Matrix Schrödinger operator on the half-line: the differential equation with respect to the spectral parameter and an analog of Freud's equations.

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Semiclassical asymptotics. I

In many physical problems it is necessary to calculate semiclassical asymptotics for solutions of the Schrödinger equation

$$-\varepsilon^2 \frac{d^2}{dx^2}\psi(x) + v(x)\psi(x) = \lambda\psi(x), \quad x \in \mathbb{R}, \tag{1}$$

with respect to a small parameter ε.
While the potential $v(x)$ is smooth enough we can use the standard WKB method to find an asymptotical behavior of solution $\psi(x)$. The graph of the potential splits our phase plane (x, λ) into two regions. In the region above the potential $(\lambda > v(x))$ there are two oscillating exponents. In the region below the potential $(\lambda < v(x))$ there is one growing and one decaying exponent. In the vicinity of simple turning point

Semiclassical asymptotics. II

($\lambda \approx v(x)$, $v'(x) \neq 0$) the solutions are described in terms of Airy functions. In the vicinity of double turning point ($\lambda \approx v(x)$, $v'(x) = 0$, $v''(x) \neq 0$) the solutions are described in terms of parabolic cylinder functions.
But what if the potential $v(x)$ has singularity, for example $v'(x_0) = \infty$? How can we describe an asymptotical behavior of solution $\psi(x, \lambda)$ in the vicinity of point $(x_0, v(x_0))$ on the phase plane? If we could fix variable x and write down the differential equation with respect to the spectral parameter λ for solution $\psi(x, \lambda)$ then we could expect that this new differential equation will have no problems and WKB method can be applied.

Conjecture

(V. Buslaev)

Consider the Schrödinger operator in $L^2(\mathbb{R})$

$$L\psi = -\psi'' + v(x)\,\psi = k^2\,\psi\,, \tag{2}$$

$$v(x) \in \mathbb{R}\,, \quad \int_0^\infty (1+x^2)|v(x)|dx < \infty \tag{3}$$

Let us assume that the discrete spectrum of the operator L is absent and $k = 0$ is not a virtual level. Then

1. *One can write down the differential equation with respect to k for eigenfunctions of continuous spectrum;*
2. *There exists a nonlinear relation connecting the operator L and the kernel of its spectral measure.*

Orthogonal polynomials on the real line I

Let us consider a system of polynomials $p_n(\lambda) = \alpha_n \lambda^n + \ldots$, that are orthogonal on the axis $\mathbb{R}$ with respect to weight function $w = \exp(-q)$, where $q = q(\lambda)$ - some positive continuously differentiable function, that tends to infinity when $|\lambda| \to \infty$ as a power of λ:

$$\int_{\mathbb{R}} p_n(\lambda) p_m(\lambda) w(\lambda) d\lambda = \delta_{nm}. \tag{4}$$

The corresponding Jacobi matrix acts on complex-valued sequences $\vec{x} = \{x_n\}_{n=0}^{\infty}$ by the rule:

$$(\mathbf{J}\vec{x})_n = r_n x_{n-1} + r_{n+1} x_{n+1}, \ n > 0, \quad (\mathbf{J}\vec{x})_0 = r_1 x_1 .$$

Orthogonal polynomials on the real line II

Here $r_n = \alpha_{n-1}/\alpha_n$. Vectors $\vec{p}(\lambda)$, $(\vec{p}(\lambda))_n = p_n(\lambda)$ are generalized eigenvectors for matrix $\mathbf{J}$: $\mathbf{J}\vec{p}(\lambda) = \lambda\vec{p}(\lambda)$. They form a basis in the space $l_2(\mathbb{N}), \mathbb{N} = \{0, ...\}$.

The spectrum of matrix $\mathbf{J}$ is simple continuous and coincides with axis $\mathbb{R}$. We have also completeness relation for these eigenvectors: $\sum_{n\geq 0} p_n(\lambda)p_n(\mu)w(\mu) = \delta(\lambda - \mu)$. There are two remarkable relations for these orthogonal polynomials:

$$\frac{d}{d\lambda} p_n(\lambda) = r_n B_{n,n} p_{n-1}(\lambda) - r_n B_{n,n-1} p_n(\lambda) \,, \tag{5}$$

$$r_n(W(\mathbf{J}))_{n,n-1} = n \,, \qquad \textbf{Freud's equations} \,. \tag{6}$$

where

$$B(\lambda) = W(\mathbf{J}) \cdot (\mathbf{J} - \lambda - i0)^{-1}, \qquad W = -\frac{d}{d\lambda} \ln w(\lambda) \,.$$

History I

The conjecture for the Schrödinger operator in $L^2(\mathbb{R})$ is not proved yet, but we have the following results:

1. The conjecture for the scalar Schrödinger operator in $L^2(\mathbb{R}^+)$ with Dirichlet boundary condition was proved in our joint work [V. S. Buslaev, V. Yu. Strazdin, *One-dimensional Schrödinger operator on the half-line: The differential equation for eigenfunctions with respect to the spectral parameter and an analog of the Freud equation.* Functional Analysis and Its Applications, (2007), **41**(3), 237-240.]
2. We have considered several sample potentials for which spectral measure can be calculated explicitly. We have shown that both differential equation and nonlinear relation are valid.

History II

3. We have reduced our assumptions about the potential $v(x)$. Actually, we do not need decaying potential at least in the case of the scalar Schrödinger operator in $L^2(\mathbb{R}^+)$ with Dirichlet boundary condition. These reduced assumptions were formulated in terms of spectral measure.
4. The conjecture for the matrix Schrödinger operators in $L^2(\mathbb{R}^+)$ with Dirichlet and Neumann boundary conditions was proved in [V. Yu. Strazdin, *Matrix Schrödinger operator on the half-line: The differential equation for generalized eigenfunctions of continuous spectrum with respect to the spectral parameter and an analog of the Freud equation.* Vestnik of St.Petersburg State University, Series 4, (2009), Number 4, 49-61 (in Russian)]

Definitions

Consider the *matrix Schrödinger equation* on the half-line

$$-\Psi''(x,k)+V(x)\Psi(x,k)=k^2\Psi(x,k)\,,\qquad x\geqslant 0\,,\qquad (7)$$

where $V=\{v_{\alpha\beta}\}_{\alpha,\beta=1}^{N}$ is a Hermitian matrix such that

$$\int\limits_0^\infty (1+x^2)\cdot|V(x)|\,dx\ <\ \infty\,,\quad |V|\equiv\max_\alpha\sum_{\beta=1}^{N}|v_{\alpha\beta}|\,.\qquad (8)$$

The solutions $\Phi_1(x,k)$, $\Phi_2(x,k)$, and $E(x,k)$ of equation (7) are uniquely determined by the following conditions:

$$\Phi_1(0,k)=\mathbf{0},\quad \Phi_1'(0,k)=\mathbf{1}\,,\quad \Phi_2(0,k)=\mathbf{1},\quad \Phi_2'(0,k)=\mathbf{0}\,,\qquad (9)$$

$$\lim_{x\to\infty} e^{-ikx}E(x,k)=\mathbf{1}\,.$$

Here $\mathbf{0}\,,\mathbf{1}$ are the *zero matrix* and the *identity matrix*.

Eigenfunctions of continuous spectrum

Let us denote by L_1 (L_2) corresponding matrix Schrödinger operators with Dirichlet (Neumann) boundary conditions. We have assumed that eigenvalues are absent. The continuous spectrum of the operator L_j has multiplicity N and coincides with half-line $\mathbb{R}^+$. The columns of the matrix $\Phi_j(x,k)$ are eigenfunctions of continuous spectrum for the operator L_j, $j=1,2$. They satisfy the following *orthonormality* and *completeness* conditions:

$$\int_0^\infty \sigma_j(l)\Phi_j^*(y,l)\Phi_j(y,k)dy = \delta(\lambda-\mu)\cdot\mathbf{1}\,, \quad j=1,2\,, \tag{10}$$

$$\int_0^\infty \Phi_j(x,k)\sigma_j(k)\Phi_j^*(y,k)d\lambda = \delta(x-y)\cdot\mathbf{1}\,, \quad j=1,2\,, \tag{11}$$

The kernel of the operator $W(L_j)$

where

$$\sigma_1(k) = \frac{k}{\pi}\left[E(0,k)E^*(0,k)\right]^{-1}, \quad \sigma_2(k) = \frac{k}{\pi}\left[E'(0,k)E^{*\prime}(0,k)\right]^{-1}.$$

The kernel $W_j(x,y)$ of the operator $W(L_j)$ is given by:

$$W_j(x,y) = \int_0^\infty \Phi_j(x,l)W(\mu)\cdot\sigma_j(l)\Phi_j^*(y,l)d\mu, \quad \mu = l^2.$$

Most of these facts about solutions to the matrix Schrödinger equation were known long time ago. See, for example, book [Z. S. Agranovich, V. A. Marchenko, *The inverse problem of scattering theory.* Gordon and Breach, New York 1963.] But some of them we had to establish ourselves.

Theorem

If the potential matrix V satisfies (8) *and the operator L_j satisfies the assumption of Conjecture then*

$$\frac{\partial}{\partial\lambda}\begin{pmatrix} \Phi_j(x,k) \\ \Phi_j'(x,k) \end{pmatrix} = \mathbb{U}_j(x,\lambda)\begin{pmatrix} \Phi_j(x,k) \\ \Phi_j'(x,k) \end{pmatrix}, \tag{12}$$

$$-2\frac{d}{dx}W_j(x,x) = \mathbf{1}, \quad j = 1,2, \tag{13}$$

where

$$\mathbb{U}_j(x,\lambda) = \begin{pmatrix} -(M_j)_y(x,y,\lambda)|_{y=x} & M_j(x,x,\lambda) \\ W_j(x,x) - (M_j)_{xy}(x,y,\lambda)|_{y=x} & (M_j)_x(x,y,\lambda)|_{y=x} \end{pmatrix}$$

$$M_j(L_j) = W_j(L_j)\cdot(L_j - \lambda - i0)^{-1}, \quad W_j(\mu) = -\frac{\partial}{\partial\mu}\Big(\ln\sigma_j(I)\Big).$$

Remark

We can write the Schrödinger equation (7) in the following form:

$$\frac{\partial}{\partial x}\begin{pmatrix} \Phi_j(x,k) \\ \Phi_j'(x,k) \end{pmatrix} = \mathbb{V}(x,\lambda)\begin{pmatrix} \Phi_j(x,k) \\ \Phi_j'(x,k) \end{pmatrix}, \tag{14}$$

where

$$\mathbb{V}(x,\lambda) = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ V(x) - \lambda \cdot \mathbf{1} & \mathbf{0} \end{pmatrix}.$$

An analogue of Freud's equation was obtained as compatibility condition of two differential equations (12) and (14).

Applications

As soon as we prove Conjecture for scalar Schrödinger operator in $L^2(\mathbb{R})$ we can use it to investigate the initial value problem for the Korteweg-de Vries equation

$$u_t - 6uu_x + \varepsilon^2 u_{xxx} = 0\,, \qquad u(x,0) = v(x)\,, \tag{15}$$

in the small dispersion limit $\varepsilon \to 0$.

Although many remarkable results have been obtained for this problem, see [P. Deift, S. Venakides and X. Zhou, *An extension of the steepest descent method for Riemann-Hilbert problems: The small dispersion limit of the Korteweg-de Vries (KdV) equation*, PNAS, **95**, (1998), 450-454], there still remain several open questions. For example, what is happening with solution at the breaking time?