbb{R}$. Suppose that the generalized Weyl commutation relations

$$U_t V_s = e^{isg_t(0)} V_{sg_t'(0)} U_t, \quad t \in \mathbb{R}, \quad s \geq 0,$$

hold, with $g_t'(x) = \frac{\partial}{\partial x} g_t(x)$.

In this talk we provide a complete classification (up to unitary equivalence) of indecomposable representations of the generalized Weyl commutation relations in the case where the semi-group of contractions is close to a unitary (semi-)group.

Our result can be considered a (non-selfadjoint) variant of the Stone-von Neumann Uniqueness Theorem that classifies the irreducible unitary representations (U, V) of the Weyl Commutation Relations

$$U_t V_s = e^{ist} V_s U_t, \quad t, s \in \mathbb{R}.$$

On Links Between the Random Matrix and Random Operator Theories

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1 Introduction

We discuss certain common topics of and links between the two branches of spectral theory and mathematical physics: random matrix theory (RMT) and random operator theory (RMT). The both branches appeared in the 1950s and have been developing mostly independently. The RMT owes a lot to E. Wigner, who proposed in 1952 to use the real symmetric and hermitian matrices of large size with independent entries to describe the experimentally discovered phenomenon of the energy level repulsion of heavy nuclei. It is worth to mention, however, that another class of large random matrices, known as sample covariance matrices, was used in statistics since the early 1930s.

The ROT dates back to the seminal paper by P.Anderson of 1958, in which it was proposed to explain the absence of the particle and charge mobility in disordered solids by the pure point nature of the spectrum of corresponding finite difference and differential operators, defined in the whole space and having random coefficients.

A number of other problems and results have appeared since then in both branches but till now the RMT is concentrated mostly on the asymptotic studies of eigenvalue distributions of $n \times n$ hermitian, real symmetric, unitary, etc random matrices, for which the number ν_n of non-zero (and having as a rule the same order of magnitude) entries grows faster than the matrix size n as $n \to \infty$ (see e.g. [9])

On the other hand, the ROT deals mostly with the studies of spectral types, the pure point type first of all, of finite difference and differential operators with ergodic (and weakly dependent as a rule) coefficients, although the asymptotic properties of eigenvalues of operators, defined by the same finite difference or differential expressions in large boxes are also of interest [16].

In an archetype case of matrices with Gaussian entries (Gaussian Unitary Ensemble (GUE)) we have $\nu_n = O(n^2),\ n \to \infty$, but it could be just $\nu_n/n \to \infty,\ n \to \infty$, as in the case of band or sparse random matrices. To prevent in this situation the spectrum to escape to infinity as $n \to \infty$, one has to assume that the entries of matrix vanish in this limit with the rate, determined by ν_n. This can be compared with the mean field type models of statistical mechanics.

The simplest example of "opposite case" is the diagonal random matrix, for which $\nu_n = n$. Another, much less simple example is a three-diagonal or Jacobi matrix, where only the principal and the two adjacent diagonals are non-zero, so that $\nu_n = 3n-2$. In these cases, unlike random matrices with $\nu_n/n \to \infty,\ n \to \infty$, the non-zero entries need not to be vanishing as $n \to \infty$, and it is assumed as a rule that the entries do not depend on n. An important characteristic property of these random matrices is that for them there exist "limiting objects", selfadjoint operators in $l^2(\mathbb{Z})$, defined by the double infinite version of the corresponding finite size matrix, i.e., by a second order finite difference equation on the whole line. These and analogous self adjoint operators in $l^2(\mathbb{Z}^d)$ and $L^2(\mathbb{R}^d)$, $d > 1$, defined by finite difference and differential equations in

$\mathbb{Z}^d$ and $\mathbb{R}^d$ with random ergodic coefficients are studied in spectral theory of random operators (ROT) and its theoretical physics counterpart, known as the theory of disordered systems [7,16].

The goal of the talk is to discuss certain problems and quantities, that are of common interest in both branches. However, the corresponding results are different as a rule. Thus, it is of interest to compare them and/or to find certain links. To this end we consider several families of random ergodic operators, each family displaying certain properties pertinent for random matrices as the parameter that indexes operators of the family tends to infinity. Part of these results dates back to [6]. However we obtain them by a simpler and more transparent method worked out recently in the RMT [14] and can also be applied to other families.

The organization of the talk is as follows. We recall the most known and studied random matrices and their properties. Then we present certain families of random ergodic operators and show that their integrated density of states (IDS) converge weakly to the limiting normalized counting measure of eigenvalues (NCM) of related random matrices.

2 Most Widely Known Random Matrices

2.1 Description

2.1.1 Gaussian Unitary Ensemble

This is the hermitian random matrix, defined as

$$M_n = n^{-1/2} W_n,\ W_n = \{W_{jk}\}_{j,k=1}^n,\ \overline{W_{jk}} = W_{kj} \in \mathbb{C} \tag{30}$$

where $W_{jk},\ 1 \geq j \geq k \geq n$ are independent complex Gaussian and

$$\mathbf{E}\{W_{jk}\} = \mathbf{E}\{W_{jk}^2\} = 0,\ \mathbf{E}\{|W_{jk}|^2\} = w^2.$$

Band Version

$$\begin{aligned} M_{n,b} &= b_n^{-1/2} \varphi(|j-k|/\beta_n) W_n,\ b_n = 2\beta_n + 1,\ \beta_n \in \mathbb{N}, \\ \text{supp}\,\varphi &= [0,1],\ \int \varphi^2(t)dt = 1. \end{aligned} \tag{31}$$

Deformed Version

$$M_n = M_n^{(0)} + n^{-1/2} W_n, \tag{32}$$

where $M_n^{(0)}$ is $n \times n$ hermitian either non-random or random and independent of W_n.

2.1.2 "Wishart" Matrices

$$M_n = n^{-1} X_{m,n}^* X_{m,n},\ X_{m,n} = \{X_{\alpha j}\}_{\alpha,j}^{m,n}, \tag{33}$$

where $\{X_{\alpha j}\}_{\alpha,j}^{m,n}$ are i.i.d. complex Gaussian and

$$\mathbf{E}\{X_{\alpha j}\} = \mathbf{E}\{X_{\alpha j}^2\} = 0,\ \mathbf{E}\{|X_{\alpha j}|^2\} = a^2 \tag{34}$$

Note that in statistics one use the term Wishart matrices for those with real Gaussian random variables [11]. The case of complex Gaussian random variables is known in the RMT as the Laguerre Ensemble.

Deformed Versions (both additive and multiplicative)

$$M_n = M_n^{(0)} + n^{-1} X_{m,n}^* T_m X_{m,n} \tag{35}$$

where $M_n^{(0)}$ and T_m are either non-random or random hermitian and independent of $X_{m,n}$ and one of another and so called "signal-noise" matrix

$$M_n = (A_{m,n} + X_{m,n})^* T_m (A_{m,n} + X_{m,n}), \tag{36}$$

where $A_{m,n}$ is either non-random or random but independent of $X_{m,n}$.

2.1.3 Law of Addition

Here we have $n \times n$ hermitian

$$M_n = A_n + U_n^* B_n U_n, \tag{37}$$

where U_n is uniformly (Haar) distributed over $U(n)$ and A_n and B_n are either non-random or random hermitian and independent of U_n and one of another. One of motivation for this problem is provided by the free probability studies, see e.g. [21].

2.1.4 Wigner Matrices

Replace $W_{jk},\ 1 \le j \le k \le n$ in the GUE and its band and deformed versions by arbitrary complex random variables with the same first and second moment.

2.1.5 Sample Covariance Matrices

Replace $\{X_{\alpha j}\}_{\alpha,j}^{m,n}$ in "Wishart" and its deformed version by arbitrary complex random variables with the same first and second moment.

2.2 Basic Results

Introduce the Normalized Counting Measure N_n of eigenvalues $\{\lambda_l^{(n)}\}_{l=1}^n$ of M_n:

$$N_n(\Delta) = \sharp\{l = 1, ..., n : \lambda_l^{(n)} \in \Delta\}/n,\ \Delta \subset \mathbb{R} \tag{38}$$

and assume that the corresponding measures $N_n^{(0)}$ for $M_n^{(0)}$, σ_m of T_m, N_{A_n} of A_n, and N_{A_n} of B_n have weak limits (with probability 1 if random) as $m,\ n \to \infty,\ m/n \to c \in [0, \infty)$. Then in all above cases N_n converges weakly with probability 1 to a non-random limit N. The limit can be found via its Stieltjes transform

$$f(z) = \int \frac{N(d\lambda)}{\lambda - z}, \Im z \neq 0$$

that solves functional equations below and via the inversion formula

$$N(\Delta) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_\Delta \Im f(\lambda + i0) d\lambda.$$

2.2.1 Deformed GUE

(i) The Stieltjes transform of the limiting NCM solves the equation

$$f(z) = f^{(0)}(z + w^2 f(z)), \tag{39}$$

where $f^{(0)}$ is the Stieltjes transform of $N^{(0)}$. The equation is uniquely soluble in the class of Nevanlinna functions, i.e., analytic for non-real z and such that

$$\Im f \Im z \geq 0,\ f(z) = -\frac{1}{z} + o(1/z),\ z \to \infty. \tag{40}$$

The corresponding limiting measure is known as the *deformed semicircle law.* The same limit is for Wigner matrices (macroscopic universality) under the condition

$$\lim_{n\to\infty} n^{-2} \sum_{1\leq j\leq k\leq n} \mathbf{P}\{|W_{jk}| > \tau\sqrt{n}\} = 0,\ \forall\tau > 0,$$

reminiscent the Lindeberg condition of probability theory [13]. N is absolutely continuous and has continuous density ρ.

In particular, if $M_n^{(0)} = 0$ (GUE, Wigner), then we have the *semicircle law* by Wigner

$$\begin{aligned} f(z) &= \frac{1}{2w^2}\left(\sqrt{z^2 - 4w^2} - z\right), \\ N(d\lambda) &= \rho(\lambda)d\lambda,\ \rho(\lambda) = \mathbf{1}_{[-2w,2w]}(\lambda)\sqrt{4w^2 - \lambda^2}. \end{aligned} \tag{41}$$

The same limit is for band matrices if $b_n/n \to 0,\ n \to \infty$.

(ii) If λ_0 belongs to the interior (bulk) of the support of N and

$$E_n(s) = P\{[\lambda_0, \lambda_0 + s/\rho(\lambda_0)] \notin \lambda_l^{(n)},\ l = 1, ..., n\} \tag{42}$$

is the gap probability, then we have the Gaudin-Wigner-Dyson law for

$$E(s) = \lim_{n\to\infty} E_n(s) = \det(1 - S(s)), \tag{43}$$

where

$$(S(s)f)(x) = \int_0^s \frac{\sin\pi(x-y)}{\pi(x-y)} f(y)dy. \tag{44}$$

In particular, we have for the limiting probability density $p(s) = E''(s)$ of spacing between adjacent eigenvalues

$$p(s) = \frac{\pi}{36}s^2(1 + o(1)),\ s \to 0, \tag{45}$$

i.e., the *level repulsion* [2,4].

(iii). Eigenvectors of the GUE are uniformly (Haar) distributed over $U(n)$, an analog of complete delocalization.

Results (ii) and (iii) were recently extended to a wide class Wigner matrices (universality) [3,20].

2.2.2 Deformed Wishart Matrices

The Stieltjes transform is the unique Nevanlinna solution of

$$f(z) = f^{(0)}\left(z - a^2 c \int \frac{\tau\sigma(d\tau)}{1 + a^2\tau f(z)}\right), \tag{46}$$

where $c = lim_{n\to\infty} m/n$. In particular, for $M_n^{(0)} = 0,\ T_m = Id$

$$N_{MP}(d\lambda) = (1-c)_+\delta(\lambda)d\lambda + \rho_{MP}(\lambda)d\lambda, \tag{47}$$

where

$$\rho_{MP}(\lambda) = (2\pi a^2\lambda)^{-1}\sqrt{(a_+ - \lambda)(\lambda - a_-)}\mathbf{1}_{[a_+,a_-]},\ a_\pm = a^2(1\pm\sqrt{c})^2.$$

The same result for a wide class of sample covariance matrices [8].

The relations (43) – (45) are also valid in this case [1], manifesting an important universality property of the limiting spacing distribution of hermitian random matrices.

2.2.3 Law of Addition

Here also the limiting NCM can be found via its Stieltjes transform, solving uniquely a certain functional equation, determined by the Stieltjes transforms f_A and f_B of limiting NCM of $\{A_n\}$ and $\{B_n\}$:

$$\begin{aligned} f(z) &= f_A(h_B(z)) \\ f(z) &= f_B(h_A(z)) \\ f^{-1}(z) &= z - h_A(z) - h_B(z), \end{aligned} \tag{48}$$

uniquely soluble in the class of functions (f, h_A, h_B), analytic for non-real z and satisfying (40) and

$$h_{A,B}(z) = z + O(1),\ z \to \infty \tag{49}$$

[17,18].

3 "Corresponding" Random Operators

3.1 Description

(i). Define a symmetric random operator H_{R_G} in $l_2(\mathbb{Z}^d)$, $d \geq 1$ by its matrix $\{H_{R_G}(x,y)\}_{x,y\in\mathbb{Z}^d}$ as

$$H_{R_G}(x,y) = h(x-y) + R^{-d/2}\varphi((x-y)/R_G)W(x,y), \quad x,y \in \mathbb{Z}^d, \tag{50}$$

where $h : \mathbb{Z}^d \to \mathbb{C}$,

$$h(-x) = \overline{h(x)}, \quad \sum_{x\in\mathbb{Z}^d} |h(x)| < \infty, \tag{51}$$

$R_G > 0$, $\varphi : \mathbb{R}^d \to \mathbb{R}$ is piece-wise continuous,

$$\max_{t\in\mathbb{R}} |\varphi(t)| \leq \varphi_0 < \infty, \quad \varphi(t) = 0,\ |t| > 1, \quad \int_{\mathbb{R}^d} \varphi^2(t)dt = 1, \tag{52}$$

and

$$W(x,y)=\overline{W(y,x)},\quad x,y\in\mathbb{Z}^d, \tag{53}$$

are independent (modulo the above symmetry condition) complex Gaussian random variables (cf (30)):

$$\mathbf{E}\{W(x,y)\}=\mathbf{E}\{W(x,y)^2\}=0,\quad \mathbf{E}\{|W(x,y)|^2\}=1,\quad x,y\in\mathbb{Z}^d. \tag{54}$$

In the case $d=1$ the random part of H_{R_G} of (50) is an infinite matrix having nonzero entries only inside the band of width $(2R_G+1)$ around the principal diagonal, and can be viewed as an analog of band matrix (31).

(ii) Define a symmetric random operator $H_d=\{H_d(x,y)\}_{x,y\in\mathbb{Z}^d}$ in $l_2(\mathbb{Z}^d)$ by

$$H_d(x,y)=h_d(x-y)+(2d)^{-1/2}W_1(x,y), \tag{55}$$

where for $x=(x_1,\ldots,x_d)$

$$h_d(x)=d^{-1/2}\sum_{j=1}^{d}h_1(x_j)\prod_{k\neq i}\delta(x_k),\quad h(0)=0, \tag{56}$$

δ is the Kronecker symbol, $h_1:\mathbb{Z}\to\mathbb{C}$ satisfies (51) for $d=1$ (e.g. the discrete Laplacian) and

$$W_1(x,y)=\begin{cases} W(x,y), & |x-y|=1,\\ 0, & |x-y|\neq 1,\end{cases} \tag{57}$$

and $W(x,y)$ are as in (54).

(iii) Define a symmetric operator $H_{n_W}=\{H_{n_W}(x,\alpha;y,\beta)\}_{x,y\in\mathbb{Z}^d,\alpha,\beta=1,\ldots n_W}$ in $l_2(\mathbb{Z}^d)\otimes\mathbb{C}^{n_W}$ as

$$H_{n_W}(x,\alpha;y,\beta)=h(x-y)\delta_{\alpha\beta}+n_W^{-1/2}\delta(x-y)W_{\alpha\beta}(x) \tag{58}$$

where $x,y\in\mathbb{Z}^d$, $\alpha,\beta=1,\ldots,n_W$, h is the same as H_{R_G}, and

$$W_{\alpha\beta}(x)=\overline{W_{\beta\alpha}(x)},\quad x\in\mathbb{Z}^d,\ \alpha,\beta=1,...,n_W, \tag{59}$$

are independent (modulo the symmetry condition) complex Gaussian random variables:

$$\mathbf{E}\{W_{\alpha\beta}(x)\}=\mathbf{E}\{W^2_{\alpha\beta}(x)\}=0,\quad \mathbf{E}\{|W_{\alpha\beta}(x)|^2\}=1,\quad x\in\mathbb{Z}^d,\ \alpha,\beta=1,...,n_W. \tag{60}$$

H_{n_W} is a special case of operators introduced by Wegner [22]. It can be regarded as the n_W-component analog of the discrete Schrodinger operator (the Anderson model (72)) or as the Hamiltonian of a disordered system in the dimension $d+n_W$, in which the random potential in n_W "transverse" dimensions is written in the "mean field" form.

(iv). The random part of operator H_R of (50) is the infinite random matrix $\{R^{-d/2}\varphi((x-y)/R)W(x,y)\}_{x,y\in\mathbb{Z}^d}$, resembling the GUE matrix $\{n^{-1/2}W_{jk}\}_{j,k=1}^n$. Recalling that one more random matrix constructed from complex Gaussian random variables is the Laguerre Ensemble (33) – (34), we can introduce the analog of H_{R_G} with the Laguerre type random part:

$$H^{(L)}_{R_L}(x,y)=h(x-y)+R_L^{-d}\varphi((x-y)/R_L)\sum_{\alpha=1}^{m}\overline{X_\alpha(x)}X_\alpha(y),\quad x,y\in\mathbb{Z}^d, \tag{61}$$

where h is as in (50), $\{X_\alpha(x)\}_{1\le\alpha\le m, x\in\mathbb{Z}^d}$ are i.i.d. complex Gaussian random variables such that

$$\mathbf{E}\{X_\alpha(x)\} = \mathbf{E}\{X_\alpha^2(x)\} = 0, \quad \mathbf{E}\{|X_\alpha(x)|^2\} = 1$$

and φ is positive definite, vanishing sufficiently fast at infinity.

(v) The next ergodic operator has the random part due to unitary Haar distributed random matrices, similarly to that of (37). Namely, consider the hermitian H_{n_V} in $l^2(\mathbb{Z}^d)\otimes\mathbb{C}^{n_V}$, defined by the matrix (cf (58) – (60))

$$H_{n_V}(x,\alpha;y,\beta) = h(x-y)\delta_{\alpha\beta} + \delta(x-y)(U^*_{n_V}(x)B_{n_V}U_{n_V}(x))_{\alpha\beta}, \tag{62}$$

where h is as above, $x,y\in\mathbb{Z}^d, \alpha,\beta = 1,\ldots,n_V$, $\{U_{n_V}(x)\}_{x\in\mathbb{Z}^d}$ are i.i.d. $n_V\times n_V$ unitary matrices whose common probability law is the normalized Haar measure on $U(n_V)$, and B is $n_V\times n_V$ hermitian matrix.

Random operators H_{R_G} and H_d can be viewed as the analogs of classical lattice Hamiltonians of statistical mechanics in which R_G is the interaction radius and d is the dimensionality of the space. The limits $R_G\to\infty$ and $d\to\infty$ in these Hamiltonians lead to the mean field models which, being rather simple, provide nevertheless fairly reasonable qualitative description of corresponding systems with large interaction radius and in high dimensions. The operators H_{n_W} and H_{n_V} are analogous to Hamiltonians of classical statistics l mechanics models having an internal structure (certain number of spin components or orbitals per site), and here the limit of infinite number of spin components or orbitals is known as the spherical model.

Note also that the four operators (50) – (60) have the form of a non-random translation invariant part and a fluctuating random part explicitly containing the parameters R_G, d,R_L, n_W, n_V that we are going to send to infinity. The random parts are such that the larger these parameters are, the more "extended" and smaller the randomness is. Similar scaling of the interaction is widely used in the mean field and the spherical approximations of statistical mechanics.

3.2 Integrated Density of States

Denote Ω the infinite dimensional probability space formed by the collection $\{W(x.y)\}_{x,y\in\mathbb{Z}^d}$ in the cases $a = R, d$, $\{W_{\alpha\beta}(x,y)\}_{\alpha,\beta=1,\ldots,n,\ x,y\in\mathbb{Z}^d}$ in the case $a = n_W$, and $\{X_\alpha(x)\}_{x\in\mathbb{Z}^d,\alpha\in\mathbb{N}}$ and $\{U(x)\}_{x\in\mathbb{Z}^d}$ in cases of (61) and $a = n_V$. Let $\{T_s\}_{s\in\mathbb{Z}^d}$ be the (shift) transformations of Ω, defined as

$$W(x,y,T_s\omega) = W(x+s,y+s,\omega),\ \forall x,y\in\mathbb{Z}^d,$$

in the case of (53) – (54), as

$$W_{\alpha\beta}(x,y,T_s\omega) = W_{\alpha\beta}(x+s,y+s,\omega),\ \forall x,y\in\mathbb{Z}^d, \alpha,\beta = 1,\ldots n_W,$$

in the case of (59) – (60), as

$$X_\alpha(x,T_s\omega) = X_a(x+s),\ \alpha = 1,...m,$$

in the case of $a = R_L$, and

$$U(x,T_s\omega) = U(x+s,\omega),\ \forall x\in\mathbb{Z}^d$$

in the case of $a = n_V$. It follows from the definitions of these collections of random variables that each T_s preserves the probability measure in Ω and that $\{T_s\}_{s\in\mathbb{Z}^d}$ is the ergodic group of transformations of Ω. This implies that the random operators $H_a,\ a = R_G, d, R_L, n_W, n_V$ are ergodic symmetric operators in the sense of [16], Sections 1.D and 2.A and if $\{U_s\}_{s\in\mathbb{Z}^d}$ is the group of unitary (shift) operators in $l^2(\mathbb{Z}^d)$, defined for any $\psi \in l^2(\mathbb{Z}^d)$ as

$$(U_s\psi)(x) = \psi(x+s), \quad \forall x \in \mathbb{Z}^d$$

in the case $a = R_G, d, R_L$ and in $l^2(\mathbb{Z}^d) \otimes \mathbb{C}^a$ as

$$(U_s\psi)(x,\alpha) = \psi(x+s,\alpha), \quad \forall x \in \mathbb{Z}^d,\ \alpha = 1, ..., a$$

in the case $a = n_W, n_V$, then we have with probability 1

$$U_s H_a(\omega) U_s^* = H(T_s\omega),\ \forall s \in \mathbb{Z}^d. \tag{63}$$

We refer the reader for the book [16], Chapters I and II for general spectral properties of ergodic operators. In particular, it follows from Corollary 4.3 of the book that all four symmetric operators H_a are defined with probability 1 on the set of sequences with finite support and is essentially self adjoint on the set.

Our intention is to study the simplest, although rather important from several point of view, spectral characteristic of the above ergodic operators known as the Integrated Density of States (IDS). It is defined as follows. For each of above operators consider its "finite box" version, i.e., the restrictions $H_{a\Lambda}$ of H_a to the cube $\Lambda \subset \mathbb{Z}^d$ centered at the origin. We obtain $|\Lambda| \times |\Lambda|$ random matrices for $a = R_G, d, R_L$, $|\Lambda| n_W \times |\Lambda| n_W$ for $a = n_W$ and $|\Lambda| n_V \times |\Lambda| n_V$ for $a = n_V$. For each of these matrices we define in the usual way the Normalized Counting Measure $N_{a\Lambda}$ of their eigenvalues as the eigenvalue counting measure divided by the size of the corresponding matrix. It follows from the general results of spectral theory of ergodic operators (see [16], Chapter IV) that for each of above operators $N_{a\Lambda}$ converges weakly with probability 1 to a non-random limit N_a and for any $\Delta \subset \mathbb{R}$

$$N_a(\Delta) = \mathbf{E}\{\mathcal{E}_a(0,0;\Delta)\}, \quad a = R, d, R_L \tag{64}$$

where $\{\mathcal{E}_a(x,y;\Delta)\}_{x,y\in\mathbb{Z}^d}$ is the matrix of the resolution of identity of H_a for $a = R, d, R_L$, and

$$N_a(\Delta) = \mathbf{E}\{a^{-1}\sum_{\alpha=1}^{a} \mathcal{E}_a(\alpha,0;\alpha,0;\Delta)\}, \quad a = n_W, n_V, \tag{65}$$

where

$$\{\mathcal{E}_a(x,\alpha;y,\beta;\Delta)\}_{x,y\in\mathbb{Z}^d,\alpha,\beta=1,...a} \tag{66}$$

is the matrix of the resolution of identity of the operators H_a for $a = n_W, n_V$.

We show that as far as it concerns the IDS of $H_a,\ a = R_G, d, R_L, n_W, n_V$, their limits as $a \to \infty$ coincide with the limiting Normalized Counting Measure of certain random matrix ensembles (the deformed semicircle law for $a = R_G, d$, and n_W, the law defined by (46) for $a = R_L$, and by (48) for $a = n_V$. In addition, the limits are strongly related to certain approximations for the Integrated Density of States of elementary excitations in certain models of disordered condensed media (see [6,5,7]) for more detailed discussions and references).

Denote $N^{(0)}$ the IDS of the non-random (unperturbed) parts of operators (50) – (61). For these convolution operators that satisfies (63) for $\Omega = \{0\}$ formula (64) implies that

$$N^{(0)}(d\lambda) = \text{mes}\{k \in \mathbb{T}^d : \widehat{h}(k) \in d\lambda\}, \tag{67}$$

where $\mathbb{T}^d = [0,1]^d$ is d-dimensional torus and

$$\widehat{h}(k) = \sum_{x \in \mathbb{Z}^d} h(x) e^{2\pi i (k,x)} \tag{68}$$

is the symbol of this operator.

Note that for operator H_d of (55) the non-random part and its IDS depend also on d. Therefore, unlike H_R, H_{n_W} and H_{n_V}, in the case of H_d the limiting transition $d \to \infty$ affects also the unperturbed IDS of the convolution operator, defined by the first term on the r.h.s. part of (55) and (56). More precisely, in this case $N^{(0)}$ is given by the limit of (56) and (67) as $d \to \infty$ and is The Gaussian measure

$$N^{(0)}(d\lambda) = (2\pi h_2)^{-1/2} e^{-\lambda^2/2h_2} d\lambda, \tag{69}$$

where

$$h_2 = \sum_{x \in \mathbb{Z}} h_1^2(x). \tag{70}$$

4 Results

We are going to prove that for $a = R_G, d, R_L, n_W$ the weak limit of N_a as $a \to \infty$ exists and is the deformed semicircle law, and that for $a = n_V$ the weak limit of N_{n_V} as $n_V \to \infty$ exists and coincides with that of (48). We start from the cases $a = R_G, d, n_W, R_L$.

Theorem 1 *Let H_a, $a = R_G, d, n_W, R_L$ be the random operators defined by (50) – (60), $N^{(0)}$ be defined by (67) for $a = R, n_W$ and by (69) for $a = d$, N_a be the Integrated Density of States of H_a given by (64) – (65). Then for $a = R_G, d, n_W$ N_a converges weakly as $a \to \infty$, to the probability measure N_{dsc}, $N_{dsc}(\mathbb{R}) = 1$ (the deformed semicircle law), whose Stieltjes transform f_{dsc} is a unique solution of the functional equation (39) with $w = 1$, $N^{(0)}$ of (67) – (70) in the class of functions, analytic for $\Im z \neq 0$ and satisfying (40), and for $a = R_L$ N_{R_L} converges to the measure, whose Stieltjes transform is uniquely determined by (46) and (40) in which the role of σ plays*

$$\sigma(\Delta) = mes\{k \in \text{supp}\, \widehat{\varphi} : \widehat{\varphi}(k) \in \Delta\},$$

where $\widehat{\varphi}$ is the Fourier transform of positive definite φ of compact support.

The proofs of the theorem is essentially based on an approach to the study the NCM in the RMT. Two main ingredient of the approach are the Poincaré - Nash bound for the variance of functions of Gaussian and classical group random variables and the differential formula for them [14,18].

Remarks *(1.)* We considered above the hermitian matrices. The case of real symmetric matrices can also be treated and leads to the same limiting results as $a \to \infty$, although requires a more involved argument.

(2.) The case $a = R_G$ for $d = 1$ can also be viewed as that of $n \times n$ band matrices (31), in which we first pass to the limit $n \to \infty$ of its infinite size and then to the limit $b = 2R_G + 1 \to \infty$ of the infinite band width. Thus, the subsequent limits $n \to \infty$ and $b \to \infty$ and the simultaneous limit $n \to \infty$, $b \to \infty$ lead to the same form of the limiting Normalized Counting Measure of eigenvalues of the corresponding matrices.

We turn now to the case $a = n_V$.

Theorem 2 *Let H_{n_V} be a self-adjoint random operator, defined in (62). Assume that the Normalized Counting Measure $N_{B_{n_V}}$ of B_{n_V} satisfies the condition*

$$\sup_{n_V} \int |\lambda|^4 N_{B_{n_V}}(d\lambda) < \infty, \tag{71}$$

and converges weakly to a probability measure N_B as $n_V \to \infty$. Then the IDS (65) of H_{n_V} converges weakly as $n_V \to \infty$ to the measure, whose Stieltjes transform is a unique solution of the system (48), in which f_A is replaced by the Stieltjes transform $f^{(0)}$ of (67) and f_B is the Stieltjes transform of N_B.

As was mentioned above, the operators H_a $a = R_G, d, R_L, n_W, n_V$ are analogs of certain Hamiltonians of lattice models of statistical mechanics, where the limits of infinite interaction radius, dimensionality or the number of spin components lead to the mean field or the spherical versions of the models. On the other hand, the studies of elementary excitations and wave propagation in disordered media are essentially based on the spectral properties of the discrete Schrodinger operator with random potential (known also as the Anderson model), the sum of the lattice Laplacian $-\Delta$ and the multiplication operator V, defined by the collection of i.i.d. random variables $\{V(x)\}_{x\in\mathbb{Z}^d}$, i.e. the discrete Schrodinger operator

$$-\Delta + V. \tag{72}$$

Spectral analysis of this operator and other finite difference and differential operators with random coefficients are among the main objectives of the ROT and the branch of condensed matter theory, known as the theory of disordered systems. In particular, the theory includes several approximation schemes, analogous to the mean field approximations in statistical mechanics (see e.g. [7], Chapter 5). One may ask then about the meaning of the results of this section in the context of the ROT and the theory of disordered systems. It can be shown that the result, mentioned in Theorem 1 for $a = R_L$ with $\widehat{\varphi} = a\mathbf{1}_A,\ a > 0,\ A \subset \mathbb{R}^d$, i.e., σ of (46), having the atoms at zero and a, the latter of the mass mes$A$, corresponds to the so called modified propagator approximation, and the result of Theorem 2 corresponds to the so called coherent potential approximation. We refer the reader to the works [7], Chapter 5, and [6,12,19,22] for related discussion and references.

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Functions of normal operators under perturbations

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I am going to speak about functions $f(N)$ of normal operators N and their behaviour under small perturbations of N.

As in the case of self-adjoint operators the following formula holds:

$$f(N_1) - f(N_2) = \iint\limits_{\mathbb{C}\times\mathbb{C}} \frac{f(\zeta_1) - f(\zeta_2)}{\zeta_1 - \zeta_2}\, dE_{N_1}(\zeta_1)(N_1 - N_2)\, dE_{N_2}(\zeta_2),$$

whenever the divided difference

$$\frac{f(\zeta_1) - f(\zeta_2)}{\zeta_1 - \zeta_2}$$

is a Schur multiplier withe respect to the spectral measures of normal operstors N_1 and N_2. However, if we want that the divided difference be a Schur multiplier with respect to all spectral measures, then f must be a linear function.

This is why the case of normal operators is much more complicated than the case of self-adjoint operators.

I am going to give another formula for $f(N_1) - f(N_2)$ in terms of double operator integrals which can be applied for large classes of functions f.

I am going to discuss the following results:

1. If f is a function of two real variables that belongs to the Besov class $B^1_{\infty 1}(\mathbb{R}^2)$, then f is operator Lipscitz, i.e.,

$$\|f(N_1) - f(N_2)\| \le \text{const}\, \|N_1 - N_2\|$$

for arbitrary normal operators N_1 and N_2.

2. If f is a function of two real variables that belongs to the Hölder class $\Lambda_\alpha(\mathbb{R}^2)$, where $0 < \alpha < 1$. Then

$$\|f(N_1) - f(N_2)\| \le \text{const}\, \|N_1 - N_2\|^\alpha$$

for arbitrary normal operators N_1 and N_2.

3. Let ω be an arbitrary modulus of continuity. Put

$$\omega_*(x) = x \int_x^\infty \frac{\omega(t)}{t^2}\, dt.$$

Suppose that f is a function on $\mathbb{R}^2$ such that

$$|f(\zeta_1) - f(\zeta_2)| \le \text{const}\,\omega(|\zeta_1 - \zeta_2|).$$

Then

$$\|f(N_1) - f(N_2)\| \le \text{const}\,\omega_*(\|N_1 - N_2\|)$$

for arbitrary normal operators N_1 and N_2.

I am also going to discuss the case when $N_1 - N_2$ belongs to the Schatten–von Neumnn class $\boldsymbol{S}_p$.

The talk is based on joint work with A.B. Aleksandrov, D. Potapov, and F. Sukochev.

On a method for computing waveguide scattering matrices [3]

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Joint work with Oleg Sarafanov

Let G be a domain in $\mathbb{R}^2$ that coincides outside a large circle with the union of finitely many non-overlapping semistrips ("cylindrical ends"). A waveguide is modeled by the Dirichlet problem for the Helmholtz equation in G with spectral parameter μ. As approximation to a row of the scattering matrix $S(\mu)$, we choose the minimizer $a(R,\mu)$ of a quadratic functional $a \mapsto J^R(a,\mu)$. To define such a functional, we solve a certain auxiliary boundary value problem in the bounded domain G^R obtained by cutting off the cylindrical ends at distance R. We prove that, as $R \to \infty$, the minimizer $a(R,)$ tends to the corresponding row of $S()$ with exponential rate uniformly with respect to μ in any finite closed interval $[\mu_1, \mu_2]$ of the continuous spectrum not containing thresholds; in doing so, we do not exclude the presence of eigenvalues of the waveguide in $[\mu_1, \mu_2]$ (to the eigenvalues there correspond eigenfunctions exponentially decaying at infinity). The applicability of the method does not restricted to the above simplest model.

Let us describe the method in more detail. We consider the boundary value problem

$$-\Delta u(x) - \mu u(x) = 0), \;\; x \in G; \;\;\; u(x) = 0, \;\; x \in \partial G. \tag{73}$$

It is known that for $\mu \in [\mu_1, \mu_2]$ there exist solutions Y_j, $j = 1, \ldots, M$, of problem (73) such that

$$Y_j(x,\mu) = u_j^+(x,\mu) + \sum_{k=1}^{M} S_{jk}(\mu) u_k^-(x,\mu) + O(e^{-\gamma|x|}) \tag{74}$$

as $|x| \to \infty$, where $u_j^+(\cdot,\mu)$ ($u_j^-(\cdot,\mu)$) is an incoming (outgoing) wave, γ being sufficiently small positive number. The matrix $S(\mu) = \|S_{jk}(\mu)\|_{j,k=1}^M$ is uniquely determined for all $\mu \in [\mu_1, \mu_2]$; it is independent of the possible arbitrariness in defining $Y_j(\cdot,\mu)$ for the μ which are eigenvalues of (73). The matrix $S(\mu)$ is unitary for all μ and is called the scattering matrix.

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To construct the aforementioned quadratic functional we consider the problem

$$\begin{aligned}(-\Delta-\mu)\mathcal{X}_l^R &= 0, \quad x\in G^R;\\ \mathcal{X}_l^R &= 0, \quad x\in \partial G^R\setminus\Gamma^R;\\ (\partial_\nu+i\zeta)\mathcal{X}_l^R &= (\partial_\nu+i\zeta)(u_l^+ +\sum\nolimits_{j=1}^M a_j u_j^-),\ x\in\Gamma^R,\end{aligned} \tag{75}$$

where Γ^R is a truncation boundary, $\zeta\in\mathbb{R}\backslash\{0\}$, ν is the outward directed normal, and $a_1,\dots,a_M$ are complex numbers. As an approximation to the row $(S_{l1},\dots,S_{lM})$ of the matrix $S=S(\mu)$, we will take the minimizer $a^0(R,\mu)=(a_1^0(R,\mu),\dots,a_M^0(R,\mu))$ of the functional

$$J_l^R(a_1,\dots,a_M,\mu)=\|\mathcal{X}_l^R-u_l^+-\sum_{j=1}^M a_j u_j^-;L_2(\Gamma^R)\|^2, \tag{76}$$

where $\mathcal{X}_l^R$ satisfies (75).

Theorem *For all $R>R_0$ and $\mu\in[\mu_1,\mu_2]$ there exists a unique minimizer $a(R,\mu)=(a_1(R,\mu),\dots,\ a_M(R,\mu))$ of the functional $a\mapsto J_l^R(a,\mu)$ in (76). The estimates*

$$|a_j(R,\mu)-S_{lj}(\mu)|\leqslant c(\Lambda)e^{-\Lambda R},\quad j=1,\dots,M,$$

hold with constant $c(\Lambda)$ independent of R and μ and with any $\Lambda<\gamma$, while γ is the number in(74). The method for computing scattering matrices was suggested in Grikurov V., Heikkola E., Neittaanmäki, Plamenevskii B., On computation of scattering matrices and on surface waves for diffraction gratings, Numer. Math., 94(2003), no.2, 269-288. The outline of the proof given there is valid under the restriction that the interval $[\mu_1,\mu_2]$ contains no eigenvalues of problem (73). The justification of the method without such a restriction has been given for first time in the present work.

Averaging in scattering problems

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The presentation is based on the join work with V. Buslaev

We consider the scattering that is described by the Schrödinger operator

$$H_\varepsilon=-\Delta_x+q\left(x,\frac{x}{\varepsilon}\right),\quad x\in\mathbb{R}^d,$$

where Δ_x is the Laplacian with respect to the variable x and ε is a small positive parameter. As for the function $q=q(x,y)$, $x,y\in\mathbb{R}^d$ it is supposed that the following assumption is satisfied.

Assumption 1.

i) q is a real valued function of the class $C^\infty(\mathbb{R}^{2d})$;

ii) $q(x,y)=0$ for $|x|\geqslant R$, $y\in\mathbb{R}^d$, where R is some positive number;

iii) $q(x,y)$ is periodic with respect to y (with some periodicity cell Ω).

Let H^s_γ, $s=0,1,2,\ldots$, $\gamma\in\mathbb{R}$ be a Sobolev space of functions $\mathbb{R}^d\to\mathbb{C}$ with the following norm

$$\|\psi\|^2_{H^s_\gamma}=\sum_{|k|\leqslant s}\int\limits_{\mathbb{R}^d}\left|\frac{\partial^{|k|}\psi(x)}{\partial x^k}\right|^2\left(1+|x|^2\right)^{-\gamma}dx.$$

Now we formulate one of our main results.

Theorem 1. *Let the potential q satisfy the assumption 1, $E>0$ and $\gamma>1$. Then the estimate*

$$\|(H_\varepsilon-E-i0)^{-1}-(\hat{H}-E-i0)^{-1}\|_{H^0_{-\gamma}\to H^1_\gamma}\leqslant C\varepsilon.$$

holds for $\varepsilon>0$. Here $\hat{H}$ is the averaged operator

$$\hat{H}=-\Delta_x+\hat{q}(x),\quad \hat{q}(x)=\frac{1}{|\Omega|}\int\limits_\Omega q(x,y)\,dy$$

and the constant C does not depend on ε (but can depend on E and γ).

Theorem 1 almost immediately leads to a consequence that can be considered as the main result of the work. Let $F_\varepsilon(\hat{x},\kappa)$, $k\in\mathbb{R}^d$, $|\kappa|=E$ be the scattering amplitude of the plain wave $e^{i<x,\kappa>}$, that is defined by the equation

$$(H_\varepsilon-E)\psi(x)=0$$

and the asymptotic expansion at infinity

$$\psi(x)=e^{i<x,\kappa>}+F_\varepsilon(\hat{x},\kappa)\frac{e^{i\sqrt{E}\,|x|}}{|x|^{\frac{d-1}{2}}}+o\left(\frac{1}{|x|^{\frac{d-1}{2}}}\right),\quad |x|\to\infty.$$

We describe the asymptotic behavior of the amplitude $F_\varepsilon(\hat{x},\kappa)$ as $\varepsilon\to+0$. Here we formulate the consequence describing the leading order of the asymptotic expansion.

Theorem 2 *Let the potential q satisfy assumption 1 and $E>0$. Then the following estimate holds*

$$\sup_{\hat{x},\kappa}\left|F_\varepsilon(\hat{x},\kappa)-\hat{F}_0(\hat{x},\kappa)\right|\leqslant C\varepsilon,$$

where $\hat{F}_0(\hat{x},\kappa)$ is the scattering amplitude for the averaged operator $\hat{H}$ and the constant C does not depend on ε.

Absolutely continuous spectrum of multi-dimensional Schrödinger operators

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We consider two Schrodinger operators

$$H_+=-\Delta+V,\quad\text{and}\quad H_-=-\Delta-V.$$

For the sake of simplicity we assume that V is bounded. Let $\lambda_j^\pm$ be the negative eigenvalues of $H_\pm$, whose negative spectrum is assumed to be discrete. It turns out that the condition

$$\sum_j |\lambda_j^+|^{1/2} + \sum_j |\lambda_j^-|^{1/2} < \infty$$

implies that the absolutely continuous spectrum of both operators $H_\pm$ is essentially supported by $[0, \infty)$. We will give three examples which show that this result can not be obtained within the limits of the scattering theory. In partucular, we will establish that the absolutely continuous spectrum of $-\Delta + V$ covers the positive half-line for "almost every" potential V of the class L^{d+1} with $d > 2$. Our arguments based on the following result related to the representaion of the potentilal. If

$$\sum_j |\lambda_j^+|^{1/2} + \sum_j |\lambda_j^-|^{1/2} < \infty$$

and V is bounded, then

$$V(x) = \operatorname{div} A(x) + W(x)$$

where A and W satisfy the property

$$\int (|A|^2 + |W|)|x|^{1-d} dx < \infty.$$

Complete asymptotic expansion of the integrated density of states of multidimensional almost-periodic Schrödinger operator

Roman Shterenberg

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Joint work with Leonid Parnovski (University College London)

We consider the Schrödinger operator

$$H = -\Delta + b \tag{77}$$

acting in $\mathbb{R}^d$. The potential $b = b(\mathbf{x})$ is assumed to be real, smooth, and either periodic, or almost-periodic; in the almost-periodic case we assume that all the derivatives of b are almost-periodic as well. We are interested in the asymptotic behavior of the (integrated) density of states $N(\lambda)$ as the spectral parameter λ tends to infinity. The density of states of H can be defined by the formula

$$N(\lambda) = N(\lambda; H) := \lim_{L\to\infty} \frac{N(\lambda; H_D^{(L)})}{(2L)^d}.$$

Here, $H_D^{(L)}$ is the restriction of H to the cube $[-L, L]^d$ with the Dirichlet boundary conditions, and $N(\lambda; A)$ is the counting function of the discrete spectrum of A. If we denote by $N_0(\lambda)$ the density of states of the unperturbed operator $H_0 = -\Delta$, one can easily see that for positive λ one has

$$N_0(\lambda) = C_d \lambda^{d/2},$$

where

$$C_d = \frac{w_d}{(2\pi)^d} \quad \text{and} \quad w_d = \frac{\pi^{d/2}}{\Gamma(1+d/2)}$$

is a volume of the unit ball in $\mathbb{R}^d$. There is a long-standing conjecture that, at least in the case of periodic b, the density of states of H enjoys the following asymptotic behavior as $\lambda \to \infty$:

$$N(\lambda) \sim \lambda^{d/2}\Big(C_d + \sum_{j=1}^{\infty} e_j \lambda^{-j}\Big), \tag{78}$$

meaning that for each $K \in \mathbb{N}$ one has

$$N(\lambda) = \lambda^{d/2}\Big(C_d + \sum_{j=1}^{K} e_j \lambda^{-j}\Big) + R_K(\lambda)$$

with $R_K(\lambda) = o(\lambda^{\frac{d}{2}-K})$. In those formulas, e_j are real numbers which depend on the potential b. They can be calculated relatively easily using the heat kernel invariants; they are equal to a certain integrals of the potential b and its derivatives.

Until recently, formula (78) was proved only in the case $d = 1$. In the recent paper [1], we (jointly with L. Parnovski) proved this formula in the case $d = 2$ and periodic potential. Even in the periodic case and $d \geq 3$, only partial results are known.

In the paper [2] (joint with L. Parnovski) we prove (78) for a generic multidimensional almost-periodic operator. Let us describe the result. Since our potential b is almost-periodic, it has the Fourier series

$$b(\mathbf{x}) \sim \sum_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} a_{\boldsymbol{\theta}} e^{i\boldsymbol{\theta}\mathbf{x}},$$

where $\boldsymbol{\Theta}$ is a (countable) set of frequencies. Without loss of generality we assume that $\boldsymbol{\Theta}$ spans $\mathbb{R}^d$ and contains 0; we also put $\boldsymbol{\Theta}_k := \boldsymbol{\Theta} + \boldsymbol{\Theta} + \cdots + \boldsymbol{\Theta}$ (algebraic sum taken k times) and $\boldsymbol{\Theta}_\infty := \cup_k \boldsymbol{\Theta}_k = Z(\boldsymbol{\Theta})$, where for a set $S \subset \mathbb{R}^d$ by $Z(S)$ we denote the set of all finite linear combinations of elements in $\boldsymbol{\Theta}$ with integer coefficients. The set $\boldsymbol{\Theta}_\infty$ is countable and non-discrete (unless the potential b is periodic). The first condition we impose on the potential is:

Condition A. Suppose that $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_d \in \boldsymbol{\Theta}_\infty$. Then $Z(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_d)$ is discrete.

It is easy to see that this condition is generic. Condition A is obviously satisfied for periodic potentials, but it becomes meaningful for quasi-periodic potentials (we call a function quasi-periodic, if it is a linear combination of finitely many exponentials).

The rest of the conditions we have to impose describe how well we can approximate the potential b by means of a quasi-periodic function. These conditions are quite complicated and need additional preparations. Not coming into details we just mention that all these conditions are automatically satisfied for smooth periodic and quasi-periodic potentials and are generic in the class of almost-periodic potentials b.

Now we can formulate our main theorem.

Theorem *Let H be an operator* (77) *with a potential b such that*

i) *b is smooth periodic or*
ii) *b is quasi-periodic satisfying Condition* A *or*
iii) *b is almost-periodic satisfying some generic conditions (including Condition* A*).*
Then integrated density of states $N(\lambda)$ *enjoys asymptotic behavior* (78).

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Localization Properties of the Random Displacement Model

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Joint work with J. Baker, F. Klopp, M. Loss and S. Nakamura

We consider the random displacement model, a random Schrödinger operator given by $H_\omega = -\Delta + V_\omega(x)$ in $L^2(\mathbb{R}^d)$, where

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^d} q(x - i - \omega_i).$$

The *single site potential* q is real-valued, bounded, supported in $[-r, r]^d$ for some $r \in (0, 1/2)$ and reflection symmetric in each variable. The displacements $\omega = (\omega_i)_{i \in \mathbb{Z}^d}$ are i.i.d. random vectors in $\mathbb{R}^d$ with distribution μ supported in the cube $[-d_{max}, d_{max}]^d$, where $d_{max} = \frac{1}{2} - r$, giving non-overlapping sites in V_ω.

Several years ago, Baker, Loss and Stolz have identified a spectrally minimizing periodic configuration ω^*, i.e. a configuration with the property that $\inf \sigma(H_{\omega^*}) = \inf \Sigma =: E_0$, where Σ is the almost sure spectrum of H_ω. This configuration is characterized as

$$\omega_i^* = ((-1)^{i_1} d_{max}, \ldots, (-1)^{i_d} d_{max}), \quad \text{for all } i = (i_1, \ldots, i_d) \in \mathbb{Z}^d,$$

meaning that clusters of 2^d neighboring single-site potentials are located in adjacent corners of their supporting unit cell.

One may find examples of single-site potentials q such that $\inf \sigma(H_\omega) = \inf \Sigma$ for *all* configurations ω, thus providing situations where $\inf \Sigma$ is not a fluctuation boundary of the spectrum. In all other cases, for example if q is nontrivial and sign-definite, and under the additional assumption $r < 1/4$, it was shown that ω^* is the unique minimizing periodic configuration if $d \geq 2$. In dimension $d = 1$ there are many other minimizing periodic configurations, which are characterized by the requirement that equally many ω_i take values d_{max} and $-d_{max}$, respectively, and none of them lies in $(-d_{max}, d_{max})$.

This difference between the one and multi-dimensional cases also leads to different low energy asymptotics of the integrated density of states $N(E)$ of H_ω. An extreme case is given by the one-dimensional *Bernoulli displacement model*, where the support of μ is $\{\pm d_{max}\}$ with $\mathbb{P}(\omega_i = d_{max}) = \mathbb{P}(\omega_i = -d_{max}) = 1/2$. In this case it can be shown that

$$N(E) \geq \frac{C}{\ln^2(E - E_0)}$$

for some $C > 0$ and E near E_0. In particular, the IDS is not Hölder-continuous at E_0.

On the other hand, it was recently shown by Klopp and Nakamura that the uniqueness result for the minimizer ω^* in $d \geq 2$ implies a weak form of *Lifshits tails* for the IDS. More precisely, if $\operatorname{supp}\mu$ is finite and all 2^d corners $(\pm d_{max}, \ldots, \pm d_{max})$ are contained in $\operatorname{supp}\mu$, then

$$\limsup_{E \downarrow E_0} \frac{\log|\log N(E)|}{\log(E - E_0)} \leq -\frac{1}{2}.$$

Work in preparation, jointly with Klopp, Loss and Nakamura, will lead to further results under the assumption that the single-site potential as well as the distribution μ are sufficiently regular. It will be shown that the above Lifshitz tail bound extends to this setting (i.e. without the assumption of finiteness of $\operatorname{supp}\mu$). These authors will also establish a *Wegner estimate* of the form

$$\mathbb{E}(\operatorname{tr}\chi_I(H_{\omega,L})) \leq C_\alpha |I|^\alpha L^d.$$

Here $H_{\omega,L}$ is the Neumann-restriction of H_ω to a cube of sidelength L, α any number in $(0, 1)$ and I any subinterval of $[E_0, E_0 + \delta]$ for a sufficiently small $\delta > 0$.

Based on these ingredients, it follows by multiscale analysis that H_ω is spectrally and dynamically localized for energies near E_0. This is the first proof of localization for the multi-dimensional random displacement model, which does not require to work in a semi-classical regime, as considered previously by Klopp, or the introduction of a generic periodic background term into the potential and smallness of the displacement parameters as in work by Ghribi and Klopp.

Negative spectrum of a perturbed Anderson Hamiltonian

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Joint work with Stanislav Molchanov

The Anderson Hamiltonian $H_0 = -\Delta + V(x, \omega)$ is considered, where V is a random potential of Bernoulli type. The operator H_0 is perturbed by a non-random, continuous potential $-w(x) \leq 0$, decaying at infinity. It will be shown that the borderline between finitely, and infinitely many negative eigenvalues of the perturbed operator, is achieved with a decay of the potential $-w(x)$ at infinity as $O(\ln^{-2/d}|x|)$.

The random potential we consider has the following structure: consider the partition of R^d into unit cubes

$$Q_n = \{x: \ ||x-n||_\infty \le \frac{1}{2}\}, \quad n = (n_1, ...n_d) \in Z^d,$$

and put

$$V(x,\omega) = \sum_{n \in Z^d} \varepsilon_n I_{Q_n}(x). \tag{79}$$

Here ε_n are i.i.d. Bernoulli r.v., namely

$$P\{\varepsilon_n = 1\} = p > 0, \quad P\{\varepsilon_n = 0\} = q = 1-p > 0 \tag{80}$$

on the probability space (Ω, F, P).

We call a domain $D \in R^d$ a clearing if $V = 0$ when $x \in D$. Since P-a.s. realizations of the potential V contain cubic clearings of arbitrary size $l \gg 1$, we have $Sp(H_0) = [0, \infty)$.

Consider a perturbation of H_0 by a non-random continuous potential:

$$H = -\Delta + hV(x,\omega) - w(x), \quad w(x) \ge 0, \quad w \to 0 \text{ as } |x| \to \infty. \tag{81}$$

The operator H is bounded from below, and its negative spectrum $\{\lambda_i\}$ is discrete. Put $N_0(w,\omega) = \#\{\lambda_i \le 0\}$. The following theorem presents the main result of the paper.

Theorem 1 *There are two constants $c_1 < c_2$ which depend only on d and independent of h and p, such that*

a) the condition

$$w(x) \le \frac{c_1}{\ln^{\frac{2}{d}}(2+|x|)\ln 1/q}, \quad |x| \to \infty,$$

implies $N_0(w,\omega) < \infty$ P-a.s.,

b) the condition

$$w(x) \ge \frac{c_2}{\ln^{\frac{2}{d}}(2+|x|)\ln 1/q}, \quad |x| \to \infty,$$

implies $N_0(w,\omega) = \infty$ P-a.s..

Remarks 1. Similar result is valid for the lattice Anderson model with the Bernoulli potential.

2. Together with J. Holt we proved more general results in 1-D case.

The proof is based on percolation theory and Dirichlet-Neumann bracketing. The percolation theory allows us to describe sets in R^d where $V = 1$. Let us present one of the key statements which follows from the percolation theory. We will tell that a set of cubes $\{Q_n\}$ is $\sqrt{d}$-connected if any two cubes in the set can be connected by a sequence of cubes where the neighbors have at least one common point (a vertex or an edge of the dimension $k \le d-1$, i.e., the distance between their centers does not exceed $\sqrt{d}$). Assume that $V = 0$ on the cube Q_0 which contains the origin. Let C be the maximal $\sqrt{d}$-connected set of cubes where $V = 0$ which contains Q_0, and let $|C|$ be the number of cubes in the set C.

Lemma 1 *(exponential tails). If $q < \frac{1}{3^d-2}$ then there exists a constant $c_0 = c_0(d,q)$ such that*

$$P\{|C| \ge s\} \le c_0 e^{-\gamma s}, \quad \gamma = \ln \frac{1}{q(3^d-2)} > 0. \tag{82}$$

This and similar statements allow us to reduce the problem to a study of the eigenvalues of the Schrödinger operator in bounded domains with a potential supported near the boundary.

Trapped modes in elastic media for zero Poisson ratio

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Joint work with Clemens Förster

We consider the operator

$$A = -\Delta \otimes I - divergence \quad \text{on} \quad L^2(G, \mathbb{C}^d)$$

associated with the quadratic form

$$a[u,v] = 2 \int_{G_d} \langle \epsilon(u), \epsilon(v) \rangle_{\mathbb{C}^{d\times d}} \, dx, \quad u, v \in H^1(G_d; \mathbb{C}^d)\,,$$

where

$$\epsilon(u) = \frac{1}{2} \left((\nabla u) + (\nabla u)^T \right) \,.$$

This corresponds to the elastic Hamiltonian on $G \subset \mathbb{R}^d$ with stress-free (Neumann type) boundary conditions at ∂G in the case of a Young's modulus $E = 2$ and a Poisson ratio $\nu = 0$.

Put $J = (-\pi/2, \pi/2)$. In [5] it has been shown, that in case of the semi-strip $G = \mathbb{R}_+ \times J$ the operator A has at least one embedded eigenvalue on top of the continuous spectrum. This illustrates the well-known physical effect of the elastic edge resonance, a localized oscillation near the free face of G.

In this talk we discuss the appearance of trapped modes on strip-like or plate-like domains $G \subset \mathbb{R}^d$ in the dimensions $d = 2$ and $d = 3$.

In particular, let $\Omega \subset \mathbb{R}^2$ be a non-empty bounded Lipschitz domain and put $G = (\mathbb{R}^2 \setminus \Omega) \times J$. Then A has infinitely many embedded eigenvalues ν_k accumulation at a certain positive spectral threshold Λ at the exponential rate

$$\ln(\Lambda - \nu_k) = -2k \ln k + o(k \ln k) \quad \text{as} \quad k \to \infty\,.$$

Similary, a suitable local perturbation of Young's modulus on $G = \mathbb{R}^2 \times J$ yields an infinite set of trapped modes accumulating at the same energy level Λ.

We give a review of the related results in [2,1] and state several open problems related to observations by Zernov et al [6] and Pagneux [4].

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Abstracts of the young scientists talks

The dynamical inverse problem for the Maxwell system

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We deal with the dynamical inverse problem for the Maxwell system in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary. The system is of the form

$$\begin{array}{ll} e_t = \varepsilon^{-1}\operatorname{curl} h, \quad h_t = -\mu^{-1}\operatorname{curl} e & \text{in } \Omega \times (0,T) \\ e|_{t=0} = 0, \quad h|_{t=0} = 0 & \text{in } \Omega \\ e_\theta = f & \text{in } \partial\Omega \times [0,T] \end{array}$$

where $(\,\cdot\,)_\theta$ is a tangent component of a vector at the boundary, $e = e^f(x,t)$ and $h = h^f(x,t)$ are the electric and magnetic components of the solution. With the system one associates the response operator $R^T : f \mapsto -\nu \wedge h^f|_{\partial\Omega\times(0,T)}$, where ν is an outward normal to $\partial\Omega$.

The *time-optimal* setup of the inverse problem is: given $\{R^{2T}, c\,|_{\partial\Omega}, \frac{\partial c}{\partial\nu}\,|_{\partial\Omega}\}$ to recover the electromagnetic wave velocity $c = (\varepsilon\mu)^{-1/2}$ in the subdomain $\Omega^T := \{x \in \Omega \,|\, \operatorname{dist}_c(x, \partial\Omega) < T\}$ (dist_c is a distance in the optical metric).

In [1], by the use of the boundary control method, the uniqueness of determination of $c\,|_{\Omega^T}$ was established for small enough T. Here we show that the uniqueness holds for *arbitrary* $T > 0$.

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Positive polynomials and mapping of pseudospectra

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Joint work with Nikolay Filonov

Let H be a Hilbert space and let $A \in B(H)$ be a bounded operator such that $\|A\| = 1$ and $\|[A, A^*]\| = \delta$. Equivalently, instead of $B(H)$ an arbitrary C^*-algebra can be considered. We study the following "polynomial calculus": for a polynomial p of the form

$$p(z, \overline{z}) = \sum_{0 \leq k+l \leqslant d} p_{kl} z^k \overline{z}^l, \tag{83}$$

let

$$p(A, A^*) = \sum_{0\le k+l\leqslant d} p_{kl} A^k (A^*)^l. \tag{84}$$

This calculus is additive and involutive (with respect to p), and it is also clear that it is "almost multiplicative" up to terms of order δ.

We obtain the following two results.

Theorem 1 *Let p be a polynomial* (83). *Then there exist such δ_0 and $C(p)$, that*

$$\|p(A, A^*)\| \leqslant p_{\max} + C(p)\delta, \quad \delta < \delta_0 \tag{85}$$

for all $A \in B(H)$, satisfying $\|A\| \leqslant 1$, $\|[A, A^*]\| = \delta$. Here $p_{\max} = \max_{|z|\leqslant 1} |p(z, \overline{z})|$.

If A is normal, then this estimate obviously follows from the spectral theorem. In the case of a holomorphic polynomial $p(z)$, the inequality holds with $C(p) = 0$, and is known as von Neumann inequality.

Theorem 2 *Let $R_j > 0$, $j = 1, \dots, m$. Consider*

$$S = \{z \in \mathbb{C} \colon |z| \leqslant 1, |z - \lambda_j| \geqslant R_j, j = 1, \dots, m\}.$$

Let p be a polynomial (84) *and $\mu \notin S$. Let, finally,*

$$\chi_\mu = \mathrm{dist}(\mu, \mathrm{p(S)}).$$

Then for any $\varepsilon > 0$ there exist $\delta_0 > 0$ and $C(\varepsilon, \mu, p)$, such that

$$\|(p(A, A^*) - \mu)^{-1}\| \leqslant \chi_\mu^{-1} + \varepsilon + C(\varepsilon, \mu, p)\delta, \quad \delta < \delta_0$$

for all $A \in B(H)$, satisfying

$$\|A\| = 1, \quad \|[A, A^*]\| = \delta, \quad \|(A - \lambda_j)^{-1}\| \leqslant R_j^{-1}.$$

Recall that *ε-pseudospectrum* of an operator A is the set

$$\sigma_\varepsilon(A) = \{\lambda \in \mathbb{C} \colon \|(A - \lambda I)^{-1}\| > 1/\varepsilon\}.$$

Let $R_j = \varepsilon$ for all j. If the conditions of Theorem 2 are fulfilled then $\sigma_\varepsilon(A) \subset S$. So, in this case $\sigma_{\chi'}(p(A, A^*)) \subset B_\chi(p(S))$, where B_χ is the χ-neighbourhood, and $(\chi')^{-1} = \chi^{-1} + \varepsilon + C(\varepsilon, p)\delta$.

On spectral perturbations of bounded Jacobi operators

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Let $\mathcal{S}$ be the class of all bounded semi-infinite Jacobi operators such that: (a) for every $A \in \mathcal{S}$ the essential spectrum $\sigma_{ess}(A) = E$, where $E \subset \mathbb{R}$ is a finite gap set; (b) the weight function of A satisfies the Szegö condition on E; (c) the pure point part of the spectrum of A consists of a (not more than) countable number of mass-points $\{z_k^*\}_{k=1}^N$, $z_k^* \in \mathbb{R} \setminus E$, $N \in \mathbb{N} \cup \infty$, satisfying a Blaschke-type condition.

Let $A \in \mathcal{S}$ be a perturbation of $A_0 \in \mathcal{S}$ produced by a change of its spectral measure. We are interested in the following question: what are the conditions on the spectral measures for the operator $A - A_0$ to be compact? The answer is not obvious even in the case of adding one mass-point. To investigate this question we use the well-known connection between the Jacobi operators and orthogonal polynomials. The asymptotical behavior of the orthogonal polynomials corresponding to the operators from the class $\mathcal{S}$ can be discribed using two main approaches: the first one is due to Widom, and the second one is due to Sodin, Peherstorfer and Yuditskii. Using these approaches we get a necessary and sufficient condition of the compactness of the operator $A - A_0$.

To state our main result we need some notaion. Let $\Omega := \bar{\mathbb{C}} \setminus E$. For an operator $A \in \mathcal{S}$ with a weight function $\rho(\zeta)$ we introduce the complex function $R_\rho(z)$ which is locally analytic in Ω, has nontangential limit values on $\partial\Omega$, and such that $|R_\rho(\zeta)|_{\zeta \in \partial\Omega} = \rho(\zeta)$. By $\omega_\nu(z)$, $\nu = 1, ..., p$ (where p is the number of intervals in E) we denote the harmonic measure, i.e. the function harmonic in Ω and satisfying the boundary condition $\omega_k(\zeta)|_{\zeta \in E_j} = \delta(j,k)$. Let $\Delta_{E_k} f$ denote the increment of the function f along a curve encircling the interval E_k. To each operator $A \in \mathcal{S}$ we associate the following vector-valued characteristic $\mathcal{J}(A) = (\mathcal{J}_1(A), \mathcal{J}_2(A), ..., \mathcal{J}_p(A))$

$$\mathcal{J}_\nu(A) = \frac{1}{4\pi}\Delta_{E_\nu} \arg R_\rho(z) + \sum_{j=1}^{N} \omega_\nu(z_j^*), \quad \nu = 1, ..., p.$$

Theorem *For the operators A, A_0 from the class $\mathcal{S}$ the operator $A - A_0$ is compact if and only if*

$$\mathcal{J}_\nu(A) - \mathcal{J}_\nu(A_0) \in \mathbb{Z}, \quad \nu = 1, 2, \ldots, p.$$

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Lyapunov exponent and integrated density of states for the slowly oscillating perturbations of the periodic Schrödinger operators

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In this talk I present the formulas for the Lyapunov exponent and the integrated density of states for the slowly oscillating perturbations of the periodic Schrödinger operators in dimension 1. Operators under consideration are $H_\theta = -\frac{d^2}{dx^2} + V(x) + W(x^\alpha)$ on $D(H_\theta) = \{f \in H^2(\mathbb{R}_+) \mid f(0)\cos\theta + f'(0)\sin\theta\}$ associated to $\theta \in [0, \pi)$. The assumptions on V, W and α are:

- $V : \mathbb{R} \to \mathbb{R}$, $V \in L_{2,loc}(\mathbb{R})$, $V(x+1) = V(x)$.
- $W : \mathbb{R} \to \mathbb{R}$ smooth, $W(x + 2\pi) = W(x)$.
- $\alpha \in (0, 1)$

The main spectral results for this type of operators go back to G. Stolz , who proved that the spectrum of this operators has the intervals of purely absolutely continuous spectrum and those of purely singular. This singular spectrum is dense pure point for almost every θ.
In my work I continue the study of H_θ. I prove the existence of the Lyapunov exponent and the integrated density of states for this problem. I show the formula relating the integrated density of states and the Lyaponov exponent to those of the periodic problem.
To define the integrated density of states lets first define the counting function of the eigenvalues E_k of the restrictions $H_\theta(l)$ of H_θ to the finite intervals $[0, l]$ ($l \in \mathbb{N}$) with Dirichlet boundary condition in l:

$$N(E, \theta, l) = \#\{E_k(H_\theta(l)) \mid E_k(H_\theta(l)) < E\}$$

Lets call the integrated density of states the following limit when it exists:

$$k(E) = \lim_{l \to +\infty} \frac{N(E, \theta, l)}{l}$$

Remark: When exists the integrated density of states does not depend on θ.
Consider the fundamental solution $T(x, E)$ of the corresponding Schrödinger equation $(H_\theta \Psi)(x, E) = E\Psi(x, E)$, $T : \mathbb{R} \times \mathbb{C} \to SL_2(\mathbb{C})$ satisfying:

$$T(0, E) = I \qquad \frac{d}{dx} T(x, E) = \begin{pmatrix} 0 & 1 \\ V(x) + W(x^\alpha) - E & 0 \end{pmatrix} T(x, E)$$

Lets call the Lyapounov exponent the following limits when it exists:

$$\gamma(E) = \lim_{x \to +\infty} \frac{\ln \|T(x, E)\|}{x}$$

Theorem *For all $E \in \mathbb{R}$ the integrated density of states exists and is given by the formula:*

$$k(E) = \frac{1}{2\pi} \int_0^{2\pi} k_0(E - W(x))dx$$

where $k_0(E) = \frac{1}{\pi}\Re k_p(E+i0)$ *with* k_p *denoting the main branch of the Bloch quasi-momentum. For almost all* $E \in \mathbb{R}$ *the Lyapunov exponent exists and is given by the formula:*

$$\gamma(E) = \frac{1}{2\pi}\int_0^{2\pi} \gamma_0(E - W(x))dx$$

where $\gamma_0(E) = \Im k_p(E+i0)$*.*

I use two different methods in the study of this problem. One is the Dirichlet to Neumann bracketing method useful to derive the integrated density of states formula. Then the Lyapunov exponent formula is obtained by proving the Thouless formula. This method works for all $\alpha \in (0,1)$ and follows the ideas of B. Simon and Y. Zhu .
The other method is based on quasi-periodic (periodic) approximations of H_θ. This method allows to study the asymptotic of the fundamental solution (so the Lyapunov exponent) for $\alpha > \frac{1}{2}$. The information about the approximating equation is obtained by the methods of A. Fedotov and F. Klopp (complex WKB method for the adiabatic perturbations of periodic Schrödinger operators).

Spectral inequalities for a class of non-elliptic operators

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Joint work with A.Laptev

We prove spectral inequalities for the moments of the eigenvalues of the operator $D_x^2 D_y^2 - V(x,y)$, $V \geq 0$, with Dirichlet boundary conditions in $L^2(\mathbb{R}^2_{++})$.

Zeros of the spectral density of the discrete Schrödinger operator with Wigner-von Neumann potential

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Joint work with J. Janas and S. Naboko

We consider discrete Schrödinger operator

$$\mathcal{J} := \begin{pmatrix} b_1 & 1 & 0 & \cdots \\ 1 & b_2 & 1 & \cdots \\ 0 & 1 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with

$$b_n := \frac{c\sin(2\omega n + \delta)}{n} + q_n,$$

where $\{q_n\}_{n=1}^{\infty} \in l^1$. This operator has absolutely continuous spectrum on the interval $[-2; 2]$. At two points inside this interval (critical, or resonance points),

$$\pm 2\cos\omega,$$

the operator $\mathcal{J}$ can have eigenvalues. The talk is devoted to the behavior of the spectral density (the derivative of the spectral measure) of the operator $\mathcal{J}$ near the resonance points and is based on works [1], [2], [4]. In the general situation there exist two one-side limits

$$\lim_{\lambda \to \nu_{cr} \pm 0} \frac{\rho'(\lambda)}{|\lambda - \nu_{cr}|^{\frac{|c|}{2|\sin\omega|}}},$$

(where ρ' is the spectral density of $\mathcal{J}$ and ν_{cr} is one two points $\pm 2\cos\omega$). This result is a part of a wider study of operators with Wigner-von Neumann potentials including one-dimentional Schrödinger operators with periodic background potential [3], [4].

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Matrix Schrödinger operator on the half-line: the differential equation with respect to the spectral parameter and an analog of Freud's equations

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In many physical problems it is necessary to calculate semiclassical asymptotics for solutions of the Schrödinger equation

$$-\hbar^2 \frac{d^2}{dx^2}\psi(x) + v(x)\psi(x) = k^2\psi(x) \tag{86}$$

with respect to a small parameter $\hbar$. Using the WKB method one can see an asymptotical behavior of solutions as functions of variable x.

Buslaev was the first who proposed to write down the differential equation with respect to k for solution $\varphi(x,k)$ of the *scalar Schrödinger equation* on the half-line $x \geqslant 0$ satisfying the Dirichlet boundary condition $\varphi(0,k)=0$. Using this equation one can investigate an asymptotical behavior of $\varphi(x,k)$ as function of variable k. We can also write down a nonlinear relation connecting the Schrödinger operator and the kernel of its spectral measure. An analogue of this relation in the theory of orthogonal polynomials is known as **discrete string equation** or **Freud's equations**.

Here we generalize this result for the *matrix Schrödinger equation* on the half-line:

$$-\Psi''(x,k)+V(x)\Psi(x,k)=k^2\Psi(x,k)\,, \qquad x \geqslant 0\,, \tag{87}$$

where $V(x)=\{v_{\alpha\beta}(x)\}_{\alpha,\beta=1}^{N}$ is a Hermitian matrix such that

$$\int\limits_0^\infty |V(x)|\,(1+x^2)\cdot\, dx \;<\; \infty\,, \quad |V| \equiv \max_\alpha \sum_{\beta=1}^{N} |v_{\alpha\beta}|\,. \tag{88}$$

Let us denote by L_j the matrix Schrödinger operators on the half-line with Dirichlet ($j=1$) and Neumann ($j=2$) boundary conditions. We assume that eigenvalues and virtual level in zero for the operators L_j are absent. Then the kernel $W_j(x,y)$ of the operator $W(L_j)$ can be written as:

$$W_j(x,y)=\int\limits_0^\infty \Phi_j(x,l)W(\mu)\cdot\sigma_j(l)\Phi_j^*(y,l)d\mu\,, \quad \mu=l^2\,.$$

where $\Phi_j(x,l)\sigma_j(l)\Phi_j^*(y,l)d\mu$ is the kernel of the spectral measure for the operator L_j, $\Phi_j(x,k)$ are solutions of equation (87) satisfying initial conditions:

$$\Phi_1(0,k)=\mathbf{0}, \quad \Phi_1'(0,k)=\mathbf{1}\,, \quad \Phi_2(0,k)=\mathbf{1}, \quad \Phi_2'(0,k)=\mathbf{0}\,.$$

We obtain the following differential equation for the matrix-function $\vec{\Phi}_j(x,\lambda)=\begin{pmatrix}\Phi_j(x,k)\\ \Phi_j'(x,k)\end{pmatrix}$:

$$\frac{\partial}{\partial\lambda}\vec{\Phi}_j(x,\lambda)=\mathbb{U}_j(x,\lambda)\vec{\Phi}_j(x,\lambda)\,, \tag{89}$$

where

$$\mathbb{U}_j(x,\lambda)=\begin{pmatrix} -(M_j)_y(x,y,\lambda)|_{y=x} & M_j(x,x,\lambda) \\ W_j(x,x)-(M_j)_{xy}(x,y,\lambda)|_{y=x} & (M_j)_x(x,y,\lambda)|_{y=x}\end{pmatrix},$$

$M_j(x,y,\lambda)$ is the kernel of the operator $M_j=W_j(L_j)\cdot(L_j-\lambda-i0)^{-1}$, $W_j(\mu)=-\dfrac{\partial}{\partial\mu}\Big(\ln\sigma_j(l)\Big)$.

We can write down the Schrödinger equation (87) in the similar form:

$$\frac{\partial}{\partial x}\,\vec{\Phi}_j(x,\lambda)=\begin{pmatrix}\mathbf{0} & \mathbf{1}\\ V(x)-\lambda\cdot\mathbf{1} & \mathbf{0}\end{pmatrix}\cdot\vec{\Phi}_j(x,\lambda)\,. \tag{90}$$

The compatibility condition of two differential equations (89) and (90) leads to the following relation (**an analogue of Freud's equations**):

$$-2\frac{d}{dx}\,W_j(x,x)=\mathbf{1}\,.$$

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