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SPECTRAL PROBLEMS IN LIPSCHITZ DOMAINS

1. Introduction. We consider a bounded domain Ω in $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary Γ: locally, in appropriate coordinates, it is a graph of a function $x_n = \phi(x')$ satisfying the Lipschitz condition $|\phi(x') - \phi(y')| \leq C|x' - y'|$. The tangent plane exists at almost every point of Γ.

Examples: polyhedrons (not all), cones, cylinders, and their images under Lipschitz diffeomorphisms.

In Ω, we consider a second-order system $Lu = f$ in the divergent form. $Lu :=$

$$-\sum \partial_j a_{j,k}(x)\partial_k u(x) + \sum b_j(x)\partial_j u(x) + c(x)u(x).$$

Here $a_{j,k}$, b_j, c_j are $m \times m$ matrices, u and f are columns of hight m. All functions are complex-valued. The smoothness of coefficients is minimized. In particular, $a_{j,k} \in C^1(\overline{\Omega})$.

The main assumption is *the strong ellipticity*: the principal symbol $a_0(x,\xi) = \sum a_{j,k}(x)\xi_j\xi_k$ has a positively definite real part:

$$\tfrac{1}{2}[a_0(x,\xi) + a_0^*(x,\xi)] \geq C_0|\xi|^2 I, \quad \xi \in \mathbb{R}^n.$$

In the main cases, $a_{j,k} = a'_{k,j}$ and are real, $\Rightarrow a_0(x,\xi)$ is real symmetric.

Examples.

1. The Laplace or Beltrami-Laplace equation $\Delta u + \ldots = f$ with $m \geq 1$. Nonsmooth problems in acoustics and electrodynamics.

2. The Lamé system in isotropic elasticity

$$-\mu\Delta u - (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\, u + \ \ldots\ = f.$$

It is strongly elliptic if $\mu > 0$, $\lambda + 2\mu > 0$.

3. Systems of elasticity for non-homogeneous anisotropic bodies: with real $a_{j,k}(x) = (a_{j,k}^{r,s}(x))$,

$$a_{j,k}^{r,s} = a_{k,j}^{r,s} = a_{r,s}^{j,k} = a_{r,k}^{j,s}.$$

In 2 and 3, $m = n = 2$ or 3.

4. Multidimensional analogs.

2. Spaces H^s. For all $s \in \mathbb{R}$,

$H^s(\mathbb{R}^n) = F^{-1}(1+|\xi|^2)^{-s/2} F L_2(\mathbb{R}^n)$.

Sobolev–Slobodetskii sp. $W_2^s(\mathbb{R}^n)$ if $s > 0$.

$H^s(\Omega)$: restrictions to Ω of elements of $H^s(\mathbb{R}^n)$ with norm inf. There exists an extension operator $\mathcal{E}$ bounded *for all* s (Rychkov, 1999).

$\widetilde{H}^s(\Omega)$: the subspace in $H^s(\mathbb{R}^n)$ of elements supported in $\overline{\Omega}$. Can be identified with the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ if $s > 0$ and $s + 1/2 \notin \mathbb{N}$.

$H^s(\Omega)$ and $\widetilde{H}^{-s}(\Omega)$ are *mutually adjoint* with respect to the extension of the form $(\mathcal{E}u, v)_{\mathbb{R}^n}$.

$H^s(\Gamma)$ is defined by a partition of unity and norms in $H^s(\mathbb{R}^{n-1})$, $|s| \le 1$. The spaces $H^s(\Gamma)$ and $H^{-s}(\Gamma)$ are mutually adjoint with respect to the extension of the inner product in $L_2(\Gamma)$.

The trace operator $u \to u^+ = \gamma^+ u$ acts from $H^{s+1/2}(\Omega)$ to $H^s(\Gamma)$ and is *bounded for* $0 < s < 1$. It has a bounded right inverse.

In Ω and on Γ, we have compact dense embeddings $H^s \subset H^t$ for $s > t$. $H^s(\Omega)$ and $\widetilde{H}^s(\Omega)$ can be identified for $-1/2 < s < 1/2$, and we can pass from one scale to another.

3. Dirichlet and Neumann problems.

The Neumann problem: $Lu = f$ in Ω, $T^+u = h$. Here T^+u is *the conormal derivative.* If $u(x)$ is smooth, then

$$T^+u(x) = \gamma^+ \sum \nu_j(x) a_{j,k}(x) \partial_k u(x)$$

on Γ, where ν is the unit exterior normal.

In our situation, we need *the variational definitions.* Introduce the form $\Phi_\Omega(u,v) =$

$$\int_\Omega \left[\sum a_{j,k} \partial_k u \cdot \partial_j \overline{v} + \sum b_j \partial_j u \cdot \overline{v} + cu \cdot \overline{v} \right] dx.$$

Here $u, v \in H^1(\Omega)$. The following *first Green formula* can be proved for u with additional smoothness: if $u \in H^s(\Omega)$, $s > 3/2$, or if $Lu \in L_2(\Omega)$:

$$(Lu, v)_\Omega = \Phi_\Omega(u,v) - (T^+u, v^+)_\Gamma.$$

If $u \in H^1(\Omega)$, then Lu is defined by u only as a distribution in Ω, and to define it as an element of $\widetilde{H}^{-1}(\Omega)$ we have to choose and add an element of $H^{-1}(\mathbb{R}^n)$ supported on Γ. Because of this we simply write the formula with given $f \in \widetilde{H}^{-1}(\Omega)$ and *define* T^+u by it. In general, it is only a distribution from $H^{-1/2}(\Gamma)$.

The same formula is *the variational, or weak, definition* of the Neumann problem. In it, $u, v \in H^1(\Omega)$, $f = Lu \in \widetilde{H}^{-1}(\Omega)$, $h = T^+u \in H^{-1/2}(\Gamma)$. Here h can be replaced by 0 by changing f.

The Dirichlet problem: $Lu = f$ in Ω, $u^+ = g$. If $g = 0$, the variational definition is

$$(Lu, v)_\Omega = \Phi_\Omega(u, v)$$

with $u, v \in \widetilde{H}^1(\Omega)$, $f \in H^{-1}(\Omega)$. The case of $g \neq 0$, $g \in H^{1/2}(\Gamma)$ is easily reduced to the case $g = 0$.

Classical results for the Dirichlet problem:

Strong ellipticity $\Rightarrow$ *the Gårding inequality*

$$\Phi_\Omega(u, u) \geq C_1 \|u\|^2_{\widetilde{H}^1(\Omega)} - C_2 \|u\|^2_{L_2(\Omega)}.$$

It is true with $C_2 = 0$ if $\operatorname{Re} c(x)$ is sufficiently large. *We always assume this.*

$\Rightarrow$ *The unique solvability.*

For the Neumann problem, the Gårding inequality

$$\Phi_\Omega(u, u) \geq C_3 \|u\|^2_{H^1(\Omega)} - C_4 \|u\|^2_{L_2(\Omega)}$$

follows from the strong ellipticity for a *scalar* equation with $a_{j,k} = a_{k,j}$. *Sufficient conditions* for systems are known. The following condition can be applied to all systems of elasticity:

$$\sum a_{j,k}^{r,s} \zeta_k^s \overline{\zeta_j^r} \geq C_5 \sum |\zeta_k^s + \zeta_s^k|^2.$$

Again, the Gårding inequality is then true with $C_4 = 0$ if $\operatorname{Re} c(x)$ is sufficiently large. *We assume that we have such inequality.*

$\Rightarrow$ *The unique solvability.*

All *a priory* estimates are *two-sided*, as in smooth problems. E.g., for the Neumann problem

$\|u\|_{H^1(\Omega)} \leq C_6[\|f\|_{\widetilde{H}^{-1}(\Omega)} + \|h\|_{H^{-1/2}(\Gamma)}] \leq C_7\|u\|_{H^1(\Omega)}$.

Thus, the variational setting of our problems is natural and convenient.

4. Spectral Dirichlet and Neumann problems. Now we consider two problems

$$Lu = \lambda u \text{ in } \Omega, \quad u^+ = 0 \text{ or } T^+u = 0 \text{ on } \Gamma.$$

In the Dirichlet problem, $u \in \widetilde{H}^1(\Omega)$. In the Neumann problem, $u \in H^1(\Omega)$.

Note that the operator formally adjoint to L is

$\widetilde{L}v = -\sum \partial_k a^*_{k,j}(x)\partial_j v(x) - \sum b^*_j(x)\partial_j v(x) + \left[c^*(x) - \sum \partial_j b^*_j(x)\right]v(x).$

The corresponding first Green formula, also postulated, is

$$(u, \widetilde{L}v)_\Omega = \Phi_\Omega(u, v) - (u, \widetilde{T}^+ v)_\Gamma$$

with the same form Φ_Ω. Here $u, v \in H^1(\Omega)$. From two first Green formulas *the second Green formula* follows:

$$(Lu, v)_\Omega - (u, \widetilde{L}v)_\Omega = (u^+, \widetilde{T}^+ v)_\Gamma - (T^+ u, v^+).$$

Case 1. *L is formally selfadjoint*: $L = \widetilde{L}$. Sufficient conditions:

$$a^*_{j,k} = a_{k,j}, \quad b_j = 0, \quad c^* = c.$$

The second Green formula shows that then

$$(Lu, v)_\Omega = (u, Lv)_\Omega$$

both under Dirichlet and Neumann homogeneous boundary conditions, hence we can consider the corresponding selfadjoint operators L_D and L_N.

Usually, $L_2(\Omega)$ is considered as basic Hilbert space, and the operators are considered as acting in it. They have discrete spectra. If the boundary and coefficients are sufficiently smooth, the domains lye in $H^2(\Omega)$. However, in our situation we do not know the domains exactly.

More convenient is to consider, say, L_D as a bounded invertible operator from $\widetilde{H}^1(\Omega)$ to $H^{-1}(\Omega)$ and *take* $H^{-1}(\Omega)$ *as the basic Hilbert space* with *the inner product*

$$\langle u, v \rangle_{H^{-1}(\Omega)} = (L_D^{-1} u, v)_\Omega.$$

The corresponding norm is equivalent to the original one. *The operator* L_D *remains to be selfadjoint and has the same eigenfunctions and eigenvalues.*

In $H^{-1}(\Omega)$ it has an *orthonormal basis* of eigenfunctions. They *belong to* $\widetilde{H}^1(\Omega)$ and form there a basis orthogonal with respect to the inner product $(L_D u, v)_\Omega$. The result is extended to the intermediate spaces.

The spectral asymptotics was investigated using the variational method by Birman-Solomyak and by Métivier (1974) who proved for the counting function $N(\lambda)$ the formula

$$N(\lambda) = \mu\lambda^{n/2} + O(\lambda^{(n-1/2)/2})$$

with the usual coefficient: in the scalar case,

$$\mu = \tfrac{1}{(2\pi)^n} \int_\Omega \int_{a_0(x,\xi)<1} d\xi.$$

Case 2. Only *the principal part* L_0 *of* L *is formally selfadjoint*. Then L_D is a weak perturbation of a selfadjoint operator in $H^{-1}(\Omega)$. The spectrum is discrete and lies in the angle

$$\Theta_\theta = \{\lambda : \ |\arg\lambda| < \theta\}$$

with any small θ, except finite number of eigenvalues, and they preserve the same asymptotics.

The root functions are *complete* in the same spaces. The Fourier series with respect to these functions admit *the Abel–Lidskii summability* of order $\beta > n/2 - 1/2$. Moreover, if $b_j = 0$, then for $n = 2$ the root functions form a Riesz basis with brackets and $\beta > n/2 - 1$ for other n.

Case 3. *No selfadjointness.* The spectrum is discrete, we obtain $s_j(L_D^{-1}) \leq Cj^{2/n}$ (using our choice of the basic Hilbert space), and the root functions preserve the same smoothness. Assume that the eigenvalues of the principal symbol lie in Θ_θ. Then $\theta < \pi/2$ and we have *the optimal estimate of the resolvent*

$$\|(L_D - \lambda I)^{-1}\| \leq C(1 + |\lambda|)^{-1}$$

outside $\Theta_{\theta+\varepsilon}$ with any small ε for sufficiently large $|\lambda|$. $\Rightarrow$ *The same results* on completeness and Abel-Lidskii summability of order $\beta > n/2$ if $\theta < \pi/n$.

For the operator L_N, the results are exactly the same with natural change of spaces.

5. Problems with spectral parameter in conditions on Γ. The most important is *the Poincaré–Steklov problem*

$$Lu = 0 \ \text{ in } \ \Omega, \qquad T^+u = \lambda u^+ \ \text{ on } \ \Gamma.$$

If the Dirichlet problem is uniquely solvable, we introduce *the Dirichlet-to-Neumann operator*

$$D : u^+ \to u \to T^+u,$$

a bounded operator $H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$. The spectral problem *for eigenfunctions* is equivalent to the equation

$$D\varphi = \lambda\varphi, \text{ where } \varphi = u^+.$$

If the Neumann problem is uniquely solvable, introduce *the Neumann-to-Dirichlet operator*

$$N : T^+u \to u \to u^+.$$

If both problems are uniquely solvable, then D and N are invertible, and $D^{-1} = N$.

To go further, we need *two domains* $\Omega^\pm$ with common Lipschitz boundary Γ.

To avoid technical considerations at infinity, we assume that $\Omega = \Omega^+$ lies on *a standard torus* $\mathbb{T} = \mathbb{T}^n$ and that $\mathbb{T} = \Omega^+ \cup \Gamma \cup \Omega^-$. Let the normal ν be directed to Ω^-. Now $\widetilde{H}^s(\Omega^\pm)$ are subspaces in $H^s(\mathbb{T})$.

Assume that the system is given and strongly elliptic on $\mathbb{T}$ and that $\operatorname{Re} c$ is sufficiently large. $\Rightarrow$ $L : H^1(\mathbb{T}) \to H^{-1}(\mathbb{T})$ *is invertible.*

The inverse is an integral operator:

$$L^{-1}f(x) = \int_{\mathbb{T}} \mathcal{E}(x,y)f(y)\,dy.$$

This is *the Newtonian potential*, and $\mathcal{E}$ is a *fundamental solution.*

For $Lu = f$ on $\mathbb{T}$, we have no "Lipschitz difficulties", and an investigation of $\mathcal{E}$ by the tools of ΨDO shows that $L : H^{1+s}(\Omega) \to H^{-1+s}(\Omega)$ *is a bounded invertible operator for* $|s| < 1$ (D. Mitrea, M. Mitrea, M. Taylor, 2001).

Now we have two Poincaré–Steklov problems

$$Lu = 0 \text{ in } \Omega^{\pm}, \quad \pm T^{\pm}u = \lambda u^{\pm},$$

and set $D_{\pm}u^{\pm} = \pm T^{\pm}u, \quad N_{\pm}T^{\pm}u = \pm u^{\pm}.$

Introduce the notation for *jumps* on Γ:

$$[u] = u^{-} - u^{+}, \quad [Tu] = T^{-}u - T^{+}u.$$

We will consider two *spectral transmission problems*:

I. $Lu = 0 \text{ in } \Omega^{\pm},\ [u] = 0\ ,\ [Tu] = -\lambda u^{\pm}.$

II. $Lu = 0 \text{ in } \Omega^{\pm}\ , [Tu] = 0\ , T^{\pm}u = -\lambda[u].$

They were proposed by Moscow physicist Katsenelenbaum and his collaborators for the Helmholtz equation about 40 years ago. Physical sense: a half-transparent screen.

6. Surface potentials and integral formulas. Having the fundamental solution $\mathcal{E}$, we can introduce the classical *single layer potential*

$$\mathcal{A}\psi(x) = \int_\Gamma \mathcal{E}(x,y)\psi(y)\, dS_y \quad (x \in \mathbb{T})$$

and *double layer potential*

$$\mathcal{B}\varphi(x) = \int_\Gamma [T_y^+ \widetilde{\mathcal{E}}^*(x,y)]^* \varphi(y) dS_y \quad (x \notin \Gamma).$$

However, in the case of a Lipschitz boundary, even the boundedness of these operators is a non-trivial question. The direct investigation is possible but very difficult. At this moment, we explain another approach.

It is based on the results for the Dirichlet and Neumann problems and proposed by Nečas (1967), Costabel (1984) and McLean (2000).

The trace operator γ is bounded from $H^{1+s}(\mathbb{T})$ to $H^{1/2+s}(\Gamma)$, $|s| < 1/2$. $\Rightarrow$ The adjoint operator γ^* is bounded from $H^{-1/2-s}(\Gamma)$ to $H^{-1-s}(\mathbb{T})$. Following Costabel and McLean , we set

$$\mathcal{A} = L^{-1}\gamma^*.$$

$\Rightarrow$ *$\mathcal{A}$ is a bounded operator from $H^{-1/2+s}(\Gamma)$ to*

$H^{1+s}(\mathbb{T})$ *for* $|s| < 1/2$, in particular, for $s = 0$. Obviously, two definitions of $\mathcal{A}$ coincide.

Similar approach to $\mathcal{B}$ is more complicated. Following McLean, we set

$$\mathcal{B} = L^{-1}\widetilde{T}^{*},$$

where $\widetilde{T}$ is the "smooth" conormal derivative (for the formally adjoint operator) on functions from $H^{s}(\mathbb{T})$ with $3/2 < s < 2$. From this, we cannot obtain a good boundedness result for $\mathcal{B}$ immediately and need the following key result (in essential due to Costabel and McLean) .

Representation Theorem. *Let* $u \in H^{1}(\Omega^{\pm})$ *and* $Lu = f^{\pm}$ *belongs to* $\widetilde{H}^{-1}(\Omega^{\pm})$, *so that* $f = f^{+} + f^{-}$ *belongs to* $H^{-1}(\mathbb{T})$. *Then*

$$u = L^{-1}f + \mathcal{B}[u] - \mathcal{A}[Tu].$$

Proof. Formula for $(Lu, v)_{\mathbb{T}}$ ($v \in H^{s}(\mathbb{T})$, $3/2 < s < 2$) follows from 4 Green formulas in $\Omega^{\pm}$ for L and $\widetilde{L}$. $\Rightarrow$ Formula for $Lu \Rightarrow$ for u.

Assuming the Dirichlet problem to be uniquely solvable and comparing the left and right sides, we see that $\mathcal{B}$ *is a bounded operator from* $H^{1/2}(\Gamma)$ *to* $H^1(\Omega^\pm)$.

Identification with the classical definition of $\mathcal{B}$ is possible but requires a special work (McLean; D.Mitea–M.Mitrea–Taylor).

We also see from the definitions that $u = \mathcal{A}\psi$ and $u = \mathcal{B}\psi$ are solutions of the system $Lu = 0$ in $\Omega^\pm$.

Now we *assume that the Dirichlet and Neumann problems in* $\Omega^\pm$ *are uniquely solvable* and list important corollaries.

6.1. *The jump relations*

$$[\mathcal{A}\psi] = 0,\ [T\mathcal{A}\psi] = -\psi,\ [\mathcal{B}\varphi] = \varphi,\ [T\mathcal{B}\varphi] = 0.$$

Introduce important operators on Γ:

$$A = \gamma^\pm \mathcal{A}, \quad B = \tfrac{1}{2}[\gamma^- \mathcal{B} + \gamma^+ \mathcal{B}],$$
$$\widehat{B} = \tfrac{1}{2}[T^- \mathcal{A} + T^+ \mathcal{A}], \quad H = -T^\pm \mathcal{B}.$$

Here B is *the direct value* of the double layer potential, and H is called *the hypersingular operator*. These operators are bounded: A from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$, B in $H^{1/2}(\Gamma)$, $\widehat{B}$ in $H^{-1/2}(\Gamma)$, and H from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$.

6.2. *Duality relations* (with respect to the extension of the inner product in $L_2(\Gamma)$)

$$A^* = \widetilde{A}, \;\; H^* = \widetilde{H}, \;\; \widehat{B} = (\widetilde{B})^*.$$

6.3. Equations on Γ. From the Representation Theorem we can obtain similar statements for solutions in Ω^+ and Ω^- separately. $\Rightarrow$ Passing to the boundary, we obtain for solutions of the homogeneous systems in $\Omega^\pm$

$$(\tfrac{1}{2}I+B)u^+ = AT^+u, \quad Hu^+ = (\tfrac{1}{2}I-\widehat{B})T^+u;$$

$$(\tfrac{1}{2}I-B)u^- = -AT^-u, \quad -Hu^- = (\tfrac{1}{2}I+\widehat{B})T^-u.$$

6.4. Theorem on invertibility.

1. *The Dirichlet problem is uniquely solvable* $\Leftrightarrow$ $A : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *is invertible.*

2. *Let the Dirichlet problem be uniquely solvable. Then the Neumann problem is uniquely solvable* $\Leftrightarrow$ $H : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *is invertible.*

3. *Let the Dirichlet and Neumann problems be uniquely solvable. Then* $\frac{1}{2}I + B$ *and* $\frac{1}{2}I + \widehat{B}$ *are invertible in* $H^{1/2}(\Gamma)$ *and* $H^{-1/2}(\Gamma)$ *respectively.*

6.5. Relations between transmission problems and operators on Γ.

Proposition 1. *The first and second spectral transmission problems on eigenfunctions are equivalent to the equations at* Γ

$$A^{-1}\psi = \lambda\psi, \quad H\varphi = \lambda\varphi$$

respectively. Here $\psi = [Tu]$, $\varphi = [u]$.

Proposition 2.

$$A^{-1} = D_+ + D_-, \;\; H^{-1} = N_+ + N_-,$$
$$N_\pm = (\tfrac{1}{2}I \pm B)^{-1}A = A(\tfrac{1}{2}I \pm \widehat{B})^{-1},$$
$$H^{-1} = (\tfrac{1}{4}I - B^2)^{-1}A = A(\tfrac{1}{4}I - \widehat{B}^2)^{-1}.$$

This is useful when we search spectral asymptotics. In two last rows we use the relation $BA = A\widehat{B}$. It is among corollaries of the relation $P^2 = P$ for the Calderón projector P.

7. Spectral properties of the operators A, $N_\pm$, and H^{-1}. In the smooth case, they are strongly elliptic ΨDO of order -1.

These operators have similar spectral properties. For definiteness, consider A.

We have the Gårding type inequality:

$$\mathrm{Re}(A\psi, \psi)_\Gamma \geq C_1 \|\psi\|^2_{H^{-1/2}(\Gamma)}.$$

Case 1. *L is formally selfadjoint.* We consider $H^{-1/2}(\Gamma)$ as the basic Hilbert space and use there the inner product

$$\langle \psi_1, \psi_2 \rangle_{H^{-1/2}(\Gamma)} = (A\psi_1, \psi_2)_\Gamma.$$

There, the operator A is a compact selfadjoint operator, it has an orthonormal basis of eigenfunctions. They belong to $H^{1/2}(\Gamma)$ and form there an orthogonal basis with respect to the inner product $(A^{-1}\psi_1, \psi_2)_\Gamma$. The result is extended to intermediate spaces.

A satisfactory result for spectral asymptotics follows from the paper by Agr.-Amosov (1996). A Lipschitz surface is called there *almost smooth* if it is C^∞ outside a closed subset of measure zero. For such surfaces, the asymptotics is the same as in the smooth case (but without an estimate of the remainder):

$$N_{A^{-1}}(\lambda) = \mu\lambda^{n-1} + o(\lambda^{n-1}),$$

μ is calculated as usually. In the scalar case,

$$\mu = \tfrac{1}{(2\pi)^{n-1}} \iint_{\alpha(x',\xi')<1} dx'd\xi',$$

where α is the principal symbol of A^{-1}.

The restriction "almost smooth" was removed for A by Rozenblum and Tashchiyan (2006).

Case 2. *Only L_0 is formally selfadjoint.* Then A is a small perturbation of a selfadjoint operator. The eigenvalues lie in Θ_θ with any small θ except finite number of them, and their asymptotics is the same. The root functions are complete in $H^{\pm 1/2}(\Gamma)$ and in intermediate spaces, and a result on Abel–Lidskii summability of Fourier series in root functions is true.

Case 3. *No selfadjointess.* We have $s_j(A) \leq C_1 j^{-1/(n-1)}$. The eigenvalues lye in some Θ_θ except finite number of them. Outside of $\Theta_{\theta+\varepsilon}$ with any small ε we have the optimal estimate

$$\|((A^{-1} - \lambda I)^{-1}\| \leq C_2(1 + |\lambda|)^{-1}.$$

If $\theta < \pi/2(n-1)$, we have the completeness and Abel–Lidskii summability of order $\beta > n-1$ in $H^{\pm 1/2}(\Gamma)$ and in the intermediate spaces.

For N, the spectral asymptotics in non-smooth domains was investigated by many authors, in particular by Sandgren (1955) and Suslina (1998).

For $N_\pm$ and H, the most general result on the spectral asymptotics in apparently in the case of almost smooth boundary. Return to A: it is important that its kernel of A is the restriction to Γ of the kernel of a ΨDO in $\mathbb{R}^n$. For $N_\pm$ and H^{-1} it is not so, but each of these operators admit two representations $T_1 A$ and AT_2, where T_1 and T_2 are bounded operators.

8. The spaces H_p^s and B_p^s. They are generalizations of H^s for $p \neq 2$. Below

$$s \in \mathbb{R}, \quad 1 < p < \infty, \quad \tfrac{1}{p} + \tfrac{1}{p'} = 1.$$

The space H_p^s of Bessel potentials in $\mathbb{R}^n$ is defined by

$$H_p^s(\mathbb{R}^n) = F^{-1}(1 + |\xi|^2)^{-s/2} F L_p(\mathbb{R}^n).$$

It is the Sobolev space $W_p^s(\mathbb{R}^n)$ if $s > 0$ is an integer. *The Besov space $B_p^s(\mathbb{R}^n)$* is defined similarly starting from the Slobodetskii space $W_p^\sigma(\mathbb{R}^n)$, $0 < \sigma < 1$:

$$B_p^s(\mathbb{R}^n) = F^{-1}(1+|\xi|^2)^{(\sigma-s)/2} F W_p^\sigma(\mathbb{R}^n).$$

These two scales are very close one to another.

The spaces $H_p^s(\Omega)$ and $\widetilde{H}_p^s(\Omega)$ can be identified for $-1/p' < s < 1/p$. The spaces $H_p^s(\Omega)$ and $\widetilde{H}_{p'}^{-s}(\Omega)$ are mutually adjoint with respect to the extension of the form $(\mathcal{E}u, v)_{\mathbb{R}^n}$.

The trace operator is bounded from $H_p^{s+1/p}(\Omega)$ and $B_p^{s+1/p}(\Omega)$ to $B_p^s(\Gamma)$ for $0 < s < 1$, and there is a common right inverse operator (Jonsson ans Wallin, 1984).

9. Generalization of the variational setting of the Dirichlet and Neumann problems. Recall the formula defining the Dirichlet problem with $u^+ = 0$:

$$(Lu, v)_\Omega = \Phi_\Omega(u, v).$$

We generalize this problem as follows:

$$u \in \widetilde{H}_p^{1/2+s+1/p}(\Omega), \quad Lu \in H^{-1/2+s-1/p'}(\Omega),$$
$$v \in \widetilde{H}_{p'}^{1/2-s+1/p'}(\Omega).$$

If $u^+ \neq 0$, then $u^+ \in B_p^{1/2+s}(\Gamma)$.

Similarly generalize the Neumann problem:

$$u \in H_p^{1/2+s+1/p}(\Omega), \quad Lu \in \widetilde{H}^{-1/2+s-1/p'}(\Omega),$$
$$v \in H_{p'}^{1/2-s+1/p'}(\Omega).$$

If $T^+u \neq 0$, then $T^+u \in B_p^{-1/2+s}(\Gamma)$.
All relations of duality are important.
Everywhere, H can be replaced by B.

Let us introduce *the square* Q of admissible pairs of indices:

$$Q = \{(s,t): \ |s| < 1/2, \ \ 0 < t = 1/p < 1\}.$$

Here $|s| < 1/2$ because of the restrictions in the trace theorem, and $0 < t < 1$ since $1 < p < \infty$. For the same (s,t) we can consider the problems in $\Omega^{\pm} \subset \mathbb{T}$. However, in general, the problems are solvable not for all $(s,t) \in Q$.

10. Regularity theorems and generalizations of theorems on the unique solvability. The regularity statement is as follows. Let u be be a solution, say, of $L_D u = f$. If f is "better", then u is "better".

Such results can give more general statements on the unique solvability than before.

There are 3 approaches to the regularity.

1. *Savaré* proposed a new method of investigation of the regularity for variational problems, linear and non-linear, using the interpolation theory (1998). For the scalar equation

$$\operatorname{div} a(x) \operatorname{grad} u + \ \dots \ = f$$

with real symmetric $a(x)$ his result imply that if the Dirichlet and/or Neumann problem is uniquely solvable at the centrum of Q, then this is true at all $(s,t) \in Q$ with $t = 1/2$.

I have checked that *the same is true for systems with formally selfadjoint L_0 if, additionally,*

$$\sum a_{j,k}^{r,s}(x)\zeta_k^s\overline{\zeta_j^k} \geq 0$$

which apparently is not a heavy restriction.

2. *Shneiberg* proved a strong theorem on the extrapolation of the invertibility of an operator acting in interpolation scales of spaces (or on the stability of the invertibility) (1974). $\Rightarrow$

If we have the unique solvability at $(s, 1/2)$, $|s| < 1/2$, *than we have it in a strip*

$$Q_\delta = \{(s,t) : |s|, 1/2, \ \ |t - 1/2| < \delta\}$$

where $\delta > 0$ can be estimated from below.

Moreover, if we cannot apply the Savaré theorem, *we have the unique solvability in some neighborhood* O *of the centrum of* Q (a remark by Mazya, M. Mitrea and Shaposhnikova, 2009, for the Dirichlet problem).

All *a priory* estimates remain two-sided.

3. Long ago it was known that there are some algebraic *Rellich identities* for systems with formally selfadjoint L_0. They permit to estimate Neumann data in terms of Dirichlet data and to show the following. If the Dirichlet problem is uniquely solvable at the centrum of Q, then it is uniquely solvable at $(s, 1/2)$ for $|s| < 1/2$. E.g. see Nečas, 1967.

Here it is possible to replace "Dirichlet" by "Neumann" and vice versa in the case of a *scalar* equation with real $a_{j,k}(x) = a_{k,j}(x)$. Similar result is also true for the Lamé system (Dahlberg, Kenig, Verchota, 1988).

11. Alternative approach to problems in Lipschitz domains. Up to now, I almost did not touch the extended (non-spectral) theory of boundary value problems in Lipschitz domains constructed by many strong mathematicians during the last 30-35 years. First problems were stated and solved by Dahlberg, Calderón, Jerison, Kenig, Verchota.

From the very beginning, these investigations were oriented to solve the Dirichlet problem with $u^+ \in L_2(\Gamma)$ or $H^1(\Gamma)$, the Neumann problem with $T^+u \in L_2(\Gamma)$ and, further, with p near 2 instead of 2. These points lye at the boundary of Q, and the usual trace theorems do not work there. A "non-tangential convergence" was used. A system of equal truncated cones $K(x)$ was fixed having no common points with Ω, with vertices at $x \in \Gamma$, and the convergence was understood as $u(y) \to u^+(x)$ pointwise, $K(x) \ni y \to x$, a.e. on Γ, controlled by maximal functions $u^*(x)$. In the Dirichlet problem with $u^+ = g \in L_2(\Gamma)$, $u^*(x) = \max_{K(x)} |u(y)|$, and $\|u^*\|_{L_2(\Gamma)} \leq C\|g\|_{L_2(\Gamma)}$.

The main technical tool: surface potentials, they were carefully investigated. The direct value B of $\mathcal{B}$ on a Lipschitz Γ is in general a singular integral operator even for Δ. A problem to prove the boundedness of a singular integral operator on a Lipschitz surface arose. It was solved by Calderón (1977), Coifman–McIntosh–Meyer (1982), M. Mitrea–Taylor (1998).

Especially deep results were obtained for the Laplace and Beltrami–Laplace equation using specific tools from the harmonic analysis and interpolation theory, with work on parts of the boundary of Q. The most general results were obtained by M. Mitrea–Taylor (2000), they cover a part of Q between two lines connecting the points 1) $(-1/2, 1/2+\varepsilon)$ and $(1/2-\varepsilon, 1)$, 2) $(-1/2+\varepsilon, 0)$ and $(1/2, 1/2-\varepsilon)$.

The use of Rellich identities is essential in these papers. $\Rightarrow$ The results cover the Dirichlet problem for all strongly elliptic systems with formally adjoint principal part, but the Neumann problem was considered only in the scalar case and the case of the Lamé system.

12. Extension of spectral results to Banach spaces H_p^s and B_p^s. Assume the unique solvability of the Dirichlet and Neumann problems for $(s, t) \in Q_\delta$ or O. What remains true for L_D, L_N, A, H^{-1}, $N_\pm$ for these (s, t)?

Almost all.

Note that our spectral problems are much easier then abstract spectral problems in Banach spaces. Using embedding theorems and the spectral equation, we conclude that the spectrum and the eigen- or root functions *do not depend on* (s, t). For $t = 1/2$, they form an ON basis for all s (if L is formally selfadjoint). The *completeness* result is also generalized using embedding theorems and the equation. There is also an abstract version of the completeness theorem (Burgoyne, 1995; Agr).

The most difficult for a generalization is the Abel–Lidskii summability. Here we need to understand the abstract theory. I improved something made by A. Markus in 1966 who followed Grothendieck. The monographs by König (1986), Pietsch (1987) contain, in my opinion, too much information. The first question: which analog of s-numbers to use? The answer: approximation numbers. The necessary estimates for them are contained in Edmunds– Triebel (1996). We need uniform optimal resolvent estimates and obtain them using the interpolation theory. We then represent the resolvent as a ratio of entire analytic functions, operator-valued and numerical, with estimating of their grows.

Now for a given function f we define, say,

$$f(t) = \frac{1}{2\pi i} \int_{\gamma_\theta} (D - \lambda I)^{-1} e^{-\lambda^\beta t} d\lambda f$$

with integration along the boundary γ_θ of Θ_θ. Using deep theorems on entire functions, we prove the possibility to find arcs dividing Θ_θ into a sequence of bounded domains such that the integral $f(t)$ is equal to the sum of integrals $f_j(t)$ along boundaries γ_j of these domains, and

$$f = \lim_{t\to+0} \sum f_j(t).$$

This is the Abel–Lidskii summability in Banach spaces.

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