Decorrelation estimates for the eigenlevels of random operators in the localized regime

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Outline

The Anderson model in the localized regime

On $\ell^2(\mathbb{Z}^d)$, we consider the Anderson model $H_\omega = -\Delta + V_\omega$ where $V_\omega = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma \pi_\gamma$ and

- $-\Delta$ is the standard discrete Laplacian,
- π_γ is the orthogonal projector on δ_γ,
- the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are non trivial, i.i.d. bounded and admit a bounded density.

Well known : there exists a set, say $I \subset \mathbb{R}$, such that, in I, the spectrum of H_ω is localized.

Pick $E \in I$ and $L \in \mathbb{N}$. Let $\Lambda = \Lambda_L = [-L, L]^d \cap \mathbb{Z}^d \subset \mathbb{Z}^d$ and $H_\omega(\Lambda) = H_{\omega|\Lambda}$ (per. BC).

Denote its eigenvalues by $E_1(\omega,\Lambda) \leq E_2(\omega,\Lambda) \leq \cdots \leq E_N(\omega,\Lambda)$.

Integrated density of states: $N(E) = \lim_{N\to\infty} \frac{1}{N} \max\{j; E_j(\omega,\Lambda) \leq N\}$.

Density of states $\nu(E) = N'(E)$.

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Local level statistics near E:

$$\Xi(\xi,E,\omega,\Lambda)=\sum_{j=1}^{N}\delta_{\xi_j(E,\omega,\Lambda)}(\xi)\quad\text{where}\quad \xi_j(E,\omega,\Lambda)=|\Lambda|\,\nu(E)\,(E_j(\omega,\Lambda)-E).$$

Theorem (Molchanov,Minami,Germinet-K.)

Assume that $\nu(E)>0$. When $|\Lambda|\to+\infty$, the point process $\Xi(,\omega,\Lambda)$ converges weakly to a Poisson process on $\mathbb{R}$ with intensity the Lebesgue measure.

Question: pick $E_0\in I$ and $E_0'\in I$ such that $E_0\neq E_0'$, $\nu(E_0)>0$ and $\nu(E_0')>0$;

Are the point processes $\Xi(E_0,\omega,\Lambda)$ and $\Xi(E_0',\omega,\Lambda)$ asymptotically independent?

Not much known about this question for random Schrödinger operators.

Results for random matrices.

The answer may be model dependent:

$$\begin{pmatrix}\omega_1 & 0 & \cdots & 0\\ 0 & \omega_2 & 0 & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & \omega_{2n}\end{pmatrix}\qquad \begin{pmatrix}\omega_1 & 0 & 0 & \cdots & 0\\ 0 & \omega_1+1 & 0 & \cdots & 0\\ \vdots & 0 & \omega_2 & 0 & \vdots\\ \vdots & & & \ddots & 0\\ 0 & 0 & \cdots & 0 & \omega_n+1\end{pmatrix}$$

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Results for random matrices.

The answer may be model dependent:

$$\begin{pmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & 0 & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \omega_{2n} \end{pmatrix} \qquad \begin{pmatrix} \omega_1 & 0 & 0 & \cdots & 0 \\ 0 & \omega_1+1 & 0 & \cdots & 0 \\ \vdots & 0 & \omega_2 & 0 & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \omega_n+1 \end{pmatrix}$$

Local level statistics near E:

$$\Xi(\xi, E, \omega, \Lambda) = \sum_{j=1}^{N} \delta_{\xi_j(E,\omega,\Lambda)}(\xi) \quad \text{where} \quad \xi_j(E,\omega,\Lambda) = |\Lambda|\,\nu(E)\,(E_j(\omega,\Lambda) - E).$$

Theorem (Molchanov,Minami,Germinet-K.)

Assume that $\nu(E) > 0$. When $|\Lambda| \to +\infty$, the point process $\Xi(,\omega,\Lambda)$ converges weakly to a Poisson process on $\mathbb{R}$ with intensity the Lebesgue measure.

Question: pick $E_0 \in I$ and $E_0' \in I$ such that $E_0 \neq E_0'$, $\nu(E_0) > 0$ and $\nu(E_0') > 0$;

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The independence

Theorem (Ge-Kl,Kl)

Assume that the dimension $d=1$. When $|\Lambda|\to+\infty$, the point processes $\Xi(E_0,\omega,\Lambda)$ and $\Xi(E_0',\omega,\Lambda)$ converge weakly respectively to two independent Poisson processes on $\mathbb{R}$ with intensity the Lebesgue measure. That is, for $U_+\subset\mathbb{R}$ and $U_-\subset\mathbb{R}$ compact intervals and $\{k_+,k_-\}\in\mathbb{N}\times\mathbb{N}$, one has

$$\mathbb{P}\left(\left\{\omega;\left\{\begin{array}{c}\#\{j;\xi_j(E_0,\omega,\Lambda)\in U_+\}=k_+\\ \#\{j;\xi_j(E_0',\omega,\Lambda)\in U_-\}=k_-\end{array}\right\}\right)\underset{\Lambda\to\mathbb{Z}^d}{\to} e^{-|U_+|}\frac{|U_+|^{k_+}}{k_+!}\cdot e^{-|U_-|}\frac{|U_-|^{k_-}}{k_-!}.$$

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Pick $E_0\in I$ and $E_0'\in I$ such that $|E_0-E_0'|>2d$, $\nu(E_0)>0$ and $\nu(E_0')>0$. When $|\Lambda|\to+\infty$, the point processes $\Xi(E_0,\omega,\Lambda)$ and $\Xi(E_0',\omega,\Lambda)$ converge weakly respectively to two independent Poisson processes on $\mathbb{R}$ with intensity the Lebesgue measure.

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The decorrelation lemmas

Lemma (Kl)

For the discrete Anderson model , fix $\alpha \in (0,1)$, $\beta \in (1/2,1)$ and $\{E_0, E_0'\} \subset I$ s.t. $|E_0 - E_0'| > 2d$, for any $c > 0$, there exists $C > 0$ such that, for $L \geq 3$ and $cL^\alpha \leq \ell \leq L^\alpha/c$, one has

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Let $J_L = E_0 + L^{-d}(-1,1)$ and $J'_L = E'_0 + L^{-d}(-1,1)$.
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Basic idea: find random variables $(\omega_\gamma, \omega_{\gamma'})$ such that
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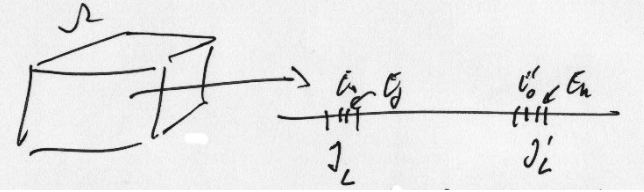

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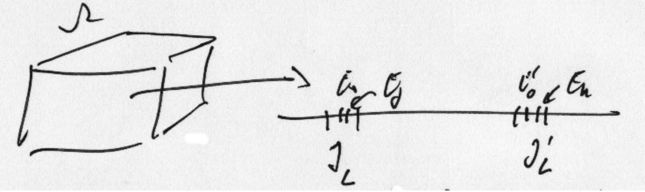

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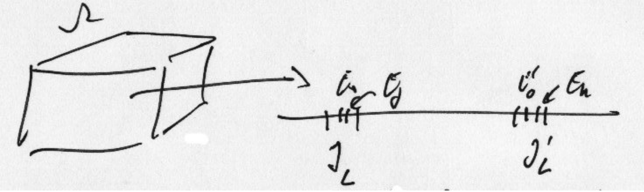

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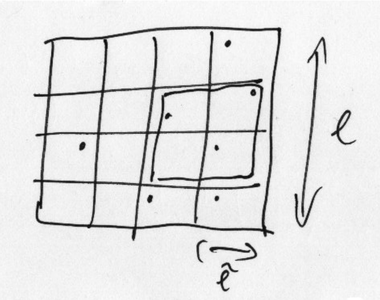

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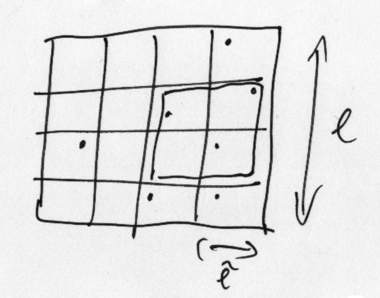

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2. *in dimension 1: fix* $E < E'$ *and* $\beta > 1/2$; *let* $\mathbb{P}$ *denote the probability that there exists* $E_j(\omega)$ *and* $E_k(\omega)$, *simple eigenvalues of* $H_\omega(\Lambda_L)$ *such that* $|E_k(\omega) - E| + |E_j(\omega) - E'| \leq e^{-L^\beta}$ *and such that*

$$\|\nabla_\omega (E_j(\omega) - E_k(\omega))\|_1 \leq e^{-L^\beta};$$

then, there exists $c > 0$ *such that* $\mathbb{P} \leq e^{-cL^{2\beta}}$.

To estimate the $\mathrm{Jac}(\psi)$, need to show that $\nabla_\omega E(\omega)$ and $\nabla_\omega E'(\omega)$ not colinear as

Lemma

Pick $(u,v) \in (\mathbb{R}^+)^{2n}$ *such that* $\|u\|_1 = \|v\|_1 = 1$. *Then* $\max_{j\neq k} \begin{vmatrix} u_j & u_k \\ v_j & v_k \end{vmatrix}^2 \geq \frac{1}{2n^3}\|u-v\|_1^2$.

Difficulty : gradient may be colinear e.g. for $\omega = 0$.

The fundamental estimate:

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Completing the proof of the decorrelation lemma

One now has $\mathbb{P}_\varepsilon \leq \sum_{\gamma\neq\gamma'} \mathbb{P}(\Omega_{0,v}^{\gamma,\gamma'}(\varepsilon)) + \mathbb{P}_r$ where

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Summing over $(\gamma,\gamma') \in \Lambda_{\tilde{\ell}}^2$, we obtain

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Proof is complete.

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The proof of the fundamental estimate: case 1

$E_j(\omega)$ and $E_k(\omega)$ simple evs of $H_\omega(\Lambda_L)$ such that $|E_k(\omega) - E_j(\omega)| \geq \Delta E > 2d$.

Then, $\omega \mapsto E_j(\omega)$ and $\omega \mapsto E_k(\omega)$ are real analytic functions.

Let $\omega \mapsto \varphi_j(\omega)$ and $\omega \mapsto \varphi_k(\omega)$ be normalized eigenvec. ass. resp. to $E_j(\omega)$ and $E_k(\omega)$.

Differentiating the eigenvalue equation in ω, one computes

$$\begin{aligned}\omega \cdot \nabla_\omega (E_j(\omega) - E_k(\omega)) &= \langle V_\omega \varphi_j(\omega), \varphi_j(\omega)\rangle - \langle V_\omega \varphi_k(\omega), \varphi_k(\omega)\rangle \\ &= E_j(\omega) - E_k(\omega) + \langle -\Delta\varphi_k(\omega), \varphi_k(\omega)\rangle - \langle -\Delta\varphi_j(\omega), \varphi_j(\omega)\rangle.\end{aligned}$$

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The proof of the fundamental estimate: case 2

Let us now assume $d = 1$. We prove a weaker result.

Theorem

Fix $\nu > 8$. For the discrete Anderson model in dimension 1, there exists $\Delta\mathscr{E}$ of total measure such that, for $E - E' \in \Delta\mathscr{E}$, for L sufficiently large, if $E_j(\omega)$ and $E_k(\omega)$ are simple eigenvalues of $H_\omega(\Lambda_L)$ such that $|E_k(\omega) - E| + |E_j(\omega) - E'| \leq L^{-\nu}$ then $\|\nabla_\omega(E_j(\omega) - E_k(\omega))\|_1 \geq L^{-\nu}$;

Fix $E < E'$. Pick $E_j(\omega)$ and $E_k(\omega)$, simple evs s.t. $|E_k(\omega) - E| + |E_j(\omega) - E'| \leq L^{-\alpha}$. Then,

$$4L^{-2\nu} \geq \|\nabla_\omega(E_j(\omega) - E_k(\omega))\|_2^2 = \sum_{\gamma \in \Lambda_L} |\varphi_\gamma^j(\omega) - \varphi_\gamma^k(\omega)|^2 \cdot |\varphi_\gamma^j(\omega) + \varphi_\gamma^k(\omega)|^2$$

there exists a partition of Λ_L, say $\mathscr{P} \subset \Lambda_L$ and $\mathscr{Q} \subset \Lambda_L$ s.t.

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Introduce the orthogonal projectors P and Q defined by

$$P = \sum_{\gamma \in \mathscr{P}} |\gamma\rangle\langle\gamma| \quad \text{and} \quad Q = \sum_{\gamma \in \mathscr{Q}} |\gamma\rangle\langle\gamma|.$$

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Let us now assume $d = 1$. We prove a weaker result.

Theorem

Fix $\nu > 8$. For the discrete Anderson model in dimension 1, there exists $\Delta\mathscr{E}$ of total measure such that, for $E - E' \in \Delta\mathscr{E}$, for L sufficiently large, if $E_j(\omega)$ and $E_k(\omega)$ are simple eigenvalues of $H_\omega(\Lambda_L)$ such that $|E_k(\omega) - E| + |E_j(\omega) - E'| \leq L^{-\nu}$ then $\|\nabla_\omega(E_j(\omega) - E_k(\omega))\|_1 \geq L^{-\nu}$;

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One has $\|P\varphi^j - P\varphi^k\|_2 \le L^{-\nu+d/2}$ and $\|Q\varphi^j + Q\varphi^k\|_2 \le L^{-\nu+d/2}$.

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Plugging this into the eigenavalue equations yields

$$\begin{cases} [-(P\Delta Q + Q\Delta P) - \Delta E]u & = O(L^{-\alpha}) \\ [-(P\Delta P + Q\Delta Q) + V_\omega - \overline{E}]u & = O(L^{-\alpha}), \end{cases}$$

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For $\mathscr{A}\subset\Lambda_L$ we define $\mathscr{A}+1=\{p+1;\ p\in\mathscr{A}\}$ to be the shift by one of $\mathscr{A}$.

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Consider the set $\mathscr{C}:=\partial\mathscr{P}\cup\partial\mathscr{Q}$.

Partition it into its “connected components” i.e. $\mathscr{C}$ can be written a a disjoint union of intervals of integers, say $\mathscr{C}=\cup_{l=1}^{l_0}\mathscr{C}_l^c$.

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Each of these operators : Dirichlet Laplacian on interval of length, the length of C_l.

Its spectral decomposition can be computed explicitly: for segment of length n,

- the eigenvalues are simple and are given by $(2\cos(k\pi/(n+1)))_{1\leq k\leq n}$;
- for $k \in \{1,\cdots,n\}$, the eigenspace associated to $2\cos(k\pi/(n+1))$ is generated by the vector $(\sin(kj\pi/(n+1))_{1\leq j\leq n}$.

Let $\Delta\mathscr{E}_L^c = \cup_{n=0}^{L}\sigma(-C_n\Delta C_n) + [-L^{-\nu}, L^{-\nu}]$ then $|\cap_{n\geq 1}\cup_{L\geq n}\Delta\mathscr{E}_L^c| = 0$.

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$\Delta\mathscr{E} =^{c} (\cap_n \cup_{L\ge n} \Delta\mathscr{E}_L^{c})$ is of total measure.

This completes the proof.

Then

$$-P\Delta Q - Q\Delta P = -\sum_{l=1}^{l_0} C_l \Delta C_l$$

where C_l is the projector $C_l = \sum_{\gamma \in \mathscr{C}_j} |\gamma\rangle\langle\gamma|$.

The projectors C_l and $C_{l'}$ are orthogonal to each other for $l \neq l'$.

So the spectrum of $-P\Delta Q - Q\Delta P$ is given by the union of the spectra of $(C_l \Delta C_l)_{1 \le j \le J}$.

Each of these operators : Dirichlet Laplacian on interval of length, the length of C_l.

Its spectral decomposition can be computed explicitly: for segment of length n,

- the eigenvalues are simple and are given by $(2\cos(k\pi/(n+1)))_{1 \le k \le n}$;
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