KdV flow on the space of generalized reflectionless potentials

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Contents

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- **Weyl functions:**

$$m_+(E) = \frac{f_+'(0,E)}{f_+(0,E)}, \quad m_-(E) = -\frac{f_-'(0,E)}{f_-(0,E)}.$$

$m_\pm(E)$: Herglotz functions (holomorphic functions on $\mathbb{C}_+$ with positive imaginary parts)

Random Schrödinger operators

- Dynamical system $\{T_x\}_{x\in\mathbb{R}}$ on $(\Omega, \mathcal{F}, P)$:

$$\begin{cases} T_x : \Omega \to \Omega \ \text{ with } \quad T_{x+y} = T_x T_y \\ P(T_x A) = P(A) \ \text{ for } \forall x \in \mathbb{R}, \forall A \in \mathcal{F} \end{cases}$$

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Basic facts

- Classification of spectrum Σ of L:

$$\begin{cases} \Sigma = \Sigma_{ac}(\text{absolutely continuous sp.}) \cup \Sigma_s(\text{singular sp.}) \\ \Sigma_s = \Sigma_p(\text{point sp.}) \cup \Sigma_{sc}(\text{singular continuous sp.}) \end{cases}$$

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- **Identity on Σ_{ac}:** For all $\omega \in$ *supp* P,

$$m_+^\omega(E+i0) = -\overline{m_-^\omega(E+i0)} \ \text{ for a.e. } E \in \Sigma_{ac}$$

$$\Updownarrow$$

$$\text{For all } x \in \mathbb{R},\ \ \mathrm{Re}\, g_E^\omega(x,x) = 0 \ \text{ for a.e. } E \in \Sigma_{ac}$$

Especially for periodic potentials, this identity holds on Σ_{ac}, or equivallently on each interval of stability.

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- If $\Sigma = \Sigma_{ac} = \bigcup_{j=1}^{n} I_j$ (disjoint sum of intervals)
  - $\Longrightarrow q^{\omega}(x)$ is described by using $\Theta-$functions.

Reflectionless potentials

- If q satisfies $\int_{\mathbb{R}} |q(x)| (1 + |x|)\, dx < \infty$, then

$$\exists 1 f \text{ s.t. } \ Lf = k^2 f, \ \ f(x) \sim \begin{cases} e^{-ikx} & \text{as} \quad x \to -\infty \\ \frac{1}{t(k)} e^{-ikx} + \frac{r(k)}{t(k)} e^{ikx} & \text{as} \quad x \to \infty \end{cases}$$

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- Especially periodic potential is reflectionless on its spectrum.

- Space of reflectionless potentials:

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  - Ω_{cl}, {potentials with finite band spectrum} $\subset \Omega$ are dense.

Sato's Grassmannian manifold

- Let $H = L^2(|z| = 1)$ and introduce

$$H_\pm = H \cap \left\{ f = \sum_{\substack{n \geq 0 \\ (n \leq -1)}} f_n z^n \right\}.$$

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  - (iii) $P_{H_+} : W \to H_+$ bijective

Quick view of derivation of the equations

- Set $g_x(z) = e^{xz}$ and assume $g_x W \in Gr^{(2)}(H)$. Then $P_{H_+}(g_x W) = H_+$ implies

$$\exists 1 f(x,\cdot) \in W \text{ s.t. } e^{xz} f(x,z) = 1 + a_1(x) z^{-1} + a_2(x) z^{-2} + \cdots$$

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  $e^{xz}\left(f''(x,z) - z^2 f(x,z) + 2a_1'(x) f(x,z)\right) = \boxdot z^{-1} + \boxdot z^{-2} + \cdots$
  $\Rightarrow P_{H_+}\left(e^{g_x}\left(f''(x,\cdot) - z^2 f(x,\cdot) + 2a_1'(x) f(x,\cdot)\right)\right) = 0$
  $\Rightarrow f''(x,z) - z^2 f(x,z) + 2a_1'(x) f(x,z) = 0$ (P_{H_+} is bijective)
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- Set $g_{x,t}(z) = e^{xz-4tz^3}$ and assume $g_{x,t} W \in Gr^{(2)}(H)$. Similarly

$$e^{-xz+4tz^3} f(t,x,z) = 1 + a_1(t,x) z^{-1} + a_2(t,x) z^{-2} + \cdots,$$

and $q(t,x) = -2a_1'(t,x)$ fulfills KdV eq..

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$$\tau_W(g_1g_2) = \tau_W(g_1)\tau_{g_1^{-1}W}(g_2) \quad \text{for} \quad g_1, g_2 \in \Gamma$$
- Note
$$g^{-1}W \in Gr^{(2)}(H) \Longleftrightarrow \tau_W(g) \neq 0.$$

- For $\zeta \in \mathbb{C}$ such that $|\zeta| > 1$ set $q_\zeta(z) = 1 - \frac{z}{\zeta}$. Then, for $f \in H_-$ and $|z| < 1$ we have

$$\left(q_\zeta P_{H_+} q_\zeta^{-1} f\right)(z) = \left(1 - \frac{z}{\zeta}\right) \frac{1}{2\pi i} \int_{|\zeta'|=1} \frac{f(\zeta')}{\left(1 - \frac{\zeta'}{\zeta}\right)(\zeta' - z)} d\zeta' = f(\zeta).$$

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- $a_1(x) = \lim_{\zeta \to \infty} \zeta \left(\tau_{e^{x\cdot}W}(q_\zeta) - 1\right) = \lim_{\zeta \to \infty} \zeta \left(\frac{\tau_W\left(e^{-x\cdot} q_\zeta\right)}{\tau_W\left(e^{-x\cdot}\right)} - 1\right)$
$= \dfrac{\partial}{\partial x} \log \tau_W(e^{-x\cdot})$.

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- We denote by $Gr^{(2)}_{\mathrm{real}}(H)$ the set of all W satisfying the conditions of this theorem.

KdV flow

- For $W \in Gr^{(2)}(H)$, assume $g^{-1}W \in Gr^{(2)}(H)$ $(\Leftrightarrow \tau_w(g) \neq 0)$ for $g \in \Gamma$,

$$(K(g)q)(x) = -2\frac{\partial^2}{\partial x^2}\log \tau_{g^{-1}W}(e^{-x\cdot}) = -2\frac{\partial^2}{\partial x^2}\log \tau_W(e^{-x\cdot}g).$$

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  2. $u(t, x+\cdot) = \left(K\left(e^{-xz+4tz^3}\right)q\right)(\cdot)$ satisfies KdV equation:

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x}.$$

Structure of W and potentials

- For $W \in Gr^{(2)}(H)$, there exist unique functions of W such that
$$W \ni \Phi(z) = 1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \Psi(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots.$$
From the property (iii) it follows that
$$\left\{ p\left(z^2\right)\Phi(z) + q\left(z^2\right)\Psi(z),\ p, q \ \text{ polynomials} \right\}$$
generates W. Therefore we call them **characteristic system** of $W \in Gr^{(2)}(H)$ and denote them by $\{\Phi_W(z), \Psi_W(z)\}$. Set
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Theorem

If $W \in Gr^{(2)}_{\text{real}}(H)$, then $q_W \in \Omega$ and $m^{\pm}_{q_W}(z) = \mp a_1 \mp M_W(\pm\sqrt{z})$.
Moreover, for $W_1, W_2 \in Gr^{(2)}_{\text{real}}(H)$, $q_{W_1} = q_{W_2}$ $(\in \Omega)$ iff $M_{W_1}(z) = M_{W2}(z)$.

Properties of KdV flow

Theorem

Let $g \in \Gamma$ be such that $\log g(z)$ is an odd polynomial. Assume $W \in Gr^{(2)}(H)$ satisfies $e^{-x \log g(z)} W \in Gr^{(2)}(H)$ for any $x \in [0,1]$. Then there exists a unimodular matrix entire function

$$\Pi_W^g(z) = \begin{pmatrix} A(-z^2) & B(-z^2) \\ C(-z^2) & D(-z^2) \end{pmatrix}$$

such that

$$\begin{pmatrix} \Phi_{g^{-1}W}(z) \\ \Psi_{g^{-1}W}(z) \end{pmatrix} = \Pi_W^g(z) \begin{pmatrix} \Phi_W(z) \\ \Psi_W(z) \end{pmatrix}.$$

$\Pi_W^g(z)$ *satisfies a cocyle property:*

$$\Pi_W^{g_1 g_2}(z) = \Pi_{g_1^{-1}W}^{g_2}(z)\, \Pi_W^{g_1}(z).$$

- For $W \in Gr^{(2)}_{\mathrm{real}}(H)$, $m^{+}_{W}(E+i0) = -\overline{m^{-}_{W}(E+i0)} \Leftrightarrow$
$M_W(\sqrt{E+i0}) = -\overline{M_W(-\sqrt{E+i0})}$.

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- For $F \subset [-1,0]$, denote the set of all potentials which are reflectionless on F by $\mathcal{R}(F)$. Then from the identity

$$M_{g^{-1}W}(\sqrt{E}) = \frac{C(E) + D(E)M_W(\sqrt{E})}{A(E) + B(E)M_W(\sqrt{E})}$$

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Theorem

$\{K(g)\}_{g\in G}$ *preserves the reflectionless property:*

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Theorem

For any real $g \in \Gamma$ *and* $q \in \Omega$

$$L^{K(g)q} \overset{unitary}{\sim} L^q.$$

Almost periodicity

- Conjecture:
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- If Ω_0 is a closed $\{K(g)\}_{g\in G}$- invariant subset of Ω on which $\{K(g)\}_{g\in G}$ acts transitively. Then, for an element $q_0 \in \Omega_0$, its isotropic group
$$\Gamma = \{g \in G;\ K(g)q_0 = q_0\}$$
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- Problem:
  "Determine all closed $\{K(g)\}_{g \in G}$- invariant subsets of Ω on which $\{K(g)\}_{g \in G}$ acts transitively."

A sufficient condition

- For a closed set $F \subset [-1,0]$ s.t. $F = \overline{F}^{ess}$set

$$\Omega_F = \Omega \cap \mathcal{R}(F) \cap \{\Sigma(q) = F \cup [0,\infty)\}.$$

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For any Herglotz function m satisfying $\operatorname{Re} m(E+i0) = 0$ a.e. $e \in F$, the representing measure σ is purely absolutely continuous on F.

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- Sodin-Yuditskii proved that the homogeneity of F :

$\exists\delta > 0$ s.t. for $\forall\epsilon > 0$ and $\forall E \in F$, $|(E-\epsilon, E+\epsilon) \cap F| \geq \delta\epsilon$ holds

is sufficient for almost periodicity of $q \in \Omega_F$. Their proof shows that $\{K(g)\}_{g\in G}$ acts transitively on Ω_F and G/Γ is compact.

Thank you

Thank you for your attention
and
see you again !!