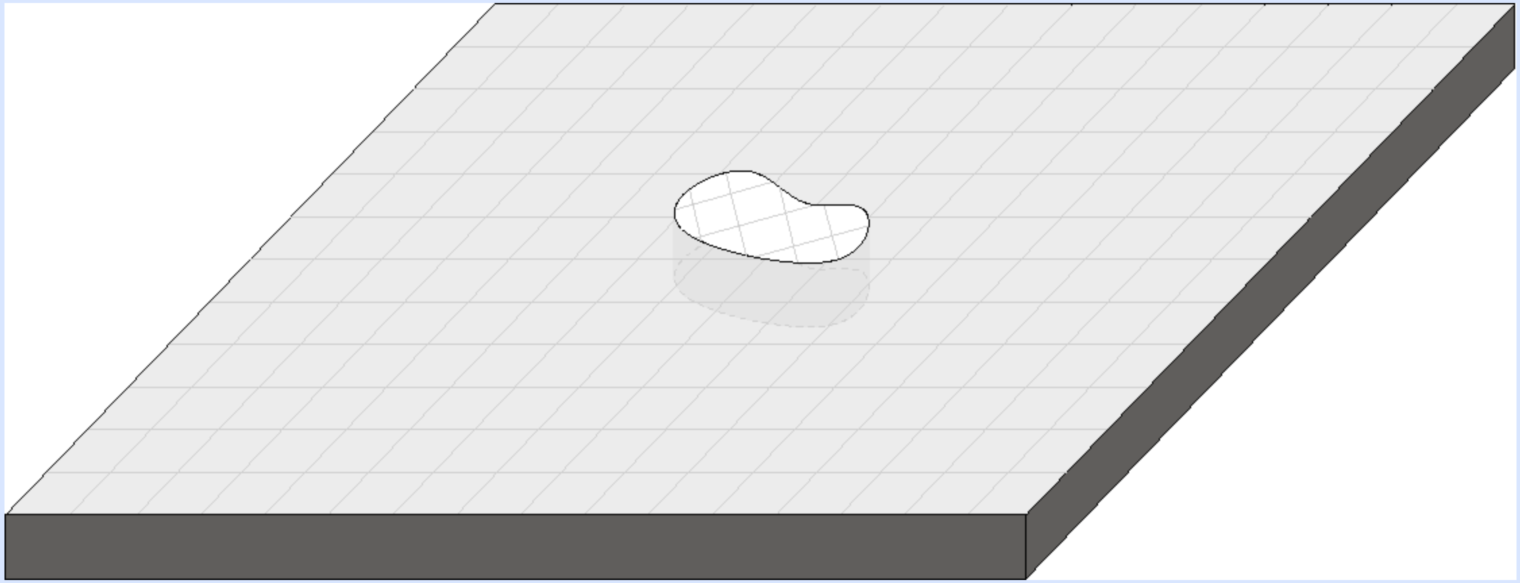
Trapped Modes in Elastic Media for Zero Poisson Coefficient



Three-dimensional elastic plate with local perturbation

Trapped modes in physics

Trapped modes are modes of oscillation which occur (in an unbounded domain) at discrete frequencies and consist of motion which is restricted to some localized region of the considered medium near some perturbation.

Water wave theory:

- Water waves in perturbed water channels

Acoustic theory:

- Acoustic resonances in waveguides with obstacles

Quantum mechanics:

- Bound states in bent, twisted and coupled waveguides

Electromagnetism:

- Trapped modes in twisted and coupled waveguides

Trapped modes correspond to (embedded) eigenvalues for systems with continuous spectrum.

Trapped modes in mathematics - a tale of two dimensions

Models for trapped modes show usually mixed dimensions:

- a global dimension - often 1 (wires) or 2 (layers)
- a local dimension d

The global dimension determines the low energy behaviour:

The Schrödinger operator $-\Delta - \alpha V(x)$ in $L^2(\mathbb{R}^d)$ has for $\int V dx > 0$ has in the limit of $\alpha \to +0$ one negative eigenvalue $-\lambda_1(\alpha)$ satisfying

$$\begin{aligned} \sqrt{\lambda_1(\alpha)} &= \tfrac{\alpha}{2} \int V dx + o(\alpha) \qquad \text{for} \quad d = 1\,, \\ \frac{1}{\ln \lambda^{-1}(\alpha)} &= \tfrac{\alpha}{4\pi} \int V dx + o(\alpha) \qquad \text{for} \quad d = 2\,. \end{aligned}$$

The local dimension determines the high energy behaviour:

Weyl type asymptotics for large $\alpha \to +\infty$.

Trapped modes in elasticity

Consider a semi-strip

$$\Omega = [0, +\infty) \times J \quad \text{with} \quad J = (-\pi/2, +\pi/2)\,.$$

Note that $-\Delta$ on this domain has no eigenvalues, neither in the Dirichlet nor in the Neumann case.

If you pass to linear elasticity with zero Poisson coefficient

$$A = -\Delta \otimes 1 - \operatorname{grad}\operatorname{div} \quad \text{on} \quad L^2(\Omega, \mathbb{C}^2) \ni u = \begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \end{pmatrix}$$

with stress-free boundary conditions (corresponds to the scalar Neumann case)

$$\begin{array}{r} \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0 \\ \frac{\partial u_2}{\partial x_2} = 0 \end{array} \quad \text{for } x_2 = \pm\frac{\pi}{2} \quad \text{and} \quad \begin{array}{r} \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0 \\ \frac{\partial u_1}{\partial x_1} = 0 \end{array} \quad \text{for } x_1 = 0\,,$$

then A has at least one positive eigenvalue embedded into the continuous spectrum.

- Shaw (1956): experiments on edge resonance in circular barium titanate disks
- many attempts to explain edge resonance by approximative analysis or numerical methods
- Roitberg, Vassiliev and W (1998): first rigorous proof for the existence of trapped modes in the elastic semi-strip
- Holst, Vassiliev (2000): Edge resonance in an elastic semi-infinite cylinder
- Gridin, Adamou, Craster (2005): Trapped modes in bent elastic rods and in curved elastic plates
- Zernov, Pichugin, Kaplunov (2006): Eigenvalue of a semi-infinite elastic plate

Main difference between elasticity and other physical systems:

- Water wave theory, Acoustics, Quantum mechanics, Electromagnetism:

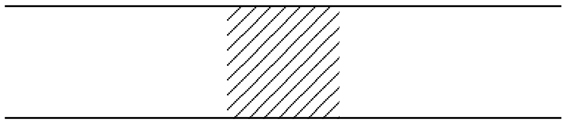
  trapped modes = eigenvalues of the Laplace operator (scalar)

- Elasticity theory:

  trapped modes = eigenvalues of the elastostatic operator (matrix)

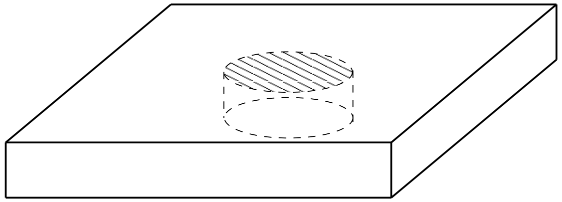
Three Model Problems

Elastic strip with local perturbation of material properties



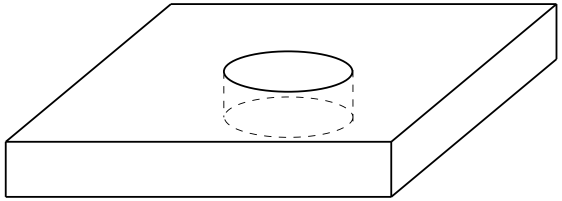
$$G_2 = \mathbb{R} \times J, \qquad J = (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

Elastic plate with local perturbation of material properties



$$G_3 = \mathbb{R}^2 \times J$$

Elastic plate with perturbation by a hole



$$G_3 \backslash \overline{\Omega}, \quad \Omega = \Omega_0 \times J, \ \Omega_0 \subset \mathbb{R}^2$$

General properties:

- isotropic, linear elastic medium
- unperturbed part has homogeneous material
- zero Poisson's ratio
- stress-free boundaries

The (unperturbed) elasticity operator

Consider the elastostatic operator with stress-free (Neumann-type) boundary conditions:

$$A_0 u = -\mathrm{div}\ \sigma(u), \quad u \in H^2(G_d; \mathbb{C}^d), \quad d = 2, 3,$$

$$\sigma(u)\mathbf{n}_{\partial G_d} = 0, \qquad \text{on} \quad \partial G_d.$$

Here we use

$$\sigma(u) = 2\mu\epsilon(u) + \lambda \mathrm{Tr}(\epsilon(u))\mathrm{I} \quad \text{is the stress matrix,}$$

$$\epsilon(u) = \frac{1}{2}\left((\nabla u) + (\nabla u)^T\right) \quad \text{is the strain matrix,}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \; \mu = \frac{E}{2(1+\nu)} \quad \text{are the Lamé constants.}$$

Special Case: Zero Poisson Coefficient

We put $E = 2$ and $\nu = 0$ and study the self-adjoint operator

$$A_0 = -\Delta \otimes I - \text{grad div} \qquad \text{in} \qquad L^2(G_d;\mathbb{C}^d)$$

associated with the Hermitian form

$$a_0[u,v] = 2\int_{G_d} \langle \epsilon(u), \epsilon(v)\rangle_{\mathbb{C}^{d\times d}}\, dx, \qquad u,v \in H^1(G_d;\mathbb{C}^d).$$

where

$$\epsilon(u) = \frac{1}{2}\left((\nabla u) + (\nabla u)^T\right) .$$

The Case of Local Changes of Material Coefficients: Additive perturbations of Young's Modulus

Let $f \in L^\infty(G_d)$ be

- compactly supported,
- independent of x_d-coordinate and
- $0 \le f(x) \le 1$ (but sometimes also just $f(x) \le 1$).

For $\beta \in (0,\infty)$ (scaling) and $\alpha \in (0,1)$ (coupling) we consider

$$A_{\alpha,\beta} u = -\mathrm{div}\,(1-\alpha f_\beta)(\nabla u + (\nabla u)^T) \qquad \text{in} \qquad L^2(G_d;\mathbb{C}^d), \quad f_\beta := f\left(\tfrac{\cdot}{\beta}\right),$$

with stress-free boundary conditions. This corresponds to

$$a_{\alpha,\beta}[u,v] = 2\int_{G_d} (1-\alpha f_\beta)\langle \epsilon(u), \epsilon(v)\rangle_{\mathbb{C}^{d\times d}}\, dx, \quad u,v \in H^1(G_d;\mathbb{C}^d).$$

Symmetries

We consider the following subspaces of $L^2(\mathbb{R} \times J; \mathbb{C}^2)$

$$H_1 \quad := \{u \in L^2(\mathbb{R} \times J; \mathbb{C}^2) \mid u_1 \text{ symmetric in } x_2,\ u_2 \text{ antisymmetric in } x_2\}$$

$$H_4 \quad := \{u \in H_1 \mid u_1(x_1, \cdot) \perp 1 \text{ in } L^2(J; \mathbb{C}) \text{ for a.e. } x_1 \in \mathbb{R}\}$$

The subspaces H_4 and $H_4^\perp$ reduce $A_{\alpha,\beta}$ and A_0. We consider

$$A^{(4)}_{\alpha,\beta} := A_{\alpha,\beta}|_{D(A_{\alpha,\beta}) \cap H_4}, \quad A^{(4)}_0 := A_0|_{D(A_0) \cap H_4}.$$

While $\sigma(A_0) = [0, +\infty)$ we have for the reduced operator $\sigma(A^{(4)}_0) = [\Lambda, +\infty)$ for a certain $\Lambda > 0$.

Let us discuss this more in detail for the unperturbed strip:

Separation of Variables for $d = 2$

Apply the unitary Fourier transform $\Phi : H_4 \to H_4$ in x_1-direction to

$$A_0^{(4)} = -\Delta - \text{grad div} = -\begin{pmatrix} 2\partial_{x_1 x_1} + \partial_{x_2 x_2} & \partial_{x_1 x_2} \\ \partial_{x_1 x_2} & \partial_{x_1 x_1} + 2\partial_{x_2 x_2} \end{pmatrix}$$

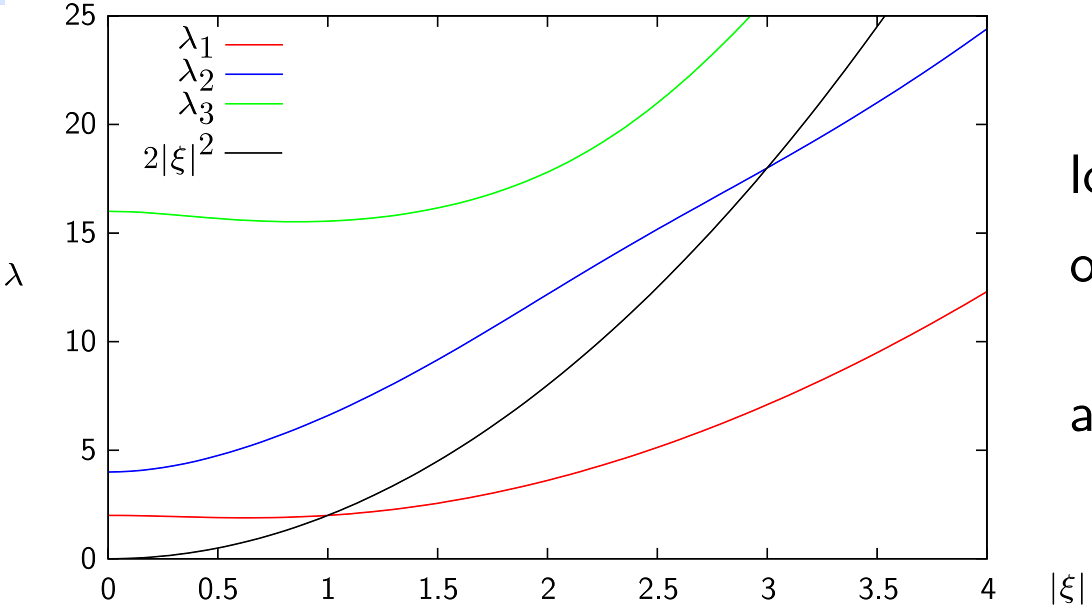
$$\partial_{x_2} u_2 = 0, \quad \partial_{x_2} u_1 + \partial_{x_1} u_2 = 0 \ \text{ for } \ x_2 = \pm\frac{\pi}{2},$$

and consider for $\xi \in \mathbb{R}$ and $\hat{u} = \Phi u$

$$A^{(4)}(\xi) := (\Phi A_0^{(4)} \Phi^*)(\xi) = \begin{pmatrix} -\partial_2^2 + 2\xi^2 & -i\xi\partial_2 \\ -i\xi\partial_2 & -2\partial_2^2 + \xi^2 \end{pmatrix},$$

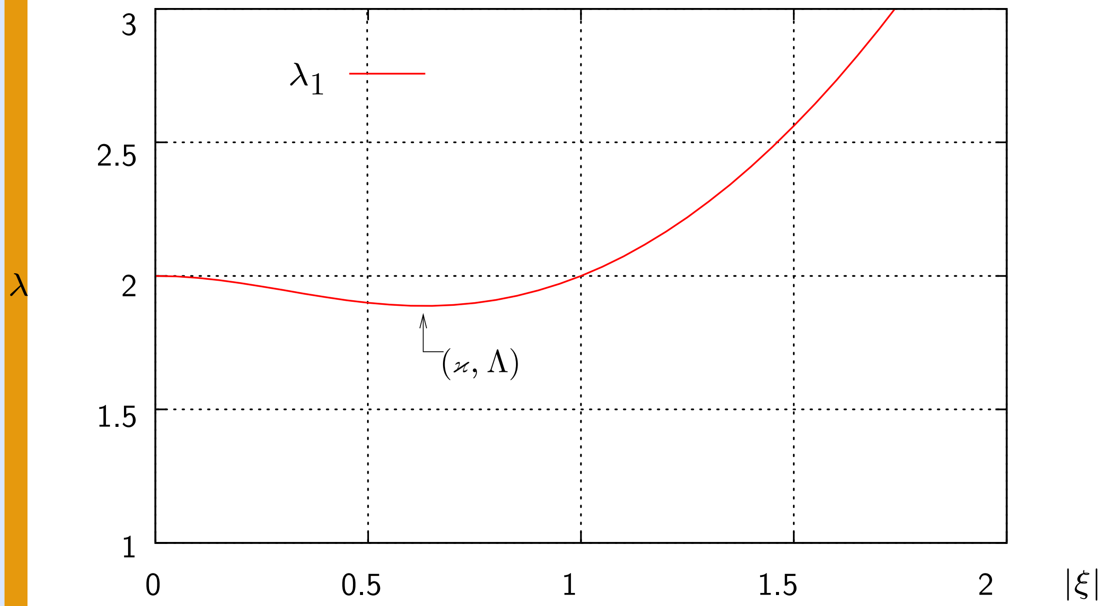
$$(\partial_{x_2} \hat{u}_2)|_{x_2 = \pi/2} = (\partial_{x_2} \hat{u}_1 + i\xi \hat{u}_2)|_{x_2 = \pi/2} = 0.$$

The symmetries in x_2-direction are preserved. For fixed ξ this Sturm-Liouville system has the (ordered) eigenvalues $\lambda_j(\xi)$ which depend continuously in ξ:



lowest three eigenvalue branches of $A^{(4)}(\xi)$

and $2|\xi|^2$ for comparison



global minimum of $\lambda_1(\xi)$ at $\varkappa \approx 0.64, \;\Lambda \approx 1.88$

minimum is non-degenerated $\lambda_1(\varkappa + \varepsilon) = \Lambda + q^2\varepsilon^2 + O(\varepsilon^3)$ for $\varepsilon \to 0$ with $q \approx 0.84$.

Results for Additive Perturbations for $d = 2$

For $\xi = \pm\varkappa$ we have two waves of minimal energy Λ:

$$-(\Delta + \text{grad div })w_\xi = \Lambda w_\xi, \quad w_\xi(x) = \begin{pmatrix} i\xi d_1(x_2) \\ d_2(x_2) \end{pmatrix} \mathrm{e}^{i\xi x_1},$$

- A local change of the material coefficients does not change the essential spectrum. Hence $\sigma(A^{(4)}_{\alpha,\beta}) = \sigma(A^{(4)}_0) = [\Lambda, +\infty)$.

- Each of the two minimal waves can give rise to a weak coupling eigenvalue for $A^{(4)}_{\alpha,\beta}$ and hence for $A_{\alpha,\beta}$ below Λ.

- The weak coupling analysis (small α - small changes of the material parameter) follows [Simon 72] for the weak coupling bound state of a 1d Schrödinger operator; here we have a 1d-PDO with the symbol $\lambda_1(\xi)$ with an additive perturbation:

Weak coupling asymptotic formulae for $d = 2$

$$\mu_j(\beta) = \beta\Lambda \int_{\mathbb{R}} f(x_1)dx_1 + (-1)^j \beta\theta \left| \int_{\mathbb{R}} e^{2i\varkappa\beta x_1} f(x_1)dx_1 \right|, \qquad j = 1, 2,$$

$\theta = 1.816478 \pm 10^{-6}$ is explicitely given in terms of the minimal waves

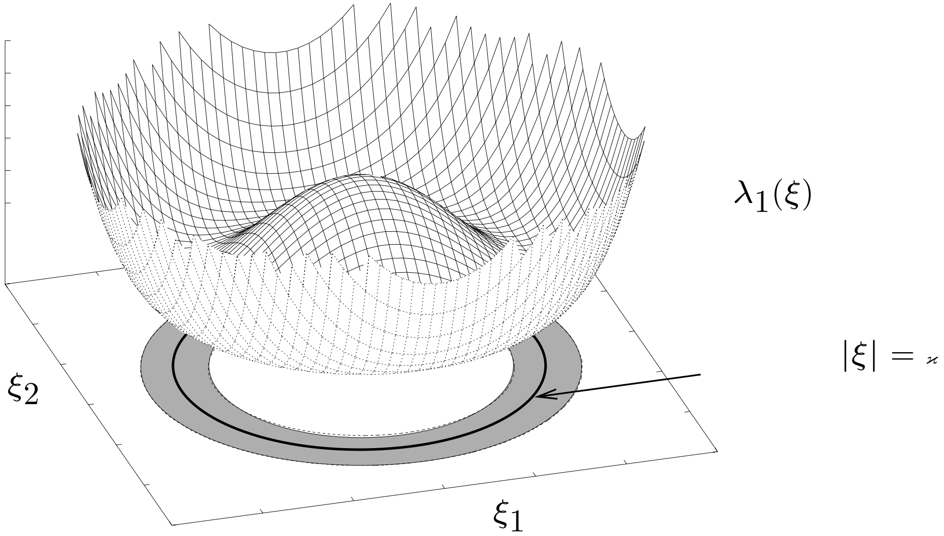
We fix $\beta > 0$.

- If $\mu_1 > 0$ and $\mu_2 > 0$, then for all sufficiently small positive α the spectrum of $A^{(4)}_{\alpha,\beta}$ below Λ consists of two eigenvalues and for $j = 1, 2$ we have

$$\nu_j(\alpha, \beta) = \Lambda - \frac{\alpha^2}{4q^2}\mu_j^2(\beta) + o(\alpha^2) \quad \text{as} \quad \alpha \to 0 \tag{1}$$

- If $\mu_1 > 0$ and $\mu_2 < 0$, then then for all sufficiently small positive α the spectrum of $A^{(4)}_{\alpha,\beta}$ below Λ consists of one eigenvalue $\nu_1(\alpha, \beta)$, satisfying (1) for $j = 1$.
- If $\mu_1 < 0$ and $\mu_2 < 0$, then then $A^{(4)}_{\alpha,\beta}$ does not have spectrum below Λ for all sufficiently small positive α.

Separation of Variables and Spectral Minimum for $d = 3$



Let Φ be the unitary Fourier transform in (x_1, x_2)-coordinates and $A^{(4)}(\xi) := (\Phi A_0^{(4)} \Phi^*)(\xi)$, $\xi \in \mathbb{R}^2$.

The lowest branch $\lambda_1(\xi)$ of the 3×3 Sturm-Liouville system is the rotation of the 2D minimal branch

We have *infinitely many* minimal waves, one for each $\xi \in \mathbb{R}^2$ with $|\xi| = \varkappa$:

$$-(\Delta + \text{grad div })w_\xi = \Lambda w_\xi, \;\; w_\xi(x) = \begin{pmatrix} i\xi_1 d_1(x_3) \\ i\xi_2 d_1(x_3) \\ d_2(x_3) \end{pmatrix} \mathrm{e}^{i\xi\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}, x \in \mathbb{R}^2 \times J.$$

Results for Additive Perturbations for $d = 3$

- Let $0 \leq f(x) \leq 1$: the additive perturbation is negatively definite; $\beta = 1$.
- Infinitely many minimal waves will give rise to infinitely many bound states
- We have to study weak coupling bound states for a PDO with the symbol $\lambda_1(\xi)$ with a strongly degenerated minimum and some additive perturbation An operator-theoretical framework for this type of problems has been given in [Laptev, Safronov, Weidl 2002]

The limit of weak parameter changes and the accumulation rate of eigenvalues:

Let $\nu_k(\alpha)$ be the eigenvalues of $A^{(4)}_{\alpha,1}$ below Λ in non-decreasing order. Let $\zeta_k(K)$ be eigenvalues of a certain compact integral operator K in non-increasing order. Then $A^{(4)}_{\alpha,1}$ has infinitely many eigenvalues below Λ and

$$\nu_k(\alpha) = \Lambda - \alpha^2(\Lambda\pi\zeta_k(K))^2 + o(\alpha^2) \quad \text{as} \quad \alpha \to 0.$$

$$\ln(\Lambda - \nu_k(\alpha)) = -2k\ln k + o(k\ln k) \quad \text{as} \quad k \to \infty.$$

Ideas of the proof

1. Step: Develop appropriate Birman-Schwinger principle

Let $U = \frac{1}{\sqrt{2}}(\nabla + \nabla^T)(A_0^{(4)})^{-\frac{1}{2}}$. The Birman-Schwinger operator is given by

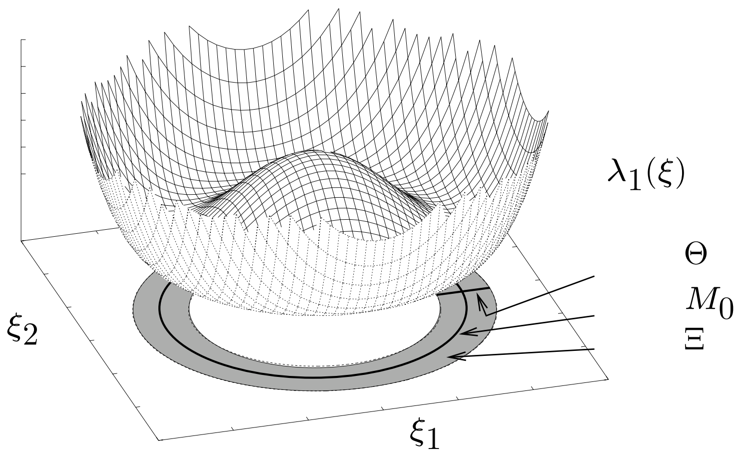
$$\begin{aligned}
\mathcal{Y}_\alpha(\tau) &= \left(\frac{\Lambda - \tau}{A_0^{(4)} - \Lambda + \tau}\right)^{\frac{1}{2}} V_\alpha \left(\frac{\Lambda - \tau}{A_0^{(4)} - \Lambda + \tau}\right)^{\frac{1}{2}}, \\
V_\alpha &= U^* \sqrt{\alpha f}(I - \alpha \sqrt{f}\, U U^* \sqrt{f})^{-1} \sqrt{\alpha f}\, U = \alpha U^* f U + \alpha^2 X_\alpha(f), \\
1 &= \zeta_j(\mathcal{Y}_\alpha(\tau)), \qquad \tau = \Lambda - \nu_j(\alpha), \qquad j \in \mathbb{N}.
\end{aligned}$$

2. Step: Count eigenvalues and use Birman-Schwinger principle

Our problem transforms into

$$\lim_{\tau \to 0} \frac{n_-(\Lambda - \tau, A_\alpha^{(4)})}{w^{-1}(\tau)} = \lim_{\tau \to 0} \frac{n_+(1, \mathcal{Y}_\alpha(\tau))}{w^{-1}(\tau)} = 1, \quad w(t) = t^{-2t}.$$

3. Step Reduce Birman-Schwinger operator to the spectral minimum



Reduce $\left(\frac{\Lambda-\tau}{A_0^{(4)}-\Lambda+\tau}\right)^{\frac{1}{2}} U^* f U \left(\frac{\Lambda-\tau}{A_0^{(4)}-\Lambda+\tau}\right)^{\frac{1}{2}}$

to $F^* \left[\left(\frac{\Lambda-\tau}{\lambda^2+\tau}\right) \otimes K\right] F \oplus \mathbb{O}$

where Π_c : spectral projection on $\lambda_1(\Xi)$

$F : \Pi_c L^2 \to L^2(\Theta) \otimes L^2(M_0, d\mu_0)$

$$\lim_{\tau\to 0} \frac{n_+(1, \mathcal{Y}_\alpha(\tau))}{w^{-1}(\tau)} = \lim_{\tau\to 0} \frac{n_+(\tau, K)}{w^{-1}(\tau^2)}, \qquad K := (F\Pi_c U^* f U F^*)|_{\lambda=0}.$$

4. Step: Estimate eigenvalues of reduced operator

An explicite computation for the reduced Birman-Schwinger operator K yields

$$\lim_{\tau\to 0} \frac{n_+(\tau, K)}{w^{-1}(\tau^2)} = 1.$$

Edge resonances in a punched Plate

Let $\Omega_0 \subset \mathbb{R}^2$ be a bounded Lipschitz domain and set $\Omega := \Omega_0 \times J$, $J = (-\frac{\pi}{2}, \frac{\pi}{2})$. Put $G_3 = \mathbb{R}^2 \times J$ and $\Omega^c := G_3 \backslash \overline{\Omega}$ as well as $\Gamma := \partial\Omega_0 \times J$. Consider

$$\begin{aligned} A_\Omega u &= -\mathrm{div}\,(\nabla u + (\nabla u)^T) \quad &\text{in} \quad & L^2(\Omega^c; \mathbb{C}^3), \\ A_{\Omega^c} u &= -\mathrm{div}\,(\nabla u + (\nabla u)^T) \quad &\text{in} \quad & L^2(\Omega; \mathbb{C}^3)\,. \end{aligned}$$

with stress-free boundary conditions. The corresponding Hermitian form is

$$\begin{aligned} a_\Omega[u,v] &= 2\int_{G_3\backslash\overline{\Omega}} \langle \epsilon(u), \epsilon(v)\rangle_{\mathbb{C}^{3\times 3}}\,dx, \quad & u,v \in H^1(\Omega^c; \mathbb{C}^3), \\ a_{\Omega^c}[u,v] &= 2\int_{G_3\backslash\overline{\Omega}} \langle \epsilon(u), \epsilon(v)\rangle_{\mathbb{C}^{3\times 3}}\,dx, \quad & u,v \in H^1(\Omega; \mathbb{C}^3). \end{aligned}$$

$$A_\Gamma = A_\Omega \oplus A_{\Omega^c} \quad \text{on} \quad L^2(G_3;\mathbb{C}^3) \quad \text{and} \quad \sigma(A_\Gamma) = \sigma(A_\Omega) \cup \sigma(A_{\Omega^c}).$$

We study

- the existence of infinitely eigenvalues below $\Lambda > 0$,
- the accumulation of eigenvalues.

Since the spectrum of the elliptic second order operator A_{Ω^c} on an bounded domain is discrete and accumulates to infinity only, A_{Ω^c} has only finitely many eigenvalues below any finite threshold! Hence we can work with A_Γ instead of A_Ω, that is A_0 perturbed by stress-free boundary conditions at Γ.

The operator $A_\Gamma^{(4)}$ has infinitely many eigenvalues ν_k below Λ. It holds

$$\ln(\Lambda - \nu_k) = -2k \ln k + o(k \ln k) \quad \text{as} \quad k \to \infty.$$

A compactly supported perturbation of a differential operator with strongly degenerated symbol yields exponential accumulation rates!

Open Problems

- Our results are just the starting point for the really interesting questions.
- Embedded eigenvalues in the zero Poisson case turn into resonances if the symmetries are broken. There should be a strong concentration of resonances near the threshold Λ.
- Experimental verification of these resonances. Note that the interesting frequency range does not much depend on the scale of the local perturbation.
- Numerical and analytical verification of the resonances. We need really 3d- and multi-scale numerics.
- Hidden cracks and enclosures.
- New symmetries for non-zero Poisson coefficients in the spirit of the [Zernov, Pichugin, Kaplunov 2006] result.
- How do the eigenfunctions look like?
- Dynamics for drilling or crack propagation?

Literature

- [RVW] I. Roitberg, D. Vassiliev and T. Weidl: Edge resonance in an elastic semi-strip. *Quart. J. Mech. Appl. Math.* **51 No. 1** 1–14 (1998)
- [GCA] D. Gridin, R. V. Craster and A. T. I. Adamou: Trapped modes in curved elastic plates. *Proc. R. Soc. A* **461** 1181–1197 (2005)
- [ZPK] V. Zernov, A. V. Pichugin and J. Kaplunov: Eigenvalue of a semi-infinite elastic plate. *Proc. R. Soc. A* **462** 1255–1270 (2006)
- [FW] C. Förster and T. Weidl: Trapped modes for an elastic strip with perturbation of the material properties. *Q. Jl. Mech. Appl. Math.* **59** 399–418 (2006)
- [F] C. Förster: Trapped modes for an elastic plate with perturbation of Young's modulus. *Preprint: Stuttgarter Mathematische Berichte* **009** (2006)
- [LSW] A. Laptev, O. Safronov, and T. Weidl: Bound states asymptotics for elliptic operators with strongly degenerating symbols. *Nonlinear problems in mathematical physics and related topics, I, In Honor of Professor O. A. Ladyzhenskaya.*, Int. Math. Ser. 1, Kluwer/Plenum, New York, 233–246, (2002)