Positive polynomials and mapping of pseudospectra

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2nd St. Petersburg Conference in Spectral Theory

dedicated to the memory of M. Sh. Birman

July 14, 2010

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Polynomial calculus

$\mathcal{A}$ — a unitary C^*-algebra (for example, $B(H)$),

$$A \in \mathcal{A}, \quad \|A\| = 1, \quad \|[A, A^*]\| = \delta.$$

$$p \in \mathbb{C}[x_1, x_2],$$

$$p(z, \overline{z}) = \sum_{k+l \leqslant d} p_{kl} z^k \overline{z}^l, \quad z = x_1 + ix_2.$$

Let $p(A, A^*) = \sum_{k+l \leqslant d} p_{kl} A^k (A^*)^l.$

Simple properties of $p \mapsto p(A, A^*)$

- $\mathbb{C}$-linearity with respect to p.
- $\overline{p}(A, A^*) = (p(A, A^*))^*$.
- $\|p(A, A^*)q(A, A^*) - (pq)(A, A^*)\| \leqslant C(p, q)\delta$.

Main question

What is the relation between $\sigma(p(A, A^*))$ and $\sigma(A)$?

If $[A, A^*] = 0$ or $p = p(z)$, then $\sigma(p(A, A^*)) = p(\sigma(A))$.

Norm estimate

Theorem 1

Let $p \in \mathbb{C}[x_1, x_2]$. Then there exists such $C(p)$, that

$$\|p(A, A^*)\| \leqslant p_{\max} + C(p)\delta$$

for all $A \in \mathcal{A}$, $\|A\| \leqslant 1$, $\|[A, A^*]\| = \delta$. Here $p_{\max} = \max\limits_{|z|\leqslant 1} |p(z, \bar{z})|$.

If $[A, A^*] = 0$ or $p = p(z)$, then $C = 0$ (von Neumann inequality).

Resolvent estimates

$$g_0(z,\bar z) = 1 - |z|^2,\ g_i(z,\bar z) = |z-\lambda_i|^2 - R_i^2,\ i = 1,\dots,m-1.$$

$$S = \{z \in \mathbb{C}\colon g_i(z,\bar z) \geqslant 0,\ i = 0,\dots,m-1\}.$$

Theorem 2

Let $p \in \mathbb{C}[x_1, x_2]$. For any $\varepsilon > 0$, $\chi > 0$ there exist such $\delta_0 > 0$, $C(\varepsilon,\chi,p)$ that

$$\|(p(A,A^*)-\mu)^{-1}\| \leqslant \chi^{-1} + \varepsilon + C(\varepsilon,\chi,p)\delta, \quad \delta < \delta_0$$

for all $A \in \mathcal{A}$ and $\mu \in \mathbb{C}$, satisfying

$$\|A\| = 1,\ \|[A,A^*]\| = \delta,\ \|(A-\lambda_j)^{-1}\| \leqslant R_j^{-1},\ \mathrm{dist}(\mu, p(S)) \geqslant \chi.$$

Proof of Theorem 1

Theorem 3 (Scheiderer, 2006)

Let $p \in \mathbb{R}[x_1, x_2]$, $p(x_1, x_2) \geqslant 0$, $|x| \leqslant 1$. *Then there exist such* $r_j, s_j \in \mathbb{R}[x_1, x_2]$, $j = 0, \dots, N$, *that*

$$p = \sum_{j=0}^{N} r_j^2 + \left(\sum_{j=0}^{N} s_j^2 \right) \left(1 - |x|^2\right).$$

$p > 0$ — Cassier, 1973.

The proofs are non-constructive.

Applied to $q(z, \overline{z}) = p_{\max}^2 - |p(z, \overline{z})|^2$.

Proof of Theorem 1

$$q(A, A^*) \geqslant \sum_{j=0}^{N} r_j(A, A^*)^2 +$$

$$+ \left(\sum_{j=0}^{N} s_j(A, A^*) \left(1 - AA^*\right) s_j(A, A^*) \right) - C_1(p)\delta \geqslant -C_1(p)\delta,$$

$$p(A, A^*)^* p(A, A^*) \leqslant p_{\max}^2 + C_2(p)\delta.$$

Proof of Theorem 2

$$g_0(x_1, x_2) = 1 - |x|^2, \; g_i(x_1, x_2) = |x - \lambda_i|^2 - R_i^2, \; i = 1, \dots, m-1.$$

$$S = \{x \in \mathbb{R}^2 \colon g_i(x_1, x_2) \geqslant 0, \; i = 0, \dots, m-1\}.$$

Theorem 4

Let $p \in \mathbb{R}[x_1, x_2]$ be positive on S. Then there exist such $r_i, r_{ij} \in \mathbb{R}[x_1, x_2]$, $i = 0, \dots, m-1$, $j = 0, \dots, N$, that

$$p = \sum_{j=0}^{N} r_j^2 + \sum_{i=0}^{m-1} \left(\sum_{j=0}^{N} r_{ij}^2 \right) g_i.$$

Notes on Theorem 4

- Theorem 4 is a particular case of Putinar's Positivstellensatz (1993).
- Partially constructive proof: Nie, Schweighofer (2006).
- In our special case it becomes completely constructive.
- Positivity cannot be replaced with non-negativity. The counterexample of the form $g_i g_j$ shows that.

Mapping of pseudospectrum

$$\sigma_\varepsilon(A) = \{\lambda \in \mathbb{C}\colon \|(A-\lambda I)^{-1}\| > 1/\varepsilon\}.$$

$$S = \{z \in \mathbb{C}\colon |z| \leqslant 1,\ |z-\lambda_i| \geqslant R,\ i = 1, \ldots, m-1\}.$$

Theorem 5

Let $p \in \mathbb{C}[x_1, x_2]$. For any $\varepsilon > 0$ and $\chi > 0$ there exist $\delta_0 > 0$ and $C(\varepsilon, \chi, p)$ such that

$$\sigma_{\chi'}(p(A, A^*)) \subset B_\chi(p(S)),\ (\chi')^{-1} = \chi^{-1} + \varepsilon + C(\varepsilon, \chi, p)\delta,\ \delta < \delta_0.$$

Proof of Theorem 4

Lojasiewicz inequality:

$$\mathrm{dist}(x, S) \leqslant -c_0 \min\{g_0(x), \dots, g_{m-1}(x)\}, \quad x \notin S.$$

$$p(x) - C_1 \sum_{i=0}^{m-1} (g_i(x) - 1)^{2k} g_i(x) > 0, \quad x \in [-1;1]^2.$$

Theorem 6 (Powers, Reznick (2001))

Let $f \in \mathbb{R}[y_1, \dots, y_n]$ *be a homogenious polynomial,* $\deg f = d$*. Let* $f(y) \geqslant f^* > 0$ *on* $\Delta_n = \{y \in \mathbb{R}^n \colon y_i \geqslant 0, \sum_i y_i = 1\}$*. Then for*

$$N > \frac{d(d-1)\|f\|}{2f^*} - d$$

all the coefficients of $(y_1 + \ldots + y_n)^N f(y_1, \dots, y_n)$ *are positive.*