Non-Weyl Asymptotics of Resonances on Graphs

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Complex Eigenvalues

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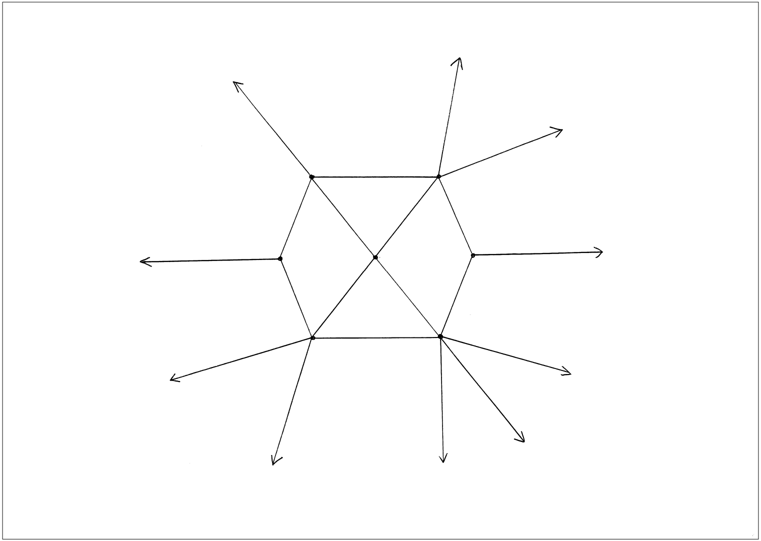
It all depends on the class of permitted solutions of the differential equation, which may be

$$-(\Delta f)(x) = k^2 f(x)$$

on some non-compact hyperbolic manifold or the Schrödinger equation in one dimension

$$-\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + V(x)f(x) = k^2 f(x).$$

Analysis on a Graph



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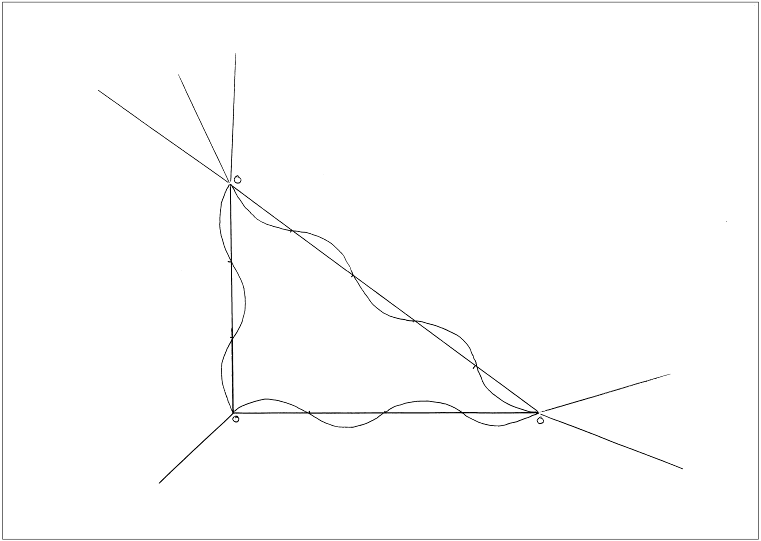
There may be eigenvalues embedded in the continuous spectrum. They are associated with very special closed loops.

Embedded eigenvalues



Definition of a Resonance

We are looking for a solution of

$$-\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = k^2 f(x).$$

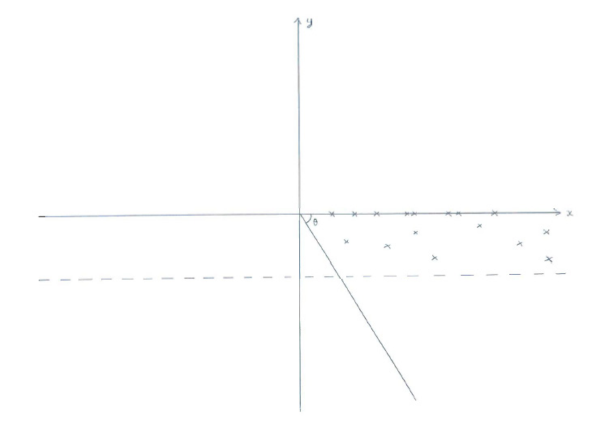
with $\mathrm{Im}\,(k) \leq 0$, that satisfies the Kirchhoff boundary condition and an outgoing wave condition.

Namely for each lead $[0, \infty)$ we require that

$$f(x) = f(0)\mathrm{e}^{ikx}$$

so f is purely exponentially increasing at infinity in every lead.

Another Definition of a Resonance



Resonances revealed by exterior complex scaling
on all of the external leads through an angle θ.

The Weyl Law

If there are no leads then the graph G has finite volume $|G|$ and discrete spectrum. If the eigenvalues are $\lambda = k^2$ where $k \geq 0$ then

$$N(r) = \#\{k : k \leq r\}$$

is given by Weyl's law

$$N(r) = \frac{|G|r}{\pi} + o(r).$$

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Can one prove a Weyl law for resonances?

Exponential Polynomials

In order to prove our main result, we need to consider the zeros of exponential polynomials, i.e. functions of the form

$$F(z) = \sum_{r=1}^{n} a_r \mathrm{e}^{i\sigma_r z} \qquad (1)$$

where $a_r \in \mathbf{C}$ and $\sigma_1 < \sigma_2 < \ldots < \sigma_n \in \mathbf{R}$.

For $r > 0$ we denote by $N(r; F)$ the number of zeros of F (counting multiplicity) in the disc $\{z \in \mathbf{C} : |z| < r\}$.

Exponential Polynomials

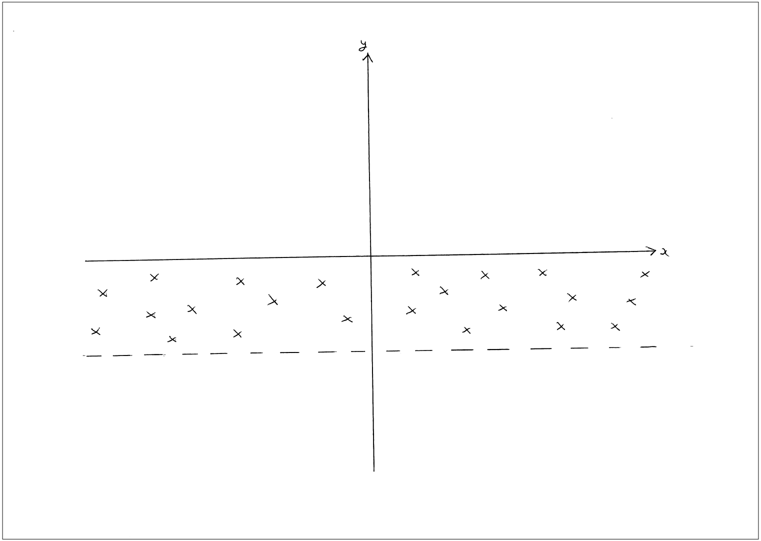
Theorem (Langer)

Let F be a function of the form (1)*, where $a_r \in \mathbf{C}$, $a_1 \neq 0$, $a_n \neq 0$ and $\sigma_1 < \sigma_2 < \ldots < \sigma_n \in \mathbf{R}$. Then there exists a constant $b < \infty$ such that all the zeros of F lie within a strip of the form $\{z : |\mathrm{Im}\,(z)| \leq b\}$. The counting function $N(r; F)$ satisfies*

$$N(r;F) = \frac{\sigma_n - \sigma_1}{\pi} r + O(1)$$

as $r \to +\infty$.

Zeros of an exponential polynomial



Getting Down to Business

Let us suppose that there are n edges, m leads, v vertices and r_i edges/leads connected to the ith vertex. Then

$$2n + m = \sum_{i=1}^{v} r_i$$

The number of variables we need to control is $2n + m$. Each edge e_i is associated with the function $\alpha_i \mathrm{e}^{ikx} + \beta_i \mathrm{e}^{-ikx}$. Each lead ℓ_i is associated with the function $\gamma_i \mathrm{e}^{ikx}$.

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The number of constraints is $\sum_{i=1}^{V} r_i$. Each vertex has $r_i - 1$ continuity conditions and 1 condition on the sum of the first derivatives.

More Business

If an edge e_i is labelled $(0, c_i)$ then at the ends of the edge the function is equal to $\alpha_i + \beta_i$ or $\alpha_i \mathrm{e}^{ikc_i} + \beta_i \mathrm{e}^{-ikc_i}$.

A similar calculation can be done for the derivatives at the ends of the edges and for the leads.

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The $2n + m$ constraints involve linear combinations of these quantities.

This leads to the problem of finding non-zero solutions of $Af = 0$ where f is a column vector with entries that involve $\alpha_i, \beta_i, \gamma_i$.

More Business

So in the end we have to evaluate a $(2n+m) \times (2n+m)$ determinant whose entries involve $\mathrm{e}^{\pm ikc_i}$.

This leads to a function of the form

$$F(k) = \sum_{r=1}^{n} a_r \mathrm{e}^{i\sigma_r k}$$

where each σ_i is a sum of some of the expressions $\pm c_j$.

The largest possible σ_i is $|G| = \sum_i c_i$ and the smallest possible is $-|G|$.

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So we appear to be finished, but we are not!

The Final Result

Theorem

The Weyl law for resonances holds if and only if every external vertex is unbalanced,i.e. the number of leads is not equal to the number of edges at that vertex.

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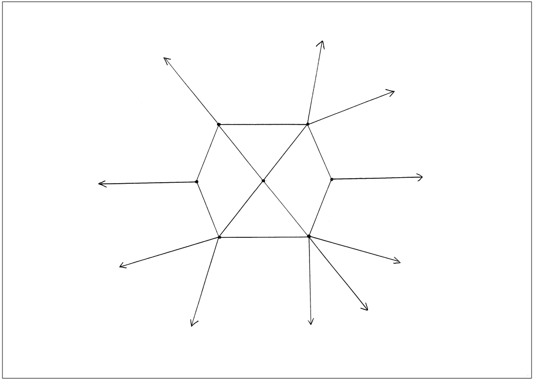
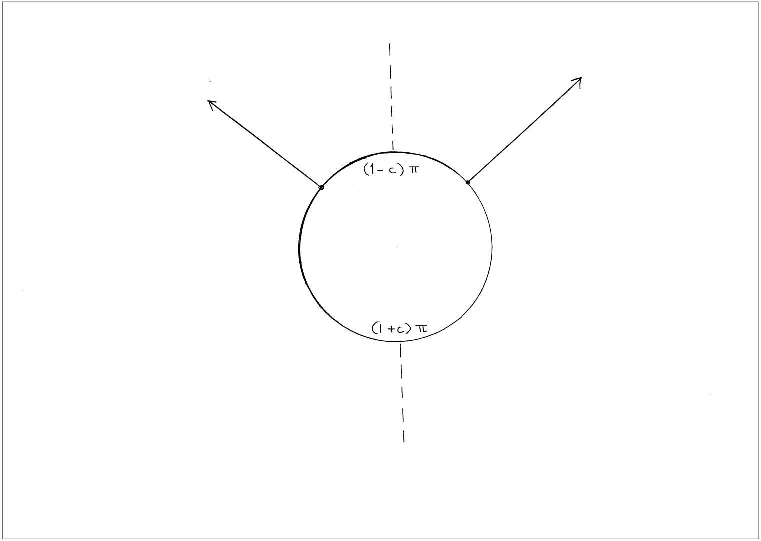


A simple example



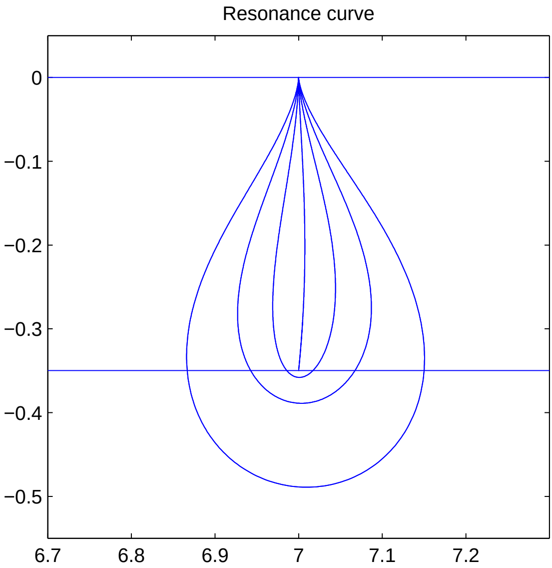
$$\det(A) = F_{\text{even}}(k,c)F_{\text{odd}}(k,c)$$

where

$$\begin{aligned} F_{\text{even}}(k,c) &= i\cos(kc\pi) + i\cos(k\pi) + 2\sin(k\pi), \\ F_{\text{odd}}(k,c) &= i\cos(kc\pi) - i\cos(k\pi) - 2\sin(k\pi). \end{aligned}$$

We will call the zeros of $F_{\text{even}}(\cdot,c)$ *(resp. of* $F_{\text{odd}}(\cdot,c)$*) the even (resp. odd) resonances.*

An odd resonances curve



An even resonances curve