Lyapunov exponent and integrated density of states for slowly oscillating perturbations of periodic Schrödinger operators

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Saint-Petersburg July 2010

The Model

Operator $H(\theta)$ in $L^2(\mathbb{R}_+)$ associated to $\theta \in [0,\pi)$:

$$H(\theta) = -\frac{d^2}{dx^2} + [V(x) + W(x^\alpha)] \tag{1}$$

on $D(H(\theta)) = \{f \in H^2(\mathbb{R}_+) \mid f(0)\cos\theta + f'(0)\sin\theta = 0\}$.

Basic assumptions :

(V) $V \in L_{2,loc}(\mathbb{R})$ and periodic $V(x+1) = V(x)$,

(W) W is smooth and periodic function with $W(x+2\pi) = W(x)$.

(α) $\alpha \in (0,1)$.

Remark :
For $\alpha \in (0,1)$ the function $W(x^\alpha)$ oscillates slowly at infinity :

$$\lim_{x\to\infty} \frac{d}{dx}(W(x^\alpha)) = \lim_{x\to\infty}(x^{\alpha-1}W'(x^\alpha)) = 0.$$

Additional assumptions : $\alpha \in (\frac{1}{2}, 1)$ and W is analytic in $S_Y = \{|\Im z| < Y\}$.

Resolvent matrix

Schrödinger equation : $H(\theta)f(x,E) = Ef(x,E)$ is equivalent to matrix equation on resolvent matrix $T(x,y,E)$:

$$\frac{d}{dx}T(x,y,E) = A(x,E)T(x,y,E) \quad \text{with} \quad T(y,y,E) = I \tag{2}$$

$$\text{and} \quad A(x,E) = \begin{pmatrix} 0 & 1 \\ V(x)+W(x^\alpha)-E & 0 \end{pmatrix} \tag{3}$$

Consider a quasi-periodic (periodic) matrix equation depending on z and ε :

$$\frac{d}{dx}T_{z,\varepsilon}(x,y,E) = A_{z,\varepsilon}(x,E)T_{z,\varepsilon}(x,y,E) \quad \text{with} \quad T_{z,\varepsilon}(y,y,E) = I \tag{4}$$

$$\text{and} \quad A_{z,\varepsilon}(x,E) = \begin{pmatrix} 0 & 1 \\ V(x)+W(\varepsilon x+z)-E & 0 \end{pmatrix} \tag{5}$$

Approximation of the resolvent matrix 1

Pose $x_n = (2\pi n)^{\frac{1}{\alpha}}$.
Claim : There exist C and n_0 such that $\forall n > n_0$ one can find (z_n, ε_n) such that :

$$\sup_{x\in[x_n,x_{n+1}]} |W(x^\alpha) - W(\varepsilon_n x + z_n)| < C\frac{1}{n} \tag{6}$$

Lemma (A. Metelkina)

Suppose basic and additional assumptions are satisfied.
$\exists(z_n, \varepsilon_n)$ *such that for n big enough the resolvent matrix* $T(x_{n+1}, x_n, E)$ *of (2) can be approched by the resolvent matrix* $T_{z_n,\varepsilon_n}(x_{n+1}, x_n, E)$ *of (4)*

Approximations of the resolvent matrix 2

Symmetries in $A_{z,\varepsilon}$.

$V(x) = V(x+1) \quad \Rightarrow \quad [V(x+1) + W(\varepsilon(x+1) + (z-\varepsilon))] = [V(x) + W(\varepsilon z + x)]$

Consistancy condition : $T_{z,\varepsilon}(x+1,E) = T_{z-\varepsilon,\varepsilon}(x,E)$

$W(\varepsilon x + (z+2\pi)) = W(\varepsilon x + z) \Rightarrow T_{z+2\pi,\varepsilon}(x,E)$ satisfies (4) if $T_{z,\varepsilon}(x,E)$ does.

Monodromy matrix $M(z,\varepsilon)$:

$$T_{z+2\pi,\varepsilon}(x,E) = T_{z,\varepsilon}(x,E)M^T(z,\varepsilon,E) \tag{7}$$

Theorem (A. Metelkina)

Suppose basic and additional assumptions are satisfied.
$\exists(z_n,\varepsilon_n)$ *such that for n big enough the resolvent matrix* $T(x_{n+1},x_n,E)$ *of (2) can be approched by the transposed monodromy matrix* $M^T(z_n,\varepsilon_n,E)$ *associated to* $T_{z_n,\varepsilon_n}(x_{n+1},x_n,E)$ *and defined in (7)*

Definitions of IDS and LE

Let $N_D(H^l, E)$ be the number of Dirichlet eigenvalues H in $L_2(0,l)$.

Definition (IDS)

We call **integrated density of states** the following limit when it exists :

$$k(E) = \lim_{l \longrightarrow \infty} \frac{N_D(H^l, E)}{l}$$

$T(x,0,E)$ be the resolvent matrix, solution of (2).

Definition (LE)

We call **Lyapunov exponent** the following limit when it exists :

$$\gamma(E) = \lim_{x \longrightarrow \infty} \frac{\ln \|T(x,0,E)\|}{x}$$

Theorem : IDS and LE

For $E \in \mathbb{C}_+$ lets $k_p(E)$ be a principal branch of Bloch quasimomentum. For $E \in \mathbb{R}$ denote $k_p(E)$ its boudary values.

Theorem (A. Metelkina)

Suppose only basic assumptions are satisfied.
For all energies $E \in \mathbb{R}$ the integrated density of states for $H(\theta)$ exists and is given by :

$$k(E) = \frac{1}{2\pi^2} \int_0^{2\pi} \Re k_p(E - W(x))dx$$

For almost all $E \in \mathbb{R}$ the Lyapunov exponent for $H(\theta)$ exists and is given by :

$$\gamma(E) = \frac{1}{2\pi} \int_0^{2\pi} \Im k_p(E - W(x))dx$$

Thank you for your attention !