Asymptotic behavior of the eigenfunctions of three-particle Schrödinger operator. II. One-dimensional charged particles.

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Introduction

Setting of the problem. Scattered plane wave

First of all, we have to describe the *configuration space* of the system. Originally, the configuration space is the euclidian space $\mathbf{R^3}$, but after the separation of the motion of the center of mass it is reduced to two-dimensional subspace

$$\Gamma = \{\mathbf{z} \in \mathbf{R^3} : z_1 + z_2 + z_3 = 0\}$$

with the naturally induced euclidian structure $\lessdot, \gtrdot$ on it.

The Schrödinger operator H has the form

$$H = -\Delta + V(\mathbf{z}), \quad \mathbf{z} \in \mathbf{\Gamma}, \tag{1}$$

where Δ is the Laplacian on the space Γ. We consider the Schrödinger equation $H\psi = E\psi$.

V is supposed to have the following structure

$$V = v(x_1) + v(x_2) + v(x_3), \tag{2}$$

where

$$x_1 = \frac{1}{\sqrt{2}}(z_3 - z_2)\,, x_2 = \frac{1}{\sqrt{2}}(z_1 - z_3)\,, x_3 = \frac{1}{\sqrt{2}}(z_1 - z_3)\,. \tag{3}$$

The variables x_j becomes equal to 0 along some axis $l_j, j = 1, 2, 3$, on Γ.

It is supposed that the pair potential $v(x)$ is a continuous positive-valued function that tends to 0 at infinity. We will distinguish, roughly speaking, two essentially different cases : fast decay pair potential,

$$xv(x) \to 0$$

, and the Coulomb type potential,

$$xv(x) \to \alpha \neq 0$$

.

Let us denote the general vector of Γ by $\mathbf{x}$, and the vector of the dual space (momentum space), naturally identified with Γ, by $\mathbf{q}$.

Let $r = |\mathbf{x}|$, $\omega = \frac{\mathbf{x}}{r}$ and $\omega_j = \frac{x_j}{r}, j = 1, 2, 3$. Consider two approximate as $r \to \infty$ solutions $R(\mathbf{x}, E)$ of the Schrödinger equation. These solutions are: for fast decaying potentials

$$R = R_0 = \frac{1}{r^{1/2}} e^{[i\sqrt{E}r]}. \tag{4}$$

For the Coulomb type potentials

$$R = R_c = \frac{1}{r^{1/2}} \exp\left[i\sqrt{E}r + i\gamma \ln r\right], \qquad \gamma = -\frac{\alpha}{2\sqrt{E}} \left(\frac{1}{\omega_1} + \frac{1}{\omega_2} + \frac{1}{\omega_3}\right). \tag{5}$$

In the last formula we suppose that $\mathbf{x} \not\subset l_j, j = 1, 2, 3$.

Now we are able to describe the boundary conditions at infinity. Let

$$n(\omega, \theta) = \sqrt{\frac{2\pi}{i\sqrt{E}}}\delta(\omega, \theta), \quad \theta = \frac{\mathbf{q}}{\sqrt{E}}, \tag{6}$$

where $\delta(\omega, \theta)$ is the delta-function on the unit circle with the standard angle measure.

Fix a vector (wave vector) $\mathbf{q}$ that lies inside one of the sectors between $l_j, j = 1, 2, 3$. Now the asymptotic behavior of the solution can be fixed by the condition

$$\psi \sim \psi(\mathbf{x}, \mathbf{q}) = n(\omega, \mathbf{q})R^* + f(\omega, \mathbf{q})R + o(r^{-1/2}). \tag{7}$$

as $r \to \infty$.

The asymptotic formula has to be considered in the weak with respect to ω sense, and both functions n and f must be treated as singular distributions. The first term of the asymptotic representation coincides with the first term of the weak asymptotic description of the plane wave $e^{i<\mathbf{q},\mathbf{x}>}$, so it is natural to call the solution $\psi(\mathbf{x},\mathbf{q})$ the scattered plane wave.

The solutions that are defined by the above asymptotic behavior can be treated as the generalized eigenfunctions of the continuous spectrum of the operator H. We have to remark that the corresponding theorem is not proved yet rigorously.

Asymptotic behavior of the scattered plane wave.

The definition of the scattered plane wave itself gives some information on the asymptotic behavior of the solution at infinity. We hope that this information is sufficient for the definition, but it is not sufficient for many other goals. We need more precise description of the asymptotic behavior, say, in the uniform norm, with respect to ω to prove rigorously the existence of the solution, to use it for physical applications, to find approaches for the numerical computations of the solution, and for some other goals. We will construct here on the heuristic level the continuous function ψ_{as} that gives the asymptotic behavior of ψ in the uniform norm.

The case of fast decreasing potential was considered in [1: V.S.Buslaev and S.B.Levin, Asymptotic behavior of the eigenfunctions of many-particle Schrödinger operator. I. One-dimensional particles, - Amer.Math.Soc.Transl. (2) v.225, pp.55-71, (2008); V.S.Buslaev, S.B.Levin, P.Neittaannmäki, T.Ojala, New approach to numerical computation of the eigenfunctions of the continuous spectrum of three-particle Schrödinger operator. I. One-dimensional particles, short-range pair potentials, - J.Phys.A: Math.Theor. 43, (2010), 285205, (pp.17)].

We represented the corresponding results a year ago at this conference.

Here we find the similar asymptotic formulas for Coulomb type potentials.

In fact, it is the first case when the asymptotic behavior (in the uniform norm) was found for a system of three particles interacting via the Coulomb pair potentials. As for the fast decreasing potentials for three dimensional particles such asymptotic behavior was obtain in famous Faddeev's work [3: L. D. Faddeev, Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory, Academy of Sciences of the USSR, Trudy Matematicheskogo Instituta, v.69, (1963)].

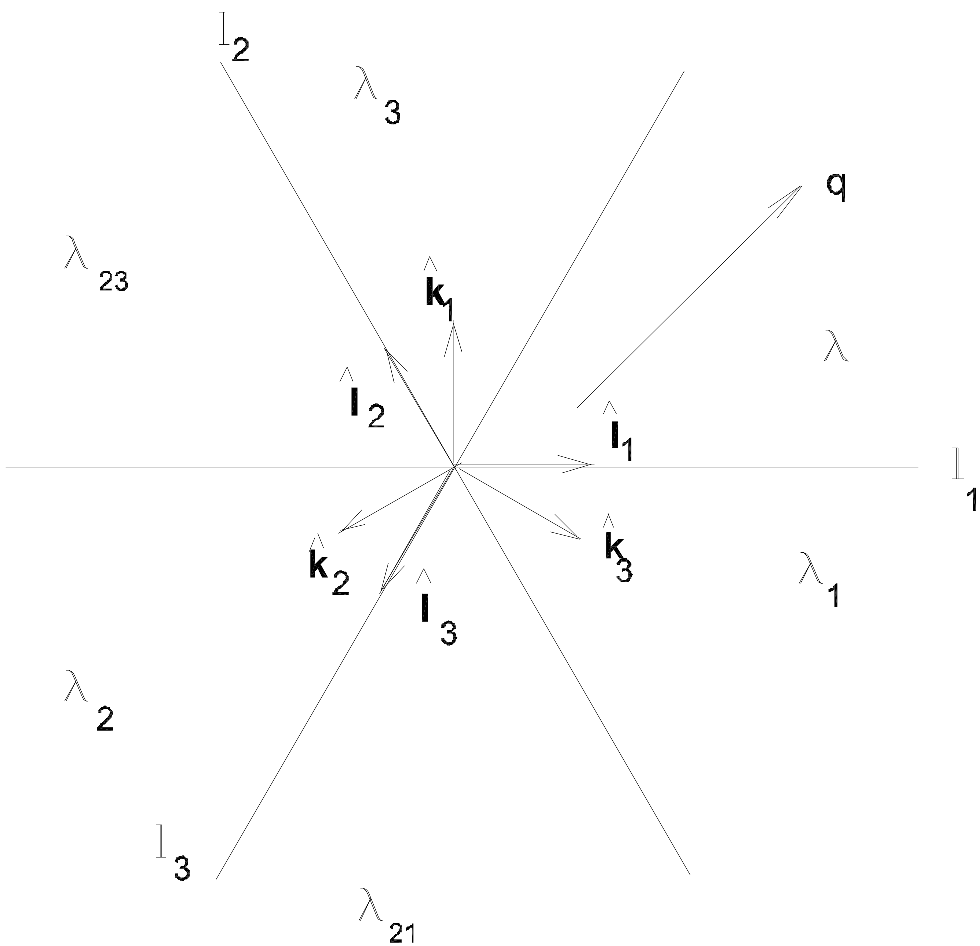
Constructing the function ψ_{as} we use two criteria:
1) The ψ_{as} satisfies the weak asymptotic behavior that was described earlier and that in our approach defines ψ, and
2) the discrepancy

$$Q[\psi_{as}] = -\Delta\psi_{as} + V\psi_{as} - E\psi_{as} \tag{8}$$

decreases at infinity faster than r^{-1}. It is worth to notice that with the present initial result we can improve the asymptotic formulas such that the discrepancy would decrease as arbitrary power of r.

Some additional geometrical remarks

Introduce on Γ three orthonormalized bases $(\mathbf{k_j}, \mathbf{l_j})$, $j = 1, 2, 3$, that have the same orientation . Let $\mathbf{l_j} \in l_j$ and the angles between any two unit vectors $\mathbf{l_j}$ are equal to $2/3\pi$. There are two such choices of the vectors $\mathbf{l_j}$, we can take any of them. The coordinates of the vector $\mathbf{x}$ with respect to these three bases will be denoted by (x_j, y_j). These are the classical Jacobi coordinates. The coordinates of the vector $\mathbf{q}$ will be denoted by (k_j, p_j). We also will consider six rays $l_j^{\pm}$ generated by the vectors $\pm\mathbf{l_j}, j = 1, 2, 3$.


l_2
λ_3
q
λ_{23}
$\hat{\mathbf{k}}_1$
λ
$\hat{\mathbf{l}}_2$
$\hat{\mathbf{l}}_1$
l_1
$\hat{\mathbf{k}}_3$
$\hat{\mathbf{k}}_2$
λ_1
$\hat{\mathbf{l}}_3$
λ_2
l_3
λ_{21}

Consider on the configuration plane the group of transformations S generated by the reflections τ_j with respect to the straight-lines l_j. It consists of six elements $I, \tau_1, \tau_2, \tau_3, \tau_2\tau_3, \tau_2\tau_1$, their general notation is σ.

The Schrödinger equation is invariant with respect to the group. Consider six sectors λ_σ that is situated between pairs of the neighboring rays $\mathbf{l}_j^{\pm}$, here σ denotes the element of S, that transfers the sector λ_I situated between l_1^+ and l_3^-, into the considered sector. The sector λ_I will be denoted also simply λ.

We will assume $\mathbf{q} \in \lambda$. Applying to it different elements of S, we obtain six other vectors $\mathbf{q}_\sigma = \sigma\mathbf{q}$, $\mathbf{q}_\sigma \in \lambda_\sigma$.

Constructing of ψ_{as}

The plan that we use to construct ψ_{as} consists of several steps:

A). First of all, we construct some generalization of the plane wave. It is easy to see that in the case of Coulomb type potentials there is no a direct analog of the elementary globally defined plane wave. We can construct the analog of the plane wave (that we call the Coulomb plane wave) only inside of any sector λ_σ.

Let us denote this analog by $\psi_c(\mathbf{x}, \mathbf{q})$ and define it in the sector λ_σ by the formula

$$\psi_\sigma(\mathbf{x}, \mathbf{q}) = \mathsf{exp}\,[i < \mathbf{q}, \mathbf{x} > +i\Delta_\sigma(\mathbf{x}, \mathbf{q})], \tag{9}$$

$$\Delta_\sigma(\mathbf{x}, \mathbf{q}) = -\alpha \left(\frac{sgn(x_1)}{2k_1} \ln|x_1| + \frac{sgn(x_2)}{2k_2} \ln|x_2| + \frac{sgn(x_3)}{2k_3} \ln|x_3| \right). \tag{10}$$

We suppose that the vector $\mathbf{q}$ is situated outside of a certain small neighborhoods of the subspaces l_j, and $\mathbf{x}$ tends to ∞ also remaining outside of these neighborhoods.

On the second step

B) We simplify the equation near the rays l_j . The simplified equations

$$-\Delta\psi + V_j(\mathbf{x})\psi = E\psi, \quad V_j(\mathbf{x}) = v(x_j) + 2v(\frac{\sqrt{3}}{2}y_j) \qquad (11)$$

have new potentials V_j such that on some vicinities of l_j the difference $V - V_j$ tends to zero at infinity faster that the Coulomb potential. At this vicinity the simplified equation allows the separation of variables.

On the step

C) We specify the solutions of the approximate equation that up to constant factors transfer at some growing distance from l_j to the Coulomb plane waves . Let us again denote such solution by

$$\chi_j(\mathbf{x}, \mathbf{q}). \tag{12}$$

Further, we combine such solutions and their continuations via the Coulomb plane waves and define the extended solution $\chi_j(\mathbf{x}, \mathbf{q})$ on the whole angle sectors K_j between the appropriate vectors $\mathbf{q}_\sigma, \mathbf{q}_{\sigma'}$ surrounding $\mathbf{l_j}$.

After that we can combine such solutions by constructing of their finite linear combination into the ray approximation ψ_R. This approximation has quickly vanishing at infinity discrepancy. It is also smooth everywhere except two rays $\mathbf{q_1} = \tau_{\mathbf{2}}\tau_{\mathbf{3}}\mathbf{q}$ and $\mathbf{q_3} = \tau_{\mathbf{2}}\tau_{\mathbf{1}}\mathbf{q}$ where the ray field has simple jumps.

On the last step

D) We modify the ray solution in neighborhoods of these two rays $\mathbf{q_1}$ and $\mathbf{q_3}$. In fact, we replace the discontinuous functions near these rays by a function smoothly connecting two different ray approximations on both sides, say, of $\mathbf{q_1}$. This function is not elementary. This completes the constructing of ψ_{as}.

Asymptotic structure of the wave field near the rays l_j. Separation of variables

Consider a neighborhood of the ray l_1^+. Let for brevity $x_1 = x, y_1 = y$. Near the ray

$$x_2 = -\frac{\sqrt{3}}{2}y - \frac{1}{2}x < 0, \quad x_3 = \frac{\sqrt{3}}{2}y - \frac{1}{2}x > 0. \tag{13}$$

It is clear that near the l_1^+ $|x| \ll y$, and the potential $V(\mathbf{x})$ can be simplified and replaced by the following expression

$$V(\mathbf{x}) = v(x_1) + v(x_2) + v(x_3) \sim v(x) + 2v(\frac{\sqrt{3}}{2}y). \tag{14}$$

Then we obtain the approximate equation

$$-\Delta\chi_0 + \left[v(x) + \frac{4\alpha}{\sqrt{3}|y|}\right]\chi_0 = E\chi_0. \tag{15}$$

It allows the separation of variables, and, as a result, has the solution of the form

$$\chi_0(\mathbf{x}, \mathbf{q}) = \xi(x, k) f(y, p), \tag{16}$$

where $\xi(x, k)$ and $f(y, p)$ are, in their turns, the solutions of the ordinary differential equations

$$-\xi'' + v\xi = k^2\xi, \qquad -f'' + \frac{4\alpha}{\sqrt{3}|y|} f = p^2 f. \tag{17}$$

Here, of course, $k^2 + p^2 = E$.

The solution of the scattering problem on the axis

Consider the solution $\xi(x,k)$ of the scattering problem for 1-dimensional Schrödinger equation $-\xi'' + v(x)\xi = k^2\xi, \quad x,k \in \mathbf{R}$.

To describe it more precisely introduce the function

$$\xi_c(x,k) = \exp\left[ikx - \frac{i\alpha}{2k} sgn(x) \ln|x|\right]. \tag{18}$$

With this notation the solution $\xi(x,k)$ can be characterized by the following asymptotic behavior as $|x| \to \infty$:

$$\xi(x,k) \sim s(k)\xi_c(x,k), \quad kx \to +\infty, \tag{19}$$

$$\chi(x,k) \sim \xi_c(x,k) + r(k)\xi_c^*(x,k), \quad kx \to -\infty. \tag{20}$$

The solution χ_1

Come back to the approximate equation .

As for ξ, we suppose that this is a solution of one-dimensional scattering problem. As for the solution f, we replace it by the leading term of the asymptotic behavior as $y \to +\infty$.

Therefore, on this stage we propose for $\chi_0(\mathbf{x}, \mathbf{q})$ the following approximation:

$$\chi_0(\mathbf{x}, \mathbf{q}) = \xi(x, k) \exp\left[ipy - i\frac{2\alpha}{\sqrt{3}p} \ln y\right]. \tag{21}$$

However, as $x \to \infty$, it does not transfer to a Coulomb plane wave.

The following approximation is more satisfactory :

$$\chi_1 = \left(\xi(x,k) + \frac{B}{y}\xi_k(x,k)\right) \exp\left[ipy + i\gamma_0 \ln y\right]. \qquad (22)$$

The numbers B, γ_0 have to be computed . In fact, they are equal to

$$B = \frac{\alpha}{2\sqrt{3}}\left(\frac{1}{k_2} + \frac{1}{k_3}\right), \qquad \gamma_0 = \alpha\frac{1}{2m_1}, \qquad (23)$$

$$\frac{1}{2m_1} = \frac{1}{2k_2} - \frac{1}{2k_3}. \qquad (24)$$

The comparison with the Coulomb plane waves

Let us explain how we come to this expression. We consider the approximate solution that is a superposition of the form

$$\chi(x,y) = \int_{\mathbf{R}} R(p^{'})\xi(x, \sqrt{E - p^{'2}})f(y, p^{'})dp^{'}. \tag{25}$$

We denote by V_ν the domain surrounding the ray $\mathbf{l_1}$ and bounded by two branches of the curve

$$|x| = y^\nu. \tag{26}$$

It is obvious that $V_\mu \in V_\nu, \nu > \mu$. We call the neighborhood of the ray the immediate neighborhood if $0 < \nu < 1$. On the immediate neighborhood the discrepancy of χ tends to 0 faster than the Coulomb potential.

Now let us seek its asymptotic behavior as $x, y \to \infty, |x| << y$. With a special choice of the density R the superposition can be transformed to the Coulomb plane wave $\psi_\sigma(\mathbf{x}, \mathbf{q})$. The comparison of the superposition with the behavior of the Coulomb plane wave as $|x| \to \infty, |x| \ll y$, defines the density of the superposition. Up to a constant factor

$$R = R_0 \left(p^{'} - p - i0\right)^{-1+i\alpha(\frac{2}{\sqrt{3}p} - 1/2(\frac{1}{k_2}+\frac{1}{k_3}))} . \qquad (27)$$

Let us introduce the symmetrical with respect to the ray some its *angle neighborhood* V. Let us consider on V a complement of $V_\mu, 1/2 < \mu < \nu$. It consists of two components V_+, V_-, on V_+ $x > 0$, on V_- $x < 0$. On this complement consider two Coulomb plane waves ψ_I, ψ_{τ_1}. Their discrepancies vanish at infinity faster than the Coulomb potential.

Let us combine two Coulomb plane waves in other almost solutions

$$\psi_+(\mathbf{x},\mathbf{q}) = s(k_1)\psi_I(\mathbf{x},\mathbf{q}), \tag{28}$$

$$\psi_-(\mathbf{x},\mathbf{q}) = \psi_{\tau_1}(\mathbf{x},\mathbf{q}) + r(k_1)\psi_{\tau_1}(\mathbf{x},\tau_1\mathbf{q}). \tag{29}$$

Here $s(k), r(k)$ are the transition and reflection coefficients for the one-dimensional equation.

Finally consider also two mutually symmetric with respect to the ray subdomains

$$D^{\pm}_{\mu,\nu} = V_{\pm} \cap V_{\nu}. \tag{30}$$

On $D^{\pm}_{\mu,\nu}$ the almost solutions χ and ψ_+, ψ_-, simultaneously have at infinity fast decaying discrepancies. Compare the solutions themselves on these sets. Let $k_1 > 0$. It is not hard to show that up to a common constant factor for $x > 0$

$$\chi_1 \sim s(k_1) e^{i<\mathbf{q},\mathbf{x}>} e^{[\frac{i\alpha}{2k_1} ln|x| + i\gamma_0 ln y]}. \tag{31}$$

As a result, on $D^{+}_{\mu,\nu}$

$$\chi_1 = \psi_+ + O(\frac{1}{x}). \tag{32}$$

Analogously, on $D^{-}_{\mu,\nu}$

$$\chi_1 = \psi_- + O((\frac{1}{x}).) \tag{33}$$

Resume: Almost solution on the angle neighborhood of l_1

Consider the covering of the angle neighborhood of the ray:

$$V = V_\nu \cap V_+ \cap V_-. \tag{34}$$

Consider the partition of the unit subordinated to this covering:

$$1 = \zeta_0 + \zeta_+ + \zeta_-. \tag{35}$$

Now we extend the almost solution χ_1 defined on the immediate neighborhood of the ray to the bigger angle neighborhood V of the ray:

$$\chi = \zeta_0 \chi_1 + \zeta_+ \psi_+ + \zeta_- \psi_-. \tag{36}$$

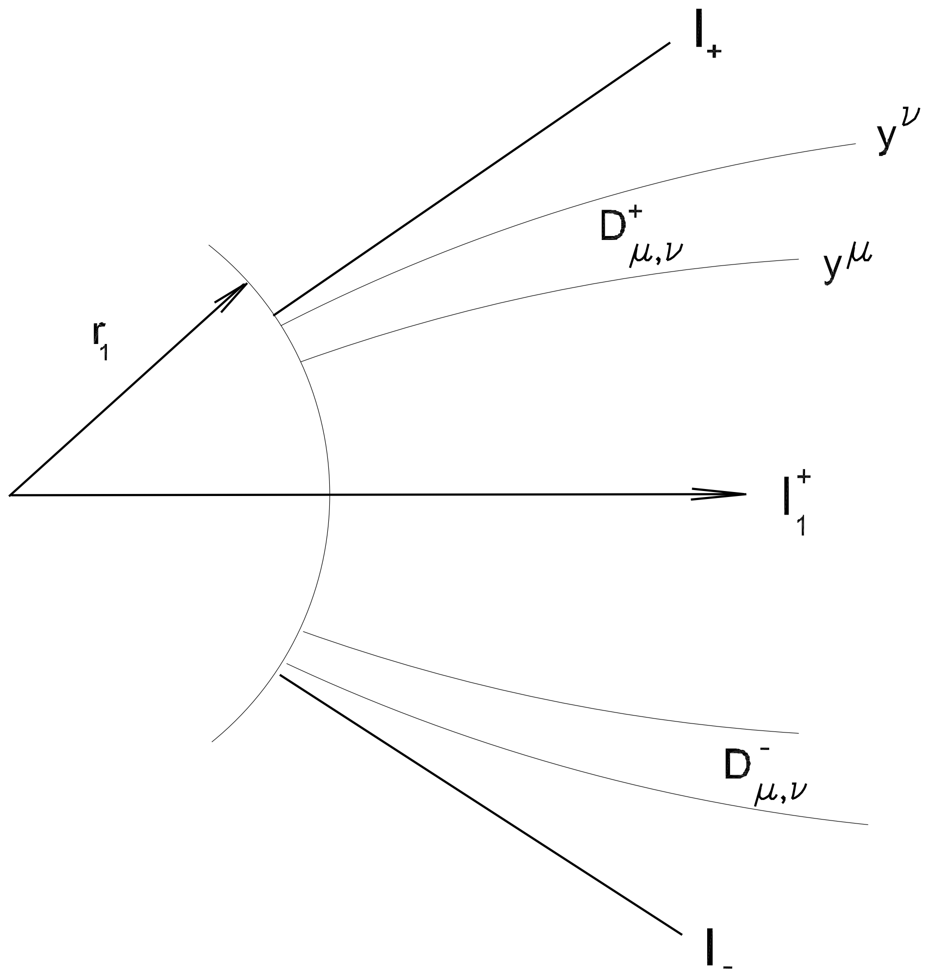
It is not hard to show now that on V

$$Q[\chi] = O(\frac{y^\mu}{y^2} + \frac{1}{y^{2\mu}}). \tag{37}$$

If we put μ, ν close to 2/3, and $\mu < \nu$, then the discrepancy will have the order

$$Q[\chi] = O(y^{-\delta}), \quad \delta < 4/3. \tag{38}$$

Thus, the discrepancy decreases at infinity faster than the Coulomb potential. The goal of the constructions is achieved.


I_+
y^ν
$D^+_{\mu,\nu}$
y^μ
r_1
I_1^+
$D^-_{\mu,\nu}$
I_-

Formulas for ψ_R

How to construct ψ_R?

In this section we will describe the ray approximation ψ_R to this function that will not have the satisfactory properties only on two from six rays . We will make the correction of the field on these two specific rays in the next section.

Let us recall that every sector $\lambda_\sigma, \sigma \in S$, contains exactly one vector of the form $\mathbf{q}_\sigma$.

Above, on the domain V, we constructed the almost solution χ. Let us extend this solution on the set $\widehat{V}$ that complements the domain V by the domain that is symmetric to V with respect to the straight line spanned by the vector $\mathbf{k}_1$.

Let us realize this extension with the help of the formula

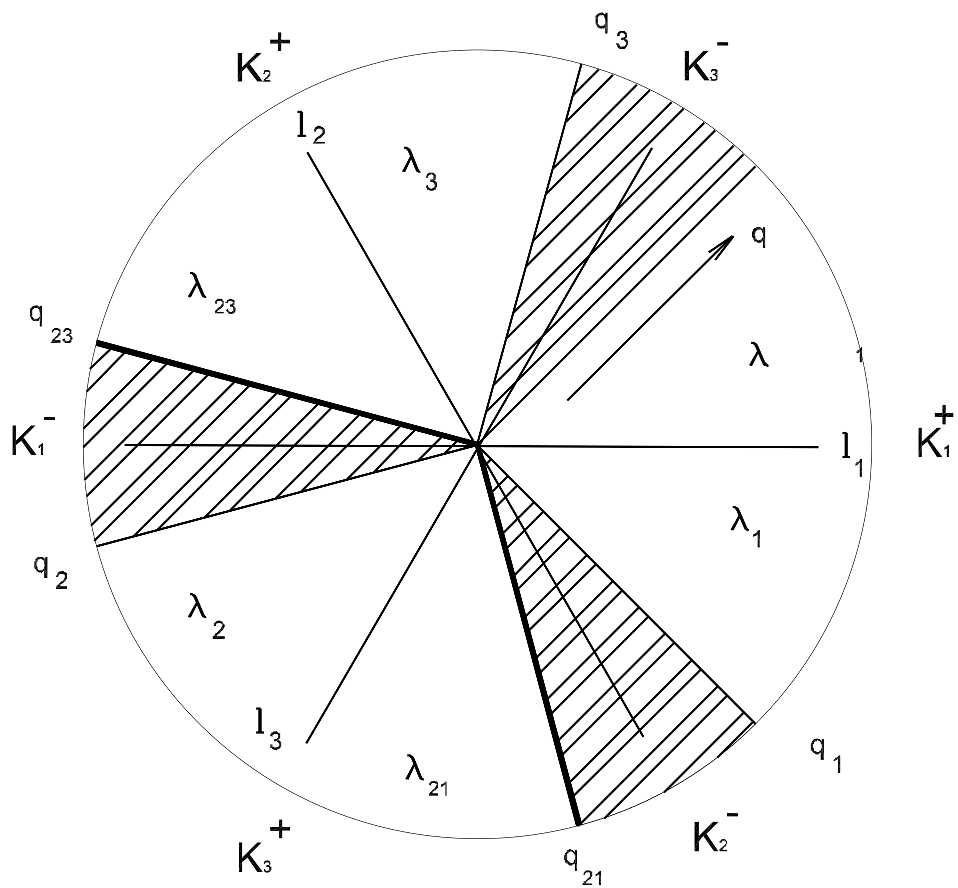
$$\chi(x, y, k_1, l_1) = \chi(x, -y, k_1, -l_1). \tag{39}$$

Let us keep the same notation χ for the field on the extended domain $\widehat{V}$.

Following the ideas of the work [1], introduce the functions

$$\chi_j(\mathbf{x}, \mathbf{q}) = \chi(x_j, y_j, k_j, l_j). \tag{40}$$

Between q_σ there lie six new sectors $K_j^\pm$.



q 3
K3 -
K2 +
l2
λ3
q
λ
1
q 23
λ 23
K1 -
l1
K1 +
λ1
q 2
λ2
l3
λ21
q 1
K3 +
q 21
K2 -

Inside each of them there is precisely one ray of the form $l_j^{\pm}$, its index is also used to denote the sector of type K. Let us describe the field ψ_R, that is the ray approximation to the asymptotic field ψ_{as}.

<u>Sector K_1^+:</u> $\psi_R = \chi_1^+$,

$$\chi_1^+(\mathbf{x}, \mathbf{q}) = \chi_1(\mathbf{x}, \mathbf{q}) s_2 s_3.$$

We used here the notation: $s_j = s(k_j), r_j = r(k_j)$.

<u>Sector K_3^-:</u> $\psi_R = \chi_3^-$,

$$\chi_3^-(\mathbf{x}, \mathbf{q}) = \chi_3(\mathbf{x}, \mathbf{q}) s_2 s_1.$$

Sector K_2^+: $\psi_R = \chi_2^+$,

$$\chi_2^+(\mathbf{x}, \mathbf{q}) = \chi_2(\mathbf{x}, \mathbf{q}) s_1 + \chi_2(\mathbf{x}, \tau_3 \mathbf{q}) s_2 r_3.$$

sector K_2^-: $\psi_R = \chi_2^-$,

$$\chi_2^-(\mathbf{x}, \mathbf{q}) = \chi_2(\mathbf{x}, \mathbf{q}) s_3 + \chi_2(\mathbf{x}, \tau_1 \mathbf{q}) s_2 r_1.$$

We won't analyze here the origin of every term, it can be easily recovered from the sequences of reflections and transitions that explicitly accompanying each term. Details can be found in [1].

Sector K_1^-: $\psi_R = \chi_1^-$,

$$\chi_1^-(\mathbf{x},\mathbf{q}) = \chi_1(\mathbf{x},\mathbf{q}) + \chi_1(\mathbf{x},\tau_2\mathbf{q}) r_2 s_1 + \chi_1(\mathbf{x},\tau_3\tau_1\mathbf{q}) r_2 r_1 + \chi_1(\mathbf{x},\tau_3\mathbf{q}) r_3.$$

sector K_3^+: $\psi_R = \chi_3^+$,

$$\chi_3^+(\mathbf{x},\mathbf{q}) = \chi_3(\mathbf{x},\mathbf{q}) + \chi_3(\mathbf{x},\tau_2\mathbf{q}) r_2 s_3 + \chi_3(\mathbf{x},\tau_1\tau_3\mathbf{q}) r_2 r_3 + \chi_3(\mathbf{x},\tau_1\mathbf{q}) r_1.$$

The full field χ_R is defined by the formula

$$\psi_R = \theta_1^+ \chi_1^+ + \theta_3^- \chi_3^- + \theta_2^+ \chi_2^+ + \theta_2^- \chi_2^- + \theta_1^- \chi_1^- + \theta_3^+ \chi_3^+.$$

We used here the notation $\theta_j^{(\pm)}$ for the characteristic function of the corresponding sector $K_j^{\pm}$,

$$\theta_1^+ + \theta_3^- + \theta_2^+ + \theta_2^- + \theta_1^- + \theta_3^+ = 1.$$

The last formula for ψ_R does not define the value of the field on the boundaries of the sectors. In the next subsection we will see that on all boundaries except two rays q_1, q_3, the ray field will be smooth.

The properties of smoothness of the field ψ_R

Notice that in the neighborhoods of all boundaries the field on both sides of the boundaries in the case of quickly decreasing potentials is reduced to finite linear combinations of the plane waves. The coefficients of these linear combinations are defined by the transition and the reflection coefficients of one-dimensional problem. These linear combinations taken on different sides of the boundary coincide on four boundaries from six. On two rest boundaries there are jumps of the coefficients before the plane waves with the wave vector oriented along the boundary of joining.

The discontinuous part of the ray field in the neighborhood of the boundary q_1, as it was shown in [1], is given by the expression

$$j_1 = (R_1\theta_2^+ + R_2\theta_1^-)e^{i<\mathbf{q}_1,\mathbf{x}>}. \qquad (41)$$

Here

$$R_1 = r_1 s_2 r_3, \quad R_2 = r_3 r_2 s_1 + s_3 r_2 r_1. \qquad (42)$$

The analogous formula is also true in the neighborhood of the boundary q_3.

In the case of the Coulomb potentials the plane waves have to be replaced by the Coulomb plane waves that we will denote, correspondingly, by $\psi_1(\mathbf{x},\mathbf{q}_1), \psi_1 = \psi_{\tau_2\tau_3}$, и $\psi_3(\mathbf{x},\mathbf{q}_3), \psi_3 = \psi_{\tau_2\tau_1}$.

Smoothing of the discontinuous solutions. The Cauchy integrals

So in a neighborhood of the ray q_1 we deal with the discontinuous almost solution

$$J_1 = (R_1\theta_2^+ + R_2\theta_1^-)\psi_1(\mathbf{x}, \mathbf{q}_1). \qquad (43)$$

Let us discuss the procedure that allows to smooth out the discontinuous almost solution J_1. First of all, let us introduce in the considered sector the polar coordinates r, η, and assume that $\eta \in [0, \pi/3], \eta = 0 \sim l_2^+$. Let the support of a function $f(\eta)$ belongs to the interval $[0, \pi/3]$.

Let us apply to such function the projection operators $P_\pm$ onto the subspaces of functions that are analytical in the upper and lower semi-planes of the complex plane :

$$(P_\pm f)(\eta) = \frac{\pm 1}{2\pi i} \int_{\mathbf{R}} \frac{d\eta'}{\eta' - (\eta \pm i0)} f(\eta'). \tag{44}$$

Our goal is to replace the discontinuous almost solutions by superpositions of the Coulomb plane waves that are almost solutions with the satisfactory discrepancy. We need also the following property of the superposition: when the point $\mathbf{x}$ recedes from the ray of the jump q_1 these superpositions interlock with the Coulomb plane waves that are the corresponding components of the ray approximation.

Consider the function

$$\phi_1^{(\pm)} = \frac{\pm 1}{2\pi i} \int_{\mathbf{R}} \frac{d\eta'}{\eta' - (\eta_0 \pm i0)} f(\eta') \psi_1(\mathbf{x}, \eta'). \tag{45}$$

As f we have to choose the function that is equal to 1 on the interval $[\alpha, \beta], 0 < \alpha < \beta < \pi/3$ and to 0 outside of the interval $[0, \pi/3]$. Further, $\psi_1(\mathbf{x}, \eta') = \psi_1(\mathbf{x}, \mathbf{q})$, and for $\mathbf{q}$ there are used the polar coordinates $(\sqrt{E}, \eta')$. These integrals are almost solutions of the Schrödinger equation with the discrepancies that satisfy to the same estimates as the Coulomb plane waves themselves.

Let us study the asymptotic behavior of these integrals as $r \to \infty$ in the angle $\eta \in (\alpha, \beta)$. The idea of the computation of the asymptotic behavior is sufficiently simple. Let compute the integrals more explicitly:

$$\phi_1^{(\pm)} = \frac{\pm 1}{2\pi i} \int_{\mathbf{R}} \frac{f(\eta')d\eta'}{\eta' - (\eta_0 \pm i0)} e^{[ir\sqrt{E}\cos(\eta'-\eta)+i\Delta(\mathbf{x},\eta')]}, \tag{46}$$

here $\mathbf{q}$ is the vector with the polar coordinates $\sqrt{E}, \eta'$. Let us make more precise the structure of the function $\Delta(\mathbf{x}, \eta')$:

$$\Delta(\mathbf{x}, \eta') = \ln r\gamma_1(\mathbf{q}(\eta')) + \gamma_2(\omega, \mathbf{q}(\eta')). \tag{47}$$

It is clear that the integrand have the stationary point η and the pole η_0. The asymptotic behavior of the integral depends on the mutual location of these points. If $r^{1/2}|\eta - \eta_0| \to \infty$, the contributions of the stationary point and the pole can be separated and generate a diverging Coulomb wave (with a certain amplitude) and the plane Coulomb wave.

More specifically, for $\eta_0 \lesseqgtr \eta$

$$\phi_1^{(\pm)} \sim \psi_1(\mathbf{x}, \mathbf{q}_1) + \frac{\pm e^{-i3/4\pi}}{\sqrt{2\pi|\mathbf{q}|}} \frac{1}{\eta - \eta_0} e^{i\gamma_2(\omega, \mathbf{q}(\eta))} R_c. \qquad (48)$$

For opposite relation of the inequalities of the solution the poles do not contribute to the asymptotic behavior, and the integrals behave asymptotically as the diverging circle Coulomb waves, that in these constructions play the role of the error : for $\eta_0 \lesseqgtr \eta$

$$\phi_1^{(\mp)} \sim \frac{\mp e^{-i3/4\pi}}{\sqrt{2\pi|\mathbf{q}|}} \frac{1}{\eta - \eta_0} e^{i\gamma_2(\omega,\mathbf{q}(\eta))} R_c. \tag{49}$$

These formulas show that for $\mathbf{x}$, that are close to q_1, one can replace $\theta_1^+ \psi_1$ by the almost solution ϕ_1^+, and $\theta_1^- \psi_1$ by the almost solution ϕ_1^-. In the next subsection we will describe the formulas that can realize this idea.

Smooth almost solution in the sectors λ_1, λ_3

Consider on the axis ω four points $\omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)}, \omega_4^{(1)}$, subordinated to the conditions:

$$0 < \delta < \omega_1^{(1)} < \omega_2^{(1)} < \omega_0 < \omega_3^{(1)} < \omega_4^{(1)} < \beta < \pi/3. \tag{50}$$

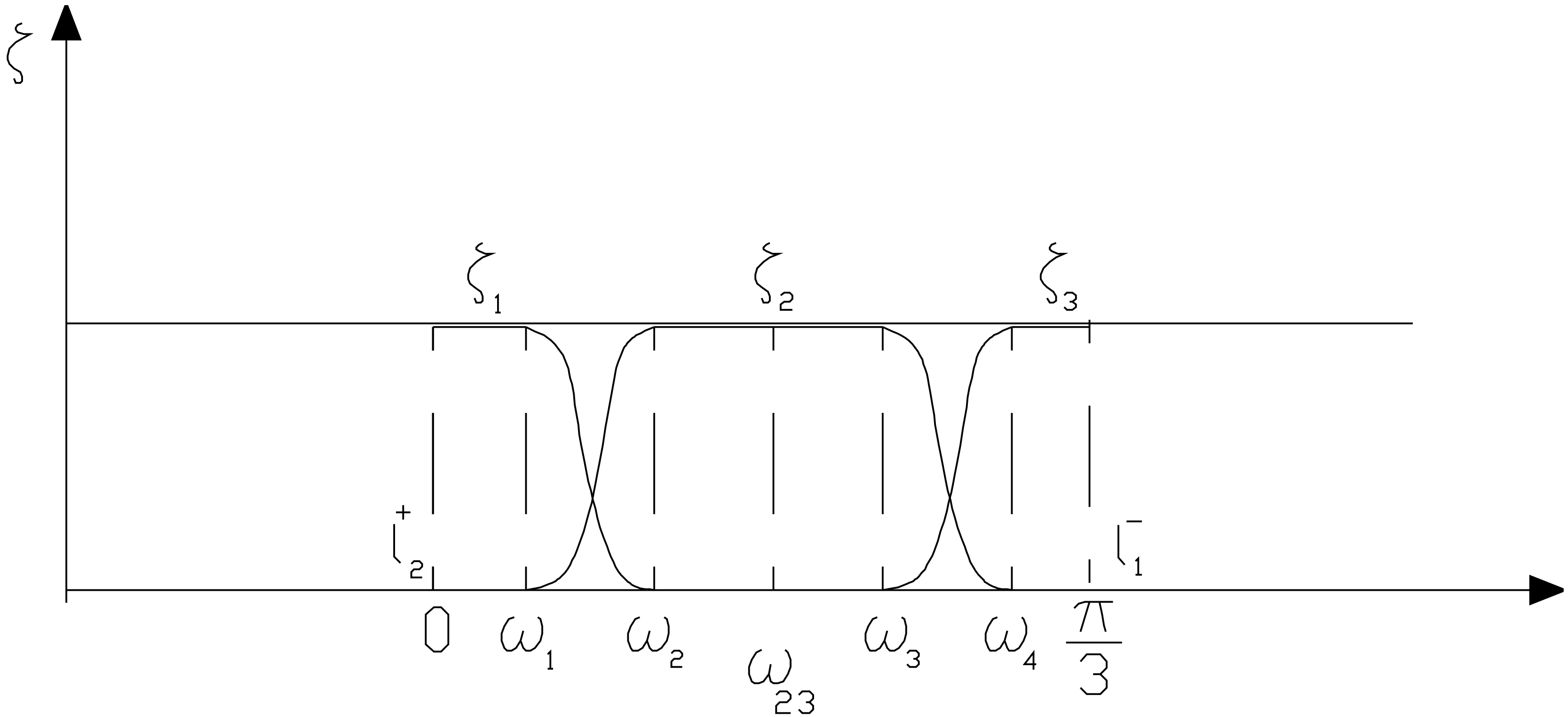
Consider the covering of $(0, \pi/3)$ by the subintervals

$$(0, \omega_2^{(1)}), (\omega_1^{(1)}, \omega_4^{(1)}), (\omega_3^{(1)}, \pi/3)$$

. Consider further the partition of the unit

$$1 = \zeta_1^{(1)} + \zeta_2^{(1)} + \zeta_3^{(1)}. \tag{51}$$

subordinated to this covering. We assume that the cut-off functions depend on the polar angle.


ζ
ζ1
ζ2
ζ3
0
ω1
ω2
ω23
ω3
ω4
π/3

Now replace the discontinuous part of the field J_1 in the considered sector λ_1 by the almost solution $\widetilde{J}_1$:

$$\widetilde{J}_1 = \zeta_1^{(1)} J_1 + \zeta_3^{(1)} J_1 + \zeta_2^{(1)} (R_1 \phi_1^{(+)} + R_2 \phi_1^{(-)}). \tag{52}$$

It is not hard to show that

$$Q[\widetilde{J}_1] = O(|\mathbf{x}|^{-2}). \tag{53}$$

We can easily repeat for the sector λ_3 the above constructions.

We now completed the constructing of the function ψ_{as}: everywhere except sectors λ_1, λ_3

$$\psi_{as} = \psi_R. \tag{54}$$

On these sectors, in their turn,

$$\psi_{as} = \psi_R + (\widetilde{J}_i - J_i). \tag{55}$$

On any directions the following uniform estimate

$$Q[\psi_{as}] = O(|\mathrm{x}|^{-\delta}), \quad \delta < 4/3, \tag{56}$$

is valid.

Let us formulate the theorem that, in fact, was proved in this paper :

Теорема 1. *The constructed function ψ_{as} possesses the following properties: 1) its discrepancy decreases at infinity faster than the Coulomb potential, and has the order $O(|\mathbf{x}|^{-\delta}), \quad \delta < 4/3$, 2) the difference*

$$\psi_{as}(\mathbf{x}, \mathbf{q}) - \psi_{\tau_2}(\mathbf{x}, \mathbf{q})\zeta(\omega), \tag{57}$$

where $\zeta(\omega)$ is a smooth cut off function concentrated in a neighborhood of the "back"wave vector $-\mathbf{q}$, asymptotically behaves as a "diverging Coulomb wave

$$\sim R_c h(\omega, \mathbf{q}). \tag{58}$$