Negative spectrum of a perturbed Anderson Hamiltonian

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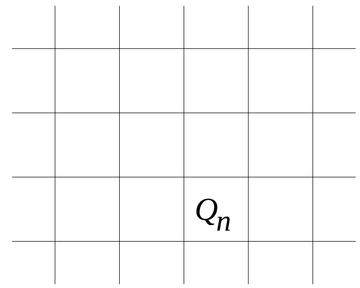
We will discuss the following problem in the spirit of the classical Cwikel-Lieb-Rozenblum estimates (CLR) for the negative spectrum of multidimensional Schrödinger operators. Let

$$H_0 = -\Delta + hV(x,\omega),\ x \in R^d,\ \omega \in (\Omega, F, P) \tag{1}$$

be the Anderson Hamiltonian on $L^2(R^d)$. The random potential we consider has the simplest Bernoulli structure: consider the partition of R^d into unit cubes



$$Q_n = \{x : \ ||x-n||_\infty \le \frac{1}{2}\}, \quad n=(n_1, ...n_d) \in Z^d.$$

Then

$$V(x,\omega) = \sum_{n \in Z^d} \varepsilon_n I_{Q_n}(x).$$

Here ε_n are i.i.d. Bernoulli r.v., namely

$$P\{\varepsilon_n = 1\} = p > 0, \quad P\{\varepsilon_n = 0\} = q = 1-p > 0$$

on the probability space (Ω, F, P).

We call a domain $D \in R^d$ a clearing if $V = 0$ when $x \in D$. Since P-a.s. realizations of the potential V contain cubic clearings of arbitrary size $l \gg 1$, we have $Sp(H_0) = [0, \infty)$.

Consider a perturbation of H_0 by a non-random continuous potential:

$$H = -\Delta + hV(x,\omega) - v(x), \quad v(x) \geq 0, \quad v \to 0 \text{ as } |x| \to \infty. \qquad (2)$$

The operator H is bounded from below, and its negative spectrum $\{\lambda_i\}$ is discrete. Put $N_0(v,\omega) = \#\{\lambda_i \leq 0\}$. The following theorem presents the main result.

Theorem 1. . *There are two constants $c_1 < c_2$ which depend only on d and independent of h and p, such that*

a) the condition

$$v(x) \leq \frac{c_1}{\ln^{\frac{2}{d}} |x| \ln 1/q}, \quad |x| \to \infty, \quad \textit{implies} \;\; N_0(v,\omega) < \infty \quad P - a.s.,$$

b) the condition

$$v(x) \geq \frac{c_2}{\ln^{\frac{2}{d}} |x| \ln 1/q}, \quad |x| \to \infty, \quad \textit{implies} \;\; N_0(v,\omega) = \infty \quad P - a.s.,$$

Remark 1. Similar result is valid for the lattice Anderson model with the Bernoulli potential. Consider $L^2(Z^d),\ d \geq 1$, and the lattice Laplacian

$$-\Delta\psi(x) = -\sum_{x':|x'-x|=1} [\psi(x') - \psi(x)], \quad Sp(-\Delta) = [0, 4d].$$

Put

$$H_0 = -\Delta\psi + h\varepsilon(x,\omega), \quad x \in Z^d,$$

where $\varepsilon(x)$ are i.i.d.r.v.; $P\{\varepsilon(x) = 1\} = p > 0$, $P\{\varepsilon(x) = 0\} = q = 1-p > 0$. Consider the perturbation

$$H = -\Delta + h\varepsilon(x,\omega) - v(x), \quad v(x) \geq 0, \quad v \to 0, \ |x| \to \infty.$$

The lattice version of Theorem 1 has the same form (with different values of c_1, c_2).

It looks natural to try to prove Theorem 1 using Cwikel-Lieb-Rozenblum (CLR) estimates together with the Donsker-Varadan estimate. I am going to describe difficulties which did not allow us to use this approach. But first let me mention that CLR approach usually leads to a power decay of the potential as a borderline between $N_0 < \infty$ and $N_0 = \infty$.

Our proof is based on percolation theory and Dirichlet-Neumann bracketing. The percolation theory allows us to describe sets in R^d where $V = 1$.

FURTHER PLAN OF MY TALK

1. Difficulties with CLR-estimates.

2. Scheme of our proof.

3. 1-D case, where a stronger results are obtained (together with J. Holt)

CLR-estimates and large deviations. The classical approach to the study of the discrete negative spectrum of Schrödinger type operators is based on Cwikel-Lieb-Rozenblum estimates. Important generalizations and references can be found in

[1] Rozenblum, G., Solomyak, M., "St. Petersburg Math. J.", 9, no. 6, pp1195-1211 (1998).

[2] Rozenblum, G., Solomyak, M., Sobolev Spaces in Mathematics. II. Applications in Analysis and Parrtial Differential Equations, International Mathematical Series, 8, Springer and T. Rozhkovskaya Publishers, pp329-354 (2008).

[3] Molchanov S., Vainberg B., in Around the research of Vladimir Maz'ya, Editor A. Laptev, Springer, 2009, pp 201-246.

In our particular case the CLR estimate can be presented in the following form. Let $p_0(t,x,y)$ be the fundamental solution for the parabolic Schrödinger problem

$$\frac{\partial p_0}{\partial t} = \Delta_x p_0 - V(x) p_0, \quad p_0(0,x,y) = \delta_y(x), \quad d \geq 3.$$

Here $V \geq 0$ and it is not essential that it is random. Consider the operator

$$H = -\Delta + V(x) - v(x), \quad v \geq 0, \quad v(x) \to 0, \; |x| \to \infty.$$

Let $N_0(v) = \#\{\lambda_j \leq 0\}$ be the number of negative eigenvalues of H. Then

$$N_0(v) \leq \frac{1}{g(1)} \int_0^\infty \int_{R^d} \frac{p_0(t,x,x)}{t} G(tv)\, dx dt, , \quad d \geq 3$$

where G is a rather general function and $g(1) = \int_0^\infty z^{-1} G(z) e^{-z} dz$.

Usually, it is enough to consider $G(z) = (z - \sigma)_+, \ \sigma > 0$, which leads to

$$N_0(v) \le \frac{1}{c(\sigma)} \int_{R^d} dx v(x) \int_{\frac{\sigma}{v(x)}} p_0(t,x,x)dt,, \quad d \ge 3 \tag{3}$$

where

$$c(\sigma) = \int_0^\infty \frac{z}{z+\sigma} e^{z+\sigma} dz.$$

The convergence of the integral (3) determines whether $N_0(v)$ is finite or infinite. This convergence connects the decay of $v(x)$ at infinity with asymptotics of $p(t,x,x)$ as $t \to \infty$. Usually $p = O(t^\gamma), \quad t \to \infty$, which leads to the borderline decay of the perturbation $v(x)$ (which separates cases of $N_0(v) < \infty$ and $N_0(v) = \infty$) which is defined by a power function. There are several examples in [3] when p decays exponentially as $t \to \infty$ (Lobachevski plane, operators on some groups). This leads to much slower borderline decay of v. In those examples a fast decay of p is a corollary of an exponential growth of the phase space.

In order to apply the estimate

$$N_0(v) \le \frac{1}{c(\sigma)} \int_{R^d} dx v(x) \int_{\frac{\sigma}{v(x)}} p_0(t,x,x) dt, , \quad d \ge 3 \tag{4}$$

to the operator with the Bernoulli piece-wise potential, one needs to have a good estimate for $p_0(t,x,x)$. A rough estimate of integral (4) (through the maximum of the integrand) leads to the following result. The presence of arbitrarily large clearings implies that P-a.s.

$$\pi(t) \equiv \sup_x p_0(t,x,x) = \frac{1}{(4\pi t)^{d/2}}.$$

which provides the standard CLR-estimate:

$$N_0(v) \le c(d) \int_{R^d} v^{d/2}(x) dx, \quad d \ge 3.$$

This estimate ignores the presence of the random potential V and therefore is very weak for the Hamiltonian $H_0 = H + V$.

Another possibility is to take the expectation (over the distribution of $V(x,\omega)$). This leads to

$$\langle N_0(v)\rangle \le \frac{1}{c(\sigma)}\int_{R^d} v(x)\int_{\frac{\sigma}{v(x)}}\langle p_0(t,x,x)\rangle dt dx.$$

The following Donsker-Varadan estimate (75) of $\langle p_0(t,x,x)\rangle$ is one of the widely known results in the theory of random operators (it is related to the concept of Lifshitz tails for the integral density of states $N(\lambda)$):

$$\ln\langle p_0(t,x,x)\rangle = \ln\langle p_0(t,0,0)\rangle \sim -c(d)t^{\frac{d}{d+2}}, \quad t\to\infty,$$

i.e., for any $\varepsilon > 0$,

$$\langle p_0(t,x,x)\rangle \le e^{-(c_1(d)-\varepsilon)t^{d/d+2}}, \quad t \ge t_0(\varepsilon).$$

This estimate and the inequality above for $\langle N_0\rangle$ lead to the following result
Theorem 2. . *If* $v(x) \le \frac{c}{\ln^{\sigma}(2+|x|)}, \quad c>0, \quad \sigma > 1+\frac{2}{d}$, *then* $\langle N_0(v)\rangle < \infty$ *(which implies, of course, that* $N_0(v) < \infty$, *P-a.s.)*

This theorem requires a stronger decay of $v(\cdot)$ than Theorem 1.

Asymptotics of mean values of random variables are known as annealed (or moment) asymptotics. Alternatively, one can use P-a.s, or quenched, asymptotics. The latter usually provides a stronger result. A quenched behavior of the kernel $p_0(t,x,x,\omega)$ was obtained by Sznitman (98). He proved that when x is fixed the following relation holds P-a.s.

$$\ln p_0(t,x,x,\omega) \sim c_1(d,p)\frac{t}{\ln^{2/d} t}. \tag{5}$$

Unfortunately, the asymptotics in (5) is highly non-uniform in x. Besides, the field $p_0(t,x,x,\omega),\ x \in R^d$, has the correlation length of order t. As a result, formula (5) can not be combined with the standard CLR-estimate

$$N_0(v) \le \frac{1}{c(\sigma)} \int_{R^d} dx v(x) \int_{\frac{\sigma}{v(x)}} p_0(t,x,x) dt, \quad d \ge 3,$$

at least directly, though the presence of the factor $\ln^{2/d} t$ indicates that (5) reflects the essence of the problem.

PERCOLATION LEMMAS

We'll prove below several results on the geometric structure of the set $X_1 \subset R^d$ where the potential

$$V(x,\omega) = \sum_{n\in Z^d} \varepsilon_n I_{Q_n}(x)$$

is equal to one. Here ε_n are i.i.d. Bernoulli r.v., and

$$P\{\varepsilon_n = 1\} = p > 0, \quad P\{\varepsilon_n = 0\} = q = 1 - p > 0.$$

I will focus mostly on statement a) of Theorem 1 (condition for $N_0 < \infty$), where estimates of the Hamiltonian H from below are needed. Thus, our goal here will be to show that set X_1 is rich enough (for any p, q). When the proof of statement b) ($N_0 = \infty$) is discussed, we will need estimates of operator H from above, and existence of large clearings where $V(x,\omega) = 0$ has to be shown there.

Let us say that a cube Q_n is *brown* if $\varepsilon_n = 1$, and *white* if $\varepsilon_n = 0$. Let us introduce the concept of connectivity for sets of cubes Q_n. Two cubes are called 1-neighbors if they have a common $(d-1)$-dimensional face, i.e., the distance between their centers is equal to one. Two cubes are called $\sqrt{d}$-neighbors if they have at least one common point (a vertex or an edge of the dimension $k \le d-1$, i.e., the distance between their centers does not exceed $\sqrt{d}$. A set of cubes is called 1-connected (or $\sqrt{d}$-connected) if any two cubes in the set can be connected by a sequence of 1-neighbors ($\sqrt{d}$-neighbors, respectively.)

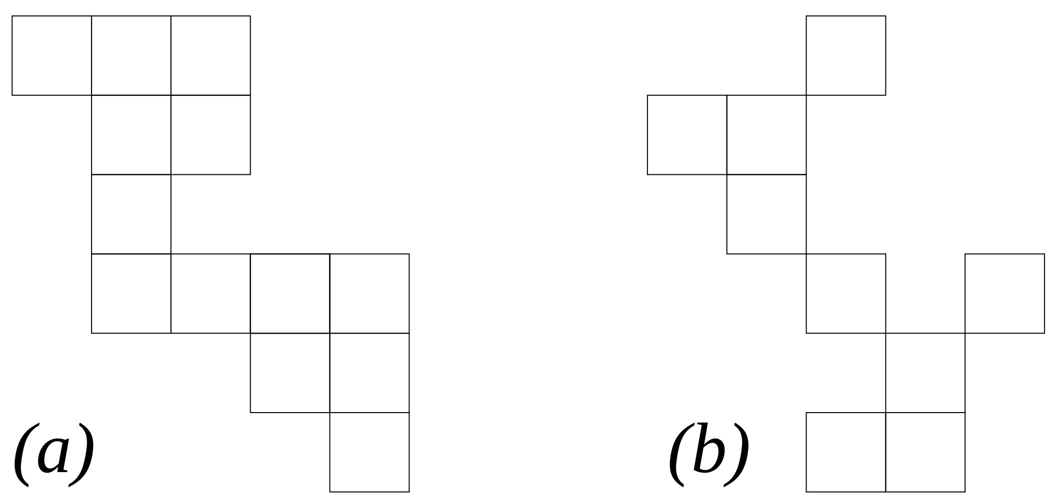


Figure 1: 1-connected and $\sqrt{2}$-connected sets

An infinite (maximal) 1-connected component of brown cubes ($V = 1$) will be called a "continent". A well known result by M. Aizenman, H. Kesten, C. M. Newman (87) states that P-a.s. there is at most one continent in R^d (even if 1-connectivity in the definition of the continent is replaced by $\sqrt{d}$-connectivity). It is also known that such a continent exists P-a.s. if $q < q_{cr}$. The continent can include $\sqrt{d}$-connected "lakes" where $\varepsilon_n = 0$, the lakes can include "islands", i.e., bounded 1-connected components where $\varepsilon_n = 1$, and so forth.

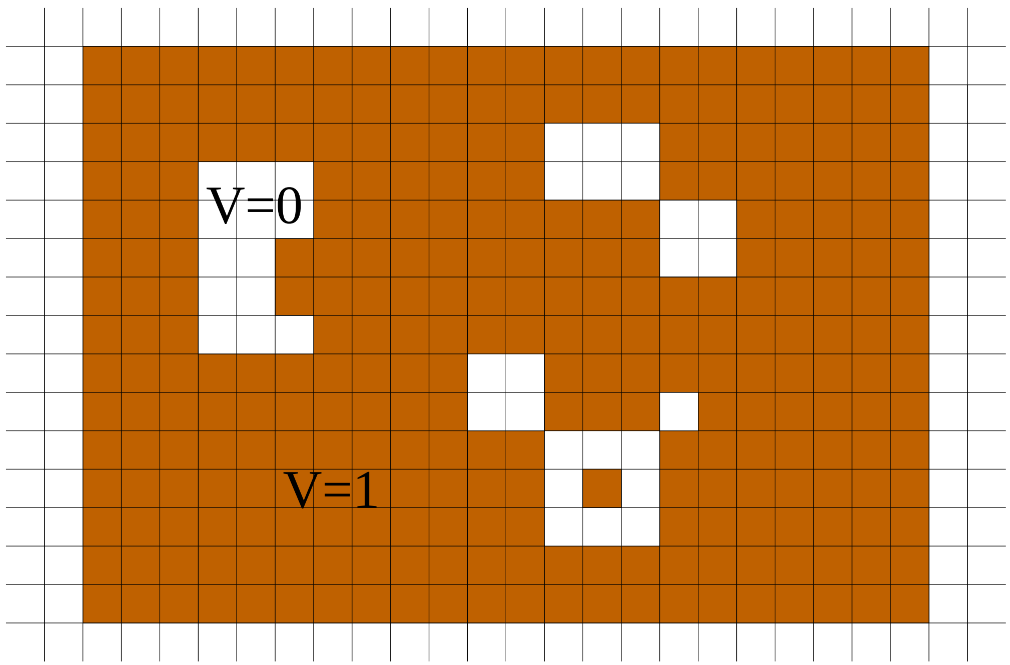


Figure 2: One continent with 3 lakes and one island

Main percolation lemma

Let

$$q < \frac{1}{3^d - 2} \tag{6}$$

Then P-a.s. there exists a unique continent and there exists a non-random constant $a = a(d,q)$ such that P-a.s. the following estimate holds for all lakes L ($\sqrt{d}$-connected sets of cubes where $V = 0$) located far enough from the origin:

$$|L| = \#\{Q_n \subset L\} < a \ln r, \quad r = \min_{x : x \in L} |x|, \quad r > r_0(\omega).$$

This main lemma follows from the Borel-Contelli lemma and the following statement.

Lemma 1. *(exponential tails). If (6) holds and L_0 is a lake containing the origin, then there exists a constant $c_0 = c_0(d,q)$ such that*

$$P\{|L_0| \geq s\} \leq c_0 e^{-\gamma s}, \quad \gamma = \ln \frac{1}{q(3^d - 2)} > 0.$$

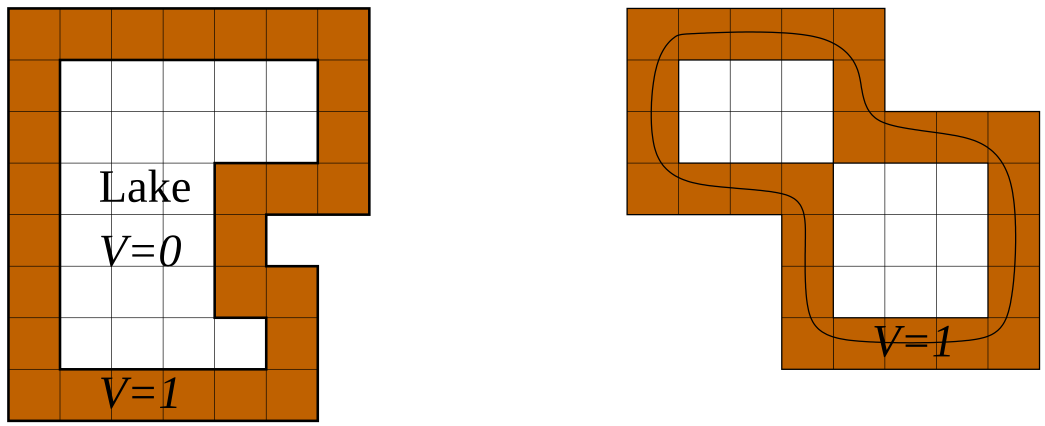
Proof. Consider all possible $\sqrt{d}$-connected sets $S = \bigcup Q_n$ of the cubes Q_n which have volume s (each of them consists of s cubes Q_n) and contain the cube Q_0 (we do not pay attention to the color of cubes in S). Grimmett called sets S "$\sqrt{d}$-animals". Let us estimate the number ν_s of all animals of volume s from above. There is only one animal of volume 1 (it consists of Q_0), and therefore, $\nu_1 = 1$. The $\sqrt{d}$-neighbors of Q_0 together with Q_0 fill out the cube of edge length 3, i.e., $\nu_2 = 3^d - 1$. Each animal of volume s can be obtained by adding a new cube to some animal of volume $s-1$. Each cube in that smaller animal has exactly $3^d - 1$ neighbors and at least one of them belongs to the animal. Thus, $\nu_s \leq \nu_{s-1}(3^d - 2), \quad s > 2$, and therefore,

$$\nu_s \leq (3^d - 1)(3^d - 2)^{s-2}, \quad s \geq 2.$$

The probability that any fixed animal of volume s has only white cubes is q^s, i.e,

$$P\{|C_w(0, \sqrt{d})| = s\} \leq q^s(3^d - 1)(3^d - 2)^{s-2} \leq c_1 e^{-\gamma s}, \quad s \geq 2.$$

Proof of statement a) of theorem ($N_0 < \infty$ if $w(x) \leq \frac{c_1}{\ln^{\frac{2}{d}} |x| \ln 1/q}$) in the case of small $q < \frac{1}{3^d - 2}$.



Consider the set of all lakes $\{L_i\}$. Let ∂L_i be the shoreline of L_i (set of cubes Q_n which do not belong to L_i, but have a common point with at least one cube from L_i). Obviously, $|\partial L_i| \leq c(d)|L_i|$. Let S_i be C^2-surfaces surrounding L_i which have the following properties:

$$S_i \subset \partial L_i, \quad \frac{1}{4} < \text{dist}(S_i, L_i) < \frac{1}{2},$$

and the main curvatures of the surfaces S_i are bounded by a constant $k < \infty$ which does not depend on i or a point on S_i.

Let $N_{0,N}$ be the number of negative eigenvalues of the operator H_N in $L^2(R^d)$ defined by $-\Delta + hV\,(x\;,\omega) - v(x)$ with the Neumann boundary condition $(\psi_\nu = 0)$ imposed on all surfaces $S_i,\; i = 1, 2, \ldots$. Then

$$N_0\,(v) \leq N_{0,N}\,(v)\,.$$

Thus, it is enough to show that $N_{0,N} < \infty$ P-a.s..

Since $V = 1$ on the continent and $v \to 0$ at infinity, the continent provides a finite number of negative eigenvalues. Each domain Ω_i with a lake L_i and $\partial\Omega_i = S_i$ is bounded and also provides a finite number of the eigenvalues. It remains to show that the Neumann problem

$$[-\Delta + hV(x,\omega) - v(x)]u = \lambda u, \quad x \in \Omega_i, \quad u_\nu|_{S_i} = 0$$

does not have negative eigenvalues when $i >> 1$. The latter is a consequence of the following lemma

Lemma 2. *Consider the operator*

$$Lu = (-\Delta + hV(x))u, \quad x \in \Omega; \qquad \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega,$$

in a bounded domain Ω *with a* C^2 *boundary,* $|\Omega| \gg 1$ *and the main curvatures of* $\partial\Omega$ *being bounded by a constant* $k < \infty$ *independent off* Ω*. Let* $V = 1$ *if* $dist(x, \partial\Omega) < 1$, $V = 0$ *if* $dist(x, \partial\Omega > 1$ *Then there is a constant* $c_0 = c_0(k, l, h, d)$ *such that the following estimate is valid for the minimal eigenvalue* λ_0 *of operator* L*:*

$$\lambda_0 \geq \frac{c_0}{|\Omega|^{2/d}}, \qquad |\Omega| \geq 1.$$

Proof of statement a) of theorem ($N_0 < \infty$ if $w(x) \le \frac{c_1}{\ln^{\frac{2}{d}}|x| \ln 1/q}$) if $q \ge \frac{1}{3^d-2}$. The following trick is used in this case. Consider the partition of R^d into bigger cubes $Q(l, nl)$ of edge length l centered at point nl : $R^d = \bigcup_{n \in Z^d} Q(l, nl)$, $l \gg 1$ is integer. Consider an individual cube Q. The realization of $V(x)$ inside Q includes l^d Bernoulli r.v. ε_s, $s = 1, 2, \cdots l^d$. Let's fix a number $0 < p^* < p$, for example, $p^* = p/2$. We will call cube Q *gray* if $\#\{s : \varepsilon_s = 1\} \ge p^* l^d$ and we will continue to call the cube Q *white* in the opposite case. Thus, Q is gray if $V(x) = 1$ on some part of this cube of at least p^* portion of its volume.

The following fact is well-known in the theory of Bernoulli experiments. It follows from the exponential Chebyshev inequality:

For each $p > 0$ there exists $l = l(p) \ge 1$ such that the majority of big cubes $Q(l, nl)$ will be gray, i.e.

$$P\{Q \text{ is yellow}\} \le e^{-c(p)l^d} < \frac{1}{3^d - 2}.$$

At the same time at least $p/2$ portion of the volume of each gray cube is covered by brown sub-cubes of edge length one where $V(x) = 1$.

After that, one can apply our previous arguments to the systems of yellow and gray cubes $Q(l, nl)$ instead of white and black cubes Q_n

1-D case

Exact constants.

Let H_0 be a 1-D Hamiltonian with a Bernoulli potential $V(x, \omega)$

$$H = -\frac{d^2}{(dx)^2} + hV(x, \omega) - v(x).$$

Theorem 3. *For any $p > 0$ and $h > 0$,*

a) If $0 \leq v(x) \leq c_1/\ln^2(|x|+1)$ with $c_1 < \ln^2_{1/p}\pi^2$ then $N_0(v) < \infty$, P-a.s.;

b) If $v(x) > c_1/\ln^2(|x|+1)$ with $c_1 > \ln^2_{1/p}\pi^2$ then $N_0(v) = \infty$, P-a.s..

More general (Kronig-Penney) potentials.

We consider $L^2(R_+)$, Dirichlet or other b.c. at $x = 0$, and

$$V_\omega(x) = \begin{cases} 1 & \text{if} \quad x_i(\omega) \le x \le y_i(\omega) \\ 0 & \text{if} \quad y_i(\omega) < x < x_{i+1}(\omega) \end{cases}$$

where x_i and y_i are random variables on a probability space with $x_i < y_i < x_{i+1}$. Let, for simplicity, $y_i - x_i = l$. We assume that the distances between the bumps $L_i = x_i - y_{i-1}$ are i.i.d.r.v., and for all $a > 0$, $\mathtt{P}\{L_i > a\} > 0$ and $E[L_i] = \mu < \infty$ with $\mu > 0$.

Results depend on the distribution of tails.

Theorem 4. *Let $\{L_k\}$ be i.i.d.r.v. with exponential tails, that is,*

$$P\{L_k > x\} \sim e^{-\eta x}, \quad \eta > 0$$

. If $v(x) \le c_0/\ln^2 x$ for all large x and $c_0 < \eta^2\pi^2$, then $N_0 < \infty$ P-a.s.. If $v(x) \ge c_0/\ln^2 x$ for all large x and $c_0 > \eta^2\pi^2$, then $\mathcal{N}_0(\omega) = \infty$ P-a.s.

Example of heavy tails Suppose

$$P(L_k \ge x) \sim \frac{c_0}{x^\alpha}$$

for some $c_0, \alpha > 0$.

a) if $v(x) < \frac{1}{x^\gamma}, \gamma > 2, \quad x \to \infty$, then $N_0(v) < \infty$ $P-$ a.s.

b) If $v(x) > \frac{1}{x^\gamma}, \gamma < 2, \quad x \to \infty$ then $N_0 = \infty$ $P-$ a.s..