Wave spectrum of symmetric semi-bounded operator and its applications

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1 Motivation

We introduce the notion of a *wave spectrum* of a symmetric semi-bounded operator in a Hilbert space. The impact comes from inverse problems of mathematical physics; the following is one of the motivating questions.

Let Ω be a smooth compact Riemannian manifold with the boundary $\partial\Omega$, $-\Delta$ the (scalar) Laplace operator, $L_0 = -\Delta|_{C_0^\infty(\Omega\backslash\partial\Omega)}$ the *minimal Laplacian* in $\mathcal{H} = L_2(\Omega)$. Assume that we are given with a unitary copy $\widetilde{L}_0 = UL_0U^*$ in $\widetilde{\mathcal{H}} = U\mathcal{H}$ (but U is unknown!). To what extent does $\widetilde{L}_0$ determine the manifold Ω? Provided the operator is unitarily equivalent to L_0, is it possible to extract Ω from $\widetilde{L}_0$? Such a question is an "invariant" version of various setups of dynamical and spectral inverse problems on manifolds [1].

Example Let $u = u_\lambda^f(x)$ solve

$$\begin{aligned} &(\Delta + \lambda)\, u = 0 \qquad \text{in } \Omega, \quad \lambda \in \mathbb{C}\backslash\sigma(L_{\mathrm{Dir}}) \\ &u = f \qquad \text{on } \partial\Omega\,. \end{aligned}$$

The *Titchmarsh-Weyl Transfer Operator Function* is $M(\lambda) : L_2(\partial\Omega) \to L_2(\partial\Omega)$, $\operatorname{Dom} M(\lambda) = H^1(\partial\Omega)$,

$$M(\lambda)f := \frac{\partial u_\lambda^f}{\partial\nu} \qquad \text{on } \partial\Omega \quad (\lambda \text{ to be admissible}).$$

Inverse Problem: given $M(\,\cdot\,)$ to recover Ω.

Lemma 1 *(V.A.Ryzhov, 2007) The TW-function M determines L_0 up to a unitary equivalence.*

Hence, one can hope for the determination $M \Rightarrow \widetilde{L}_0 \Rightarrow \Omega$.

2 Wave spectrum

2.1 Space extension

Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{L}(\mathcal{H})$ the lattice of its (closed) subspaces (i.e., $\mathcal{A}, \mathcal{B} \in \mathfrak{L}(\mathcal{H})$ implies $\mathcal{H} \ominus \mathcal{A}, \mathcal{A} \cap \mathcal{B}, \mathcal{A} \vee \mathcal{B} \in \mathfrak{L}(\mathcal{H})$).

Definition 1 *An one-parameter family $E = \{E^t\}_{t \geq 0}$ of the maps $E^t : \mathfrak{L}(\mathcal{H}) \to \mathfrak{L}(\mathcal{H})$ is said to be a space extension if*

1. $E^0 = \mathrm{id}$
2. $E^t\{0\} = \{0\}, \quad t \geq 0$
3. $t \leq t'$ *and* $\mathcal{A} \subseteq \mathcal{A}'$ *imply* $E^t\mathcal{A} \subseteq E^{t'}\mathcal{A}'$.

Definition 2 *For an extension E and a family $a \subset \mathfrak{L}(\mathcal{H})$, by $\mathfrak{L}[E, a] \subset \mathfrak{L}(\mathcal{H})$ we denote the minimal sublattice s.t.*

- $a \subset \mathfrak{L}[E, a]$
- $E^t\mathfrak{L}[E, a] \subset \mathfrak{L}[E, a] \quad t \geq 0$.

2.2 Nests

Recall that a *nest* is a family of subspaces $n \subset \mathfrak{L}(\mathcal{H})$ completely ordered w.r.t. the embedding "$\subseteq$", i.e. for any $\mathcal{A}, \mathcal{A}' \in n$ one has $\mathcal{A} \subseteq \mathcal{A}'$ or $\mathcal{A} \supseteq \mathcal{A}'$. The nests $n = \{\mathcal{A}^t\}_{t \geq 0} : \ \mathcal{A}^t \leq \mathcal{A}^{t'}$ for $t < t'$ are said to be parametrized. Parametrized nests are partially ordered: for $m = \{\mathcal{A}^t_m\}_{t \geq 0}$ and $n = \{\mathcal{B}^t_n\}_{t \geq 0}$ we put

$$\{m \preccurlyeq n\} \Longleftrightarrow \left\{\mathcal{A}^t \leq \mathcal{B}^t \quad \forall t\right\} .$$

With each $\mathcal{A} \in \mathfrak{L}[E, a]$ one associates a parametrized nest $n_{\mathcal{A}} := \{E^t\mathcal{A}\}_{t \geq 0}$. Let

$$\mathfrak{N}[E, a] := \{n_{\mathcal{A}} \mid \mathcal{A} \in \mathfrak{L}[E, a]\}$$

be the set of the nests of this kind.

A nest $m = \{\mathcal{M}^t\}_{t \geq 0} \subset \mathfrak{L}(\mathcal{H})$ is said to be *minimal* for the sublattice $\mathfrak{L}[E, a]$ if $m \preccurlyeq n$ holds for any $n \in \mathfrak{N}[E, a]$ comparable with m.

Definition 3 *The set $\Omega_{E,a}$ of minimal (for $\mathfrak{L}[E, a]$) nests is said to be a wave spectrum. The elements of $\Omega_{E,a}$ are called the points.*

So, in fact each point $x \in \Omega_{E,a}$ is a parametrized nest $x = \{\mathcal{X}^t\}_{t\geq 0}$.

Lemma 2 *Put $P^t_{\mathcal{A}} := P_{E^t\mathcal{A}}$. If there is a constant $C > 0$ s.t.*

$$\left\| \int_0^\infty t \,\mathrm{d}P^t_{\mathcal{A}} \right\| \leq C \qquad \forall \mathcal{A} \in \mathfrak{L}\,[E, a]$$

holds, then one has $\Omega_{E,a} \neq \emptyset$.

However, $\Omega_{E,a}$ can consist of a single point. The examples of $\Omega_{E,a} = \emptyset$ are not known.

2.3 Space $\Omega_{E,a}$

Let us provide the wave spectrum with relevant topology. For two of its points $x = \{\mathcal{X}^t\}_{t\geq 0}$ and $y = \{\mathcal{Y}^t\}_{t\geq 0}$ define a quasi-distance

$$\tau(x, y) := \inf\{t \geq 0 \,|\, \mathcal{X}^t \text{ is not orthogonal to}\, \mathcal{Y}^t\}$$

which is a symmetric function $\tau : \Omega_{E,a} \times \Omega_{E,a} \to [0, +\infty]$. Define the quasi-balls $B_r[y] := \{x \in \Omega_{E,a} \,|\, \tau(x, y) < r\}$, $r > 0$ and consider the family $\{B_r[y] \,|\, y \in \Omega_{E,a},\, r > 0\}$ as a prebase of a topology. *In what follows $\Omega_{E,a}$ is endowed with the topology generated by this prebase.*

Conclusion With each pair E, a one associates a topological space (wave spectrum) $\Omega_{E,a}$ by $\{E\,, a\} \Rightarrow \mathfrak{L}\,[E, a] \Rightarrow \mathfrak{N}\,[E, a] \Rightarrow \Omega_{E,a}$.

3 Space Ω_{L_0}

3.1 Extension E_L

Let

$$L = L^* \,=\, \int_0^\infty \lambda \,\mathrm{d}Q_\lambda\,; \qquad (Ly, y) \geq \varkappa\,\|y\|^2,\;\; y \in \operatorname{Dom} L \subset \mathcal{H}\,,$$

where $\mathrm{d}Q_\lambda$ is the spectral measure of L and $\varkappa$ is a positive constant. The operator L governs the evolution of a dynamical system

$$v_{tt} + Lv \,=\, h\,, \qquad t > 0 \tag{3.1}$$

$$v|_{t=0} \,=\, v_t|_{t=0} \,=\, 0\,, \tag{3.2}$$

where $h \in L_2^{\rm loc}\left((0,\infty);\mathcal{H}\right)$ is a $\mathcal{H}$-valued function of time (*control*). Its finite energy class solution (*wave*) $v = v^h(t)$ is

$$\begin{aligned} v^h(t) \ &= \ \int_0^t L^{-\frac{1}{2}} \sin\left[(t-s)L^{\frac{1}{2}}\right] \, h(s)\, {\rm d}s = \\ &= \int_0^t {\rm d}s \int_0^\infty \frac{\sin\sqrt{\lambda}(t-s)}{\sqrt{\lambda}}\, {\rm d}Q_\lambda\, h(s)\,, \qquad t \geq 0\,. \end{aligned}$$

Fix a subspace $\mathcal{A} \subset \mathcal{H}$; the set

$$\mathcal{V}^t_{\mathcal{A}} := \{v^h(t) \mid h \in L_2^{\rm loc}\left((0,\infty);\mathcal{A}\right)\}\,, \qquad t>0$$

is called *reachable* (at the moment t, from the subspace $\mathcal{A}$).

Define a family $E_L = \{E^t\}_{t\geq 0}$ of the maps $E^t : \mathfrak{L}(\mathcal{H}) \to \mathfrak{L}(\mathcal{H})$ by

$$E^0\mathcal{A} := \mathcal{A}, \quad E^t\mathcal{A} := {\rm clos}\, \mathcal{V}^t_{\mathcal{A}}\,, \quad t>0\,.$$

Lemma 3 *E_L is a space extension.*

Let L_0 be a closed densely defined symmetric positive definite operator with *nonzero* defect indexes $n_\pm = n \leq \infty$. As is easy to see, such an operator is necessarily unbounded. Let L be the extension of L_0 by Friedrichs, so that

$$L_0 \subset L \subset L_0^*$$

holds. Also, note that

$$1 \leq \dim {\rm Ker}\, L_0^* = n \leq \infty.$$

The operator L_0 determines the space extension E_L.

3.2 Family a_{L_0}

Take a smooth ${\rm Ker}\, L_0^*$-valued function $h \in C_0^\infty\left((0,\infty);{\rm Ker}\, L_0^*\right)$ and put

$$\begin{aligned} & u^h(t) \ := \ h(t) - v^{h''}(t) \ = \\ & h(t) - \int_0^t L^{-\frac{1}{2}} \sin\left[(t-s)L^{\frac{1}{2}}\right] \, h''(s)\, {\rm d}s\,, \qquad t \geq 0\,. \end{aligned}$$

A set

$$\mathcal{U}^t := \left\{ u^h(t) \mid h \in C_0^\infty\left((0,\infty); \operatorname{Ker} L_0^*\right)\right\}$$

is called *reachable from boundary* (at the moment t). These sets increase as t grows.

A family

$$a_{L_0} := \left\{\operatorname{clos}\mathcal{U}^t\right\}_{t\geq 0}$$

is said to be the *boundary nest*; by construction it is determined by the operator L_0.

3.3 Space Ω_{L_0}

Each L_0 determines the pair E_L, a_{L_0}, so that

$$\Omega_{E_L,a_{L_0}} =: \Omega_{L_0}$$

is a well-defined topological space that we call the *wave spectrum* of the operator L_0.

The subset

$$\partial\Omega_{L_0} := \{x \in \Omega_{L_0} \mid x \preccurlyeq a_{L_0}\}$$

(so that $\mathcal{X}^t \subset \mathcal{U}^t \ \forall t$ holds) is the *boundary* of the wave spectrum.

Definition 4 *Class $\mathcal{S}$ of simple manifolds.*

Theorem 1 *Let $\Omega \in \mathcal{S}$, L_0 the minimal Laplacian in Ω, Ω_{L_0} its wave spectrum. There is a bijection $\beta : \Omega_{L_0} \to \Omega$ s.t.*

- $\tau(x,y) = \operatorname{dist}_\Omega(\beta(x),\beta(y))$, *so that the quasi-distance τ is a metric on Ω_{L_0} and β is an isometry of metric spaces*
- $\beta(\partial\Omega_{L_0}) = \partial\Omega$.

So, Ω_{L_0} turns out to be an *isometric copy* of the manifold Ω.

Solving IPs $M \Rightarrow \widetilde{\mathcal{H}}, \widetilde{L}_0 \Rightarrow \Omega_{\widetilde{L}_0} \overset{\text{isom}}{=} \Omega$
von Neumann algebras $\{P_{E^t\mathcal{A}} \mid \mathcal{A} \in \mathfrak{L}\,[E,a]\} \Rightarrow \mathfrak{N}_{L_0}$. For $\Omega \in \mathcal{S}$, the Gelfand spectrum of $\mathfrak{N}_{L_0}$ is identical to Ω_{L_0}.
Functional model $\mathcal{H} = \oplus \int_{\Omega_{L_0}} \mathcal{H}_x \, d\mu_x$; $\mathcal{H}_x$ is a fiber of germs. Open questions: $\langle a_x, b_x\rangle_x =$? and what is μ? **Graphs: ???**
Model of SBV $\{\mathcal{H},\mathcal{G}; L_0, \Gamma_1, \Gamma_2\}$ with $L_0 : \mathcal{H} \to \mathcal{H}$, $\Gamma_{1,2} : \mathcal{H} \to \mathcal{G}$ s.t. $(L_0^* u, v)_\mathcal{H} - (u, L_0^* v)_\mathcal{H} = (\Gamma_1 u, \Gamma_2 v)_\mathcal{G} - (\Gamma_2 u, \Gamma_1 v)_\mathcal{G}$.

References

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