Estimates for non-elliptic operators

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This is joint work with A. Laptev.

LT Inequalities vs. Sobolev Inequalities

Well known Lieb-Thirring inequalities for a Schrödinger operator

$$-\Delta - V,$$

$V \in L^{\gamma+d/2}(\mathbb{R}^d)$, state that for the γ - moments of its negative eigenvalues $\{-\lambda_k\}$ the estimate

$$\sum_k \lambda_k^\gamma \le L_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma+d/2}(x)\, dx \tag{2.1}$$

holds, where $V_+ = (|V| + V)/2$ is the positive part of V. The constants $L_{\gamma,d}$ in this inequality are finite if $\gamma \ge 1/2$ $(d = 1)$, $\gamma > 0$ $(d = 2)$ and $\gamma \ge 0$ $(d \ge 3)$.

LT Inequalities vs. Sobolev Inequalities

If $\gamma = 1$, E.H. Lieb and W. Thirring proved that (2.1) is equivalent to a so-called generalised Sobolev inequality for an orthonormal system of functions $\{\varphi_k\}_{k=1}^N$ in $L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} [\rho_N(x)]^{(2+d)/d}\, dx \le C_d \sum_{k=1}^{N} \int_{\mathbb{R}^d} |\nabla \varphi_k(x)|\, dx, \tag{2.2}$$

where $\rho_N(x) = \sum_{k=1}^N |\varphi_k(x)|^2$. With the help of the Fourier transform (2.2) can be rewritten as

$$\int_{\mathbb{R}^d} (\rho_N(x))^{\frac{d+2}{d}}\, dx \le C_d (2\pi)^d \sum_{k=1}^{N} \int_{\mathbb{R}^d} |\xi|^2 |\hat{\varphi}_k(\xi)|^2\, d\xi, \quad x, \xi \in \mathbb{R}^d.$$

Barsegyan's Results

Recently, D.S. Barsegyan has obtained L-T type inequalities in $\mathbb{R}^2$, where the Laplace operator (whose symbol equals $|\xi|^2$) has been substituted by the product $|D_x D_y|$, $D_x = -i\partial_x$. In this case the latter inequality takes the form

$$\int_{\mathbb{R}^2} (\rho_N(x,y))^2 \, dxdy \le C\,(\log N + 1) \sum_{k=1}^{N} \int_{\mathbb{R}^2} |\xi\eta| |\hat{\varphi}_k(\xi,\eta)|^2 \, d\xi d\eta, \tag{3.1}$$

where the constant C is independent of N.

Reformulation in Terms of an Operator

This inequality could be rewritten as an inequality for the negative eigenvalues $\{-\lambda_k\}$ of the operator

$$|D_x D_y| - V \tag{3.2}$$

acting in $L^2(\mathbb{R}^2)$. Let $-\lambda_1 \leq -\lambda_2 \leq \cdots \leq -\lambda_N \leq \ldots$ be the sequence of negative eigenvalues, then (3.1) implies that for any N,

$$\sum_{k=1}^{N} \lambda_k \leq C(\log N + 1) \int_{\mathbb{R}^2} V_+^2(x, y)\, dxdy. \tag{3.3}$$

Proof.

Indeed, if $\{\varphi_k\}$ is an orthonormal system of eigenfunctions of the operator (3.2), then by (3.1) and the Cauchy-Schwartz inequality we have

$$
\begin{aligned}
-\sum_{k=1}^{N} \lambda_k &= \int_{\mathbb{R}^2} |\xi\eta| \sum_{k=1}^{N} |\hat{\varphi}_k(\xi,\eta)|^2 \, d\xi d\eta - \int_{\mathbb{R}^2} V \sum_{k=1}^{N} |\varphi_k(x,y)|^2 \, dxdy \\
&\geq C(\log N + 1)^{-1} \int_{\mathbb{R}^2} [\rho_N(x,y)]^2 \, dxdy \\
&\quad - \left(\int_{\mathbb{R}^2} V^2 \, dxdy \right)^{1/2} \left(\int_{\mathbb{R}^2} [\rho_N(x,y)]^2 \, dxdy \right)^{1/2} .
\end{aligned}
$$

(3.3) follows when minimizing the right hand side with respect to

$$
X = \left(\int_{\mathbb{R}^2} [\rho_N(x,y)]^2 \, dxdy \right)^{1/2} .
$$

Incompleteness

Although the inequality (3.1) is sharp, it does not give a satisfactory inequality for the sum of all negative eigenvalues, because the right hand side of (3.3) depends on $\log N + 1$. When $d = 2$, estimates for the number of negative eigenvalues even for Schrödinger operators is a delicate problem. Necessary and sufficient conditions for the finiteness of the negative spectrum are so far not known.

Main Result

We consider a related problem and obtain spectral inequalities for the operator

$$D_x^2 D_y^2 u - Vu = -\lambda u, \quad u(x,0) = u(0,y) = 0. \tag{4.1}$$

in $L^2(\mathbb{R}^2_{++})$, where $\mathbb{R}^2_{++} = \mathbb{R}_+ \times \mathbb{R}_+$.

Theorem

Let $\gamma \geq 1/2$. Then for the negative eigenvalues $\{-\lambda_k\}$ of the operator (4.1) we have

$$\sum_k \lambda_k^\gamma \leq \frac{(R_{\gamma,1})^2}{4^\gamma (2\pi)^2} \mathcal{B}(1/2, \gamma+1) \int_{\mathbb{R}^2_{++}} V_+^{1/2+\gamma} \log(1 + 4xy\sqrt{V_+})\, dxdy. \tag{4.2}$$

Important Remarks

For both (3.2) and (4.1) the phase volume type estimates do not exist, because the classical phase volume is infinite. Differential parts of these operators are highly non-elliptic. Some examples of operators with infinite classical phase volume were previously considered by B. Simon, M.Z. Solomyak and M.Z. Solomyak & I.L. Vulis.

Sharpness of the Result

The sharpness of the inequality (4.2) in terms of large potentials could be confirmed by the following argument. Simon showed that for the number $N(\lambda)$ of the eigenvalues $\{\lambda_k\}$ below λ of the operator $D_x^2 + D_y^2 + x^2y^2$ there is the following asymptotic formula

$$N(\lambda) = \pi^{-1}\lambda^{3/2}\log\lambda + o(\lambda^{3/2}\log\lambda), \quad \lambda \to \infty.$$

This formula immediately implies that

$$\sum_k (\lambda - \lambda_k)_+^\gamma = \frac{1}{(\gamma + 3/2)\pi}\,\lambda^{\gamma+3/2}\log\lambda + o(\lambda^{\gamma+3/2}\log\lambda), \quad \lambda \to \infty.$$

Sharpness of the Result

Using the duality of the Fourier transform it is equivalent to study the spectrum below λ of the operator $D_x^2 D_y^2 + x^2 + y^2$.
We now reduce this problem to studying of the negative spectrum of the operator $D_x^2 D_y^2 - (\lambda - x^2 - y^2)_-$. By Theorem 4.1 we find that for $\gamma \geq 1/2$

$$\sum_k (\lambda - \lambda_k)_+^\gamma \leq \frac{1}{4^\gamma}\,(2\pi)^{-2}(R_{\gamma,1})^2\,\mathcal{B}(1/2, \gamma+1) \times$$

$$\times \int_{\mathbb{R}^2_{++}} (\lambda - x^2 - y^2)_+^{1/2+\gamma} \log(1 + 4xy\sqrt{(\lambda - x^2 - y^2)_+})\,dxdy$$

$$\leq C\lambda^{\gamma+3/2}\,(1 + \log(\lambda+1))\,,$$

where C is independent of λ.