Spaces

$$\Omega \subset\subset \mathbb{R}^3, \quad \Gamma := \partial\Omega \in C^\infty,$$

$$\Omega^s := \{x \in \Omega \,|\, \mathrm{dist}(x, \Gamma) < s\},$$

$$\Gamma^s := \{x \in \Omega \,|\, \mathrm{dist}(x, \Gamma) = s\}, \quad s \geq 0.$$

Solenoidal vector fields:

$$J := \{y \in \vec{L}_2(\Omega) \,|\, \mathrm{div}\, y = 0\},$$

$$J^s := \vec{L}_2 - \mathrm{clos}\,\{y \in J \,|\, \mathrm{supp}\, y \subset \Omega^s \cup \Gamma\}, \quad J^s \nearrow$$

P^s – orthogonal projection on J^s acting in J.
Tangential fields:

$$\vec{L}_{2,\theta} := \{u \in \vec{L}_2(\Omega) \,|\, u \cdot \nu = 0\}.$$

M-transform

$$M : J \to \vec{L}_{2,\theta}.$$

$$My \mid_{\Gamma^s} := P^s y \mid_{\Gamma^s - 0}, \quad y \in J \cap \vec{C}^\infty(\overline{\Omega}).$$

$$P^s y \in J^s \Rightarrow \quad (P^s y) \cdot \nu \mid_{\Gamma^s - 0} = 0 \Rightarrow \quad My \in \vec{L}_{2,\theta}.$$

M is similar to Fourier transform $F : \vec{L}_2(\mathbb{R}^3) \to \vec{L}_2(\mathbb{R}^3)$.

$$\operatorname{div} y = 0 \quad \Rightarrow \quad \langle F[y](k), k \rangle = 0,$$

i.e. F maps solenoidal vector fields to fields tangential to spheres centered at 0.

$$P^s y = \begin{cases} y - \nabla \varphi_y^s & in \quad \Omega^s \\ 0 & in \quad \Omega \setminus \Omega^s \end{cases} ; \quad \begin{cases} \Delta \varphi_y^s \mid_{\Omega^s} = 0, \\ \varphi_y^s \mid_\Gamma = 0, \quad \frac{\partial \varphi_y^s}{\partial \nu} \mid_{\Gamma^s} = y \cdot \nu \mid_{\Gamma^s} \end{cases}$$

$$My \mid_{\Gamma^s} = (y - \nabla \varphi_y^s) \mid_{\Gamma^s - 0} = y_\theta - (\nabla \varphi_y^s)_\theta \mid_{\Gamma^s - 0} .$$

Value of My on Γ^s depends on $y \mid_{\Gamma^s}$. Hence, J^t is mapped to $\vec{L}_{2,\theta}(\Omega^t)$ – tangential fields supported in Ω^t.

M-transform

For small enough t the constraint $M : J^t \to \vec{L}_{2,\theta}(\Omega^t)$ is unitar.
"Small enough t" means that Γ^s $(s < t)$ are smooth and diffeomorphic to Γ.
For arbitrary s surfaces Γ^s may have singularities, even the boundedness of M becomes non-trivial.
Assumption:

$$\Gamma^s \in \mathrm{Lip}, \quad a.e. \quad s \in (0, s_{max}).$$

(no assumptions concerning Lipschitz constants of Γ^s).

$$My = y_\theta - \nu \times \mathrm{curl} \int_0^T (\chi^s y - P^s y)\, ds.$$

$$My \,|_{\Gamma^s} = y_\theta - (\nabla \varphi_y^s)_\theta \,|_{\Gamma^{s-0}} \,.$$

M-transform

Theorem

M is partially isometric and

$$\mathrm{Ran}\, M = \vec{L}_{2,\theta}.$$

($\Leftrightarrow$ M^* is isometric).

The intertwining property:

$$MP^s = \chi^s M.$$

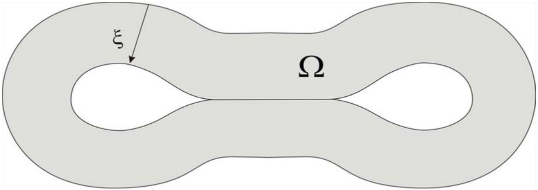
Let P^s_{sing} be a singular part of P^s.

$$P^s_{sing} J \subset \mathrm{Ker}\, M.$$

Ker M

P^s may have infinitely dimensional discontinuities:

$$\dim(P^{\xi+0} - P^{\xi}) = \infty \quad \Rightarrow \quad \dim \operatorname{Ker} M = \infty.$$



The Maxwell System

$$\varepsilon, \mu \in C^\infty(\overline{\Omega}), \quad \varepsilon, \mu > 0.$$

$e(\cdot, t), h(\cdot, t)$ – electric and magnetic fields in Ω.

$$\begin{array}{ll} e_t = \varepsilon^{-1} \mathrm{curl}\, h, \quad h_t = -\mu^{-1} \mathrm{curl}\, e & \text{in } \Omega \times (0, T) \\ e|_{t=0} = 0, \quad h|_{t=0} = 0 & \text{in } \Omega \\ e_\theta = f & \text{in } \Gamma \times [0, T] \end{array}$$

f is a smooth control on $\Gamma \times [0, T]$,
$f(\cdot, t)$ – tangential vector field on Γ,
$f = 0$ near $\Gamma \times \{t = 0\}$

$$\Rightarrow e^f, h^f \text{ – smooth classical solution in } \overline{\Omega} \times [0, T].$$

$$e^f(\cdot, t) \in J_\varepsilon, \quad h^f(\cdot, t) \in J_\mu.$$

Inverse Problem

$c = (\varepsilon\mu)^{-1/2}$ determines an optic metric $|dx|^2/c^2$ in Ω. The electromagnetic waves cover the subdomain $\Omega^T \subset \Omega$:

$$\Omega^s := \{x \in \Omega \,|\, \mathrm{dist}_c\,(x, \Gamma) < s\}, \quad s > 0.$$

The response operator:

$$R^T : f \mapsto -\nu \times h^f \,|_{\Gamma \times [0,T]} \,.$$

Due to finiteness of speed of wave propagation R^T depends on values of ε, μ only in $\Omega^{T/2}$.
Time-optimal setup of the IP:

Theorem

The data

$$\{R^T, \, c\,|_\Gamma, \, \frac{\partial c}{\partial \nu}\,|_\Gamma\}$$

determines c *in* $\Omega^{T/2}$ *uniquely.*

Belishev M.I., Glasman A.K. Dynamical inverse problem for the Maxwell system: recovering the velocity in a regular zone (the BC-method). *Algebra i Analiz*, 12 (2000), No 2, 131–182 (in Russian). English translation: *St Petersburg Math. J.*, 12 (2001), 279-316.