Stratified Monte Carlo quadrature for continuous random functions

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Outline

Suppose a continuous random field $X(\mathbf{t})$, $\mathbf{t} \in [0,1]^d$, $d \geq 1$, with finite second moment can observed in a finite number of randomly chosen points. We want to approximate

$$\int_{[0,1]^d} X(\mathbf{t}) d\mathbf{t}$$

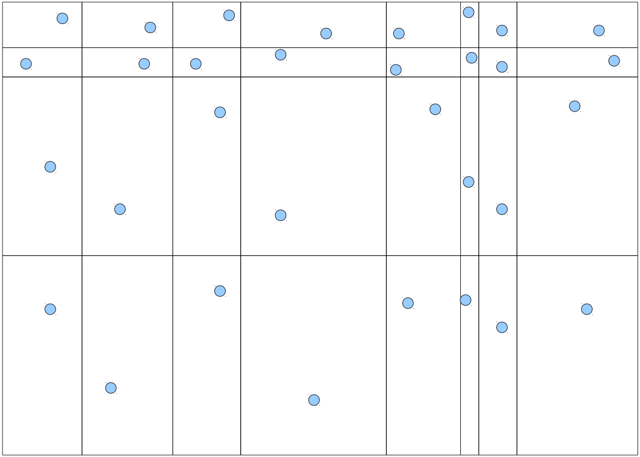
by a quadrature formula based on these observations.

Stratified Monte Carlo quadrature

Let $\mathcal{D} := [0,1]^d$ be partitioned into N stratas $\mathcal{D}_1, \ldots, \mathcal{D}_N$ by a rectangular grid. Let $|\mathcal{D}_i|$ denote the volume of the hyperrectangle $\mathcal{D}_i$, $i = 1, \ldots, N$. For a random field $X \in \mathcal{C}(\mathcal{D})$, define a *stratified Monte Carlo quadrature* (sMCq)

$$I_N(X) := \sum_{i=1}^{N} X(\boldsymbol{\eta}_i)|\mathcal{D}_i|,$$

where $\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_N$ are uniformly distributed in the strata $\mathcal{D}_1, \ldots, \mathcal{D}_N$, respectively.



Sampling grid distribution

- Interdimensional grid distribution
- Withindimensional grid distribution

Interdimensional grid distribution

The interdimensional distribution of N strata is determined by a vector function $\pi^* : \mathbb{N} \to \mathbb{N}^d$:

$$(n_1^*, n_2^*, \ldots, n_d^*) =: (\pi_1^*(N), \pi_2^*(N), \ldots, \pi_d^*(N)) =: \pi^*(N),$$

where $\lim_{N\to\infty} \pi_j^*(N) = \infty$, $j = 1, 2, \ldots, d$, and the condition

$$\prod_{j=1}^{d} \pi_j^*(N) = N$$

is satisfied.

Withindimensional grid distribution

We consider **cross regular sequences** of designs $T_N := \{\mathbf{t_i} = (t_{1,i_1}, \dots, t_{d,i_d}) : \mathbf{i} = (i_1, \dots, i_d),\ 0 \le i_k \le n_k^*,\ k = 1, \dots, d\}$ defined by the one-dimensional grids

$$\int_0^{t_{j,i}} h_j^*(v)dv = \frac{i}{n_j^*}, \quad i = 0, 1, \dots, n_j^*, \quad j = 1, \dots, d,$$

where $h_j^*(s)$, $s \in [0,1]$, $j = 1, \dots, d$, are positive and continuous density functions.

Approximation Accuracy Measure

Mean Squared Error

The accuracy of the approximation is measured by the mean squared error, i.e.,

$$e_N^2 = \mathrm{E}(I(X) - I_N(X))^2 = \|\, I(X) - I_N(X) \,\|^2 .$$

Fields of interest

For $k \leq d$, let $\mathbf{l} = (l_1, \ldots, l_k)$ be a vector of positive integers such that $\sum_{j=1}^{k} l_j = d$, and let $L_i := \sum_{j=1}^{i} l_j, i = 0, \ldots k$, $L_0 = 0$, be the sequence of its cumulative sums.

Component division

For $k \le d$, let $\mathbf{l} = (l_1, \ldots, l_k)$ be a vector of positive integers such that $\sum_{j=1}^{k} l_j = d$, and let $L_i := \sum_{j=1}^{i} l_j, i = 0, \ldots k$, $L_0 = 0$, be the sequence of its cumulative sums.

Then the vector $\mathbf{l}$ defines the *l-decomposition* of $\mathcal{D}$ into $\mathcal{D}^1 \times \mathcal{D}^2 \times \ldots \mathcal{D}^k$, with the l_j-cube $\mathcal{D}^j = [0,1]^{l_j}$, $j = 1, \ldots, k$.

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For any $\mathbf{s} \in \mathcal{D}$, we denote the coordinates vector corresponding to the j-th component of the decomposition by $\mathbf{s}^j$, i.e.,

$$\mathbf{s}^j = \mathbf{s}^j(\mathbf{l}) := (s_{L_{j-1}+1}, \ldots, s_{L_j}) \in \mathcal{D}^j, \quad j = 1, \ldots, k.$$

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Example Let $\mathcal{D} = [0,1]^3$ and $\mathbf{l} = (1,2)$. Then for any $s = (s_1, s_2, s_3) \in \mathcal{D}$, $\mathbf{s}^1 = s_1$ and $\mathbf{s}^2 = (s_2, s_3)$.

For a vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$, $0 < \alpha_j < 2$, $j = 1, \ldots, k$, and the decomposition vector $\mathbf{l} = (l_1, \ldots, l_k)$, we define

$$|| \, \mathbf{s} \, ||_{\boldsymbol{\alpha}} := \sum_{j=1}^{k} \left|\left| \, \mathbf{s}^j \, \right|\right|^{\alpha_j} \qquad \text{for all } \mathbf{s} \in \mathcal{D}$$

with the Euclidean norms $||\mathbf{s}^j||$, $j = 1, \ldots, k$.

Hölder fields

For a random field $X \in \mathcal{C}([0,1]^d)$, we say that $X \in \mathcal{C}_{\mathbf{l}}^{\boldsymbol{\alpha}}([0,1]^d, C)$ if for some $\boldsymbol{\alpha}$, $\mathbf{l}$, and a positive constant C, the random field X satisfies the Hölder condition, i.e.,

$$\| X(\mathbf{t}+\mathbf{s}) - X(\mathbf{t}) \|^2 \leq C \| \mathbf{s} \|_{\boldsymbol{\alpha}} \qquad \text{for all } \mathbf{t}, \mathbf{t}+\mathbf{s} \in [0,1]^d.$$

For a random field $X \in \mathcal{C}([0,1]^d)$, we say that $X \in \mathcal{B}_{\mathbf{l}}^{\boldsymbol{\alpha}}([0,1]^d, c(\cdot))$ if for some $\boldsymbol{\alpha}$, $\mathbf{l}$, and a vector function $c(\mathbf{t}) = (c_1(\mathbf{t}), \ldots, c_k(\mathbf{t}))$, $\mathbf{t} \in [0,1]^d$, the random field X is **locally stationary**, i.e.,

$$\frac{\| X(\mathbf{t}+\mathbf{s}) - X(\mathbf{t}) \|^2}{\sum_{j=1}^{k} c_k(\mathbf{t}) \| \mathbf{s}^j \|^{\alpha_j}} \to 1 \quad \text{as } \mathbf{s} \to 0 \text{ uniformly in } \mathbf{t} \in [0,1]^d,$$

with positive and continuous functions $c_1(\cdot), \ldots, c_k(\cdot)$.

We assume additionally that for $j = 1, \ldots, k$, the function $c_j(\cdot)$ is invariant with respect to coordinates permutation within the j-th component.

For the partition generated by a vector $\mathbf{l} = (l_1, \ldots, l_k)$, we consider cross regular designs T_N, defined by functions $h = (h_1, \ldots, h_k)$ and $\pi(N) = (n_1(N), \ldots, n_k(N))$, in the following way:

$$h_i^*(\cdot) \equiv h_j(\cdot), \quad n_i^* = n_j, \quad i = L_{j-1}+1, \ldots, L_j, \quad j = 1, \ldots, k.$$

We call functions $h_1(\cdot), \ldots, h_k(\cdot)$ and $\pi(N)$ **withincomponent densities** and **intercomponent grid distribution**, respectively. The corresponding property of a design T_N is denoted by: T_N is $cRS(h, \pi, \mathbf{l})$.

Main Results

For any $\mathbf{u} \in \mathbb{R}_+^m$, we denote

$$b_{\beta,m}(\mathbf{u}) = \frac{1}{2} \int_{[0,1]^m} \int_{[0,1]^m} ||\, \mathbf{u} * (\mathbf{t} - \mathbf{v})\, ||^{\beta}\, d\mathbf{t} d\mathbf{v}$$

where $'*'$ denotes coordinate-wise multiplication, i.e., if $x = (x_1, x_2, \ldots, x_d)'$ and $y = (y_1, y_2, \ldots, y_d)$ then $x * y = (x_1 y_1, x_2 y_2, \ldots, x_d y_d)$.

For sequences of real numbers u_n and v_n, we write $u_n \sim v_n$ if $\lim_{n\to\infty} u_n/v_n = 1$ and $u_n \lesssim v_n$ if $\lim_{n\to\infty} u_n/v_n \leq 1$.

Theorem

Let $X \in \mathcal{B}_{\mathbf{l}}^{\boldsymbol{\alpha}}(\mathcal{D}, c(\cdot))$ *be a random field and let* $I(X)$ *be approximated by sMCq* $I_N(X, T_N)$*, where* T_N *is cRS*$(h, \pi, \mathbf{l})$*. Then*

$$e_N^2 \sim \frac{1}{N} \sum_{j=1}^{k} \frac{v_j}{n_j^{\alpha_j}} \text{ as } N \to \infty,$$

where

$$v_j = \int_{\mathcal{D}} c_j(\mathbf{t}) b_{\alpha_j, l_j}(D_j(\mathbf{t}^j)) \prod_{m=1}^{d} h_m^*(t_m)^{-1} d\mathbf{t},$$

and $D_j(\mathbf{t}^j) = (1/h_j(t_{L_{j-1}+1}), \ldots, 1/h_j(t_{L_j}))$.

Intercomponent optimality

Denote by

$$\rho := \left(\sum_{i=1}^{k} \frac{l_i}{\alpha_i} \right)^{-1}, \qquad \kappa := \prod_{j=1}^{k} v_j^{l_j/\alpha_j}.$$

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Proposition

Let $X \in \mathcal{B}_{\mathbf{l}}^{\boldsymbol{\alpha}}(\mathcal{D}, c(\cdot))$ be a random field and let $I(X)$ be approximated by sMCq $I_N(X, T_N)$, where T_N is cRS$(h, \pi, \mathbf{l})$. Then

$$\| I(X) - I_N(X, T_N) \|^2 \gtrsim k \, \frac{\kappa^{\rho}}{N^{1+\rho}} \text{ as } N \to \infty.$$

Moreover, for the asymptotically optimal intercomponent grid allocation,

$$n_{j,opt} \sim \frac{v_j^{1/\alpha_j}}{\kappa^{\rho/\alpha_j}} N^{\rho/\alpha_j}, \quad j = 1, \ldots, k \text{ as } N \to \infty,$$

the equality is attained asymptotically.

For $0 < \beta < 2$, let

$$a_\beta := \frac{1}{(1+\beta)(2+\beta)}.$$

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Corollary

Let $X \in \mathcal{B}_1^\alpha([0,1], c(\cdot))$ *be a random process and let* $I(X)$ *be approximated by sMCq* $I_N(X, T_N)$, *where* T_N *is RS*(h). *Then*

$$\lim_{N\to\infty} N^{1+\alpha} \,||\, I(X) - I_N(X, T_N) \,||^2 = a_\alpha \int_0^1 c(t) h(t)^{-(1+\alpha)} dt.$$

The density minimizing the asymptotic constant is given by

$$h_{opt}(t) = \frac{c(t)^\gamma}{\int_0^1 c(\tau)^\gamma d\tau}, \qquad t \in [0,1],$$

where $\gamma := 1/(2+\alpha)$. *Furthermore, for such density, we get*

$$\lim_{N\to\infty} N^{1+\alpha} \,||\, I(X) - I_N(X, T_N) \,||^2 = a_\alpha \left(\int_0^1 c(t)^\gamma dt \right)^{1/\gamma}.$$

Proposition

Let $X \in \mathcal{C}_{\mathbf{1}}^{\boldsymbol{\alpha}}(\mathcal{D}, C)$ *be a random field and let* $I(X)$ *be approximated by sMCq* $I_N(X, T_N)$*, where* T_N *is* $cRS(h, \pi, \mathbf{l})$*. Then*

$$|| \, I(X) - I_N(X, T_N) \, ||^2 \leq \frac{C}{N} \sum_{j=1}^{k} \frac{d_j}{n^{\alpha_j}}$$

for positive constants $d_1, \ldots, d_k$*. Moreover if* $n_j \sim N^{\rho/\alpha_j}$*,* $j = 1, \ldots, k$*, then*

$$|| \, I(X) - I_N(X, T_N) \, ||^2 = \mathrm{O}\left(N^{-(\rho+1)}\right).$$

Point Singularity at the origin

We focus on the random fields consisting of one component, i.e., $k = 1$, $\mathbf{l} = d$ and $\boldsymbol{\alpha} = \alpha$, and denote the classes of corresponding Hölder and locally stationary random functions by $\mathcal{C}_d^\alpha$ and $\mathcal{B}_d^\alpha$, respectively.

Let a random function $X(\mathbf{t})$, $t \in [0,1]^d$, satisfy the Hölder condition with $\beta \in (0,2)$ for $\mathbf{t} \in [0,1]^d$. Let, additionally, X be locally stationary with parameter $\alpha > \beta$, for all points $\mathbf{t} \in (0,1]^d$. We construct sequences of grid designs with an asymptotic approximation rate $N^{-(1+\alpha/d)}$.

The definition of cRS for $k = 1$ gives that $n_j = N^{1/d}$ and $h_j^*(\cdot) = h(\cdot)$, $j = 1, \ldots, d$, for a positive and continuous density $h(t)$, $t \in [0, 1]$. For the density $h(\cdot)$, we define the related distribution functions

$$H(t) := \int_0^t h(u)du, \qquad G(t) := H^{-1}(t) = \int_0^t g(v)dv, \quad t \in [0, 1],$$

i.e., $G(\cdot)$ is a quantile function for the distribution H. Moreover, by

$$g(t) := G'(t) = 1/h(G(t)), \quad t \in [0, 1],$$

we denote the *quantile density function*.

For a random function $X \in \mathcal{C}([0,1]^d)$, we say that:

- $X \in \mathcal{C}_d^\alpha(\mathcal{A}, V(\cdot))$ if $X \in \mathcal{C}(\mathcal{A})$ and X is *locally Hölder continuous*, i.e., if for all $\mathbf{t}, \mathbf{t}+\mathbf{s}, \in \mathcal{A}$,

  $$|| X(\mathbf{t}+\mathbf{s}) - X(\mathbf{t}) ||^2 \leq V(\bar{\mathbf{t}}) || \mathbf{s} ||^\alpha, 0 < \alpha < 2,$$

  for a positive continuous function $V(\mathbf{t}), \mathbf{t} \in \mathcal{A}$, and some $\bar{\mathbf{t}} \in \{\bar{\mathbf{t}} : \bar{\mathbf{t}} = \mathbf{t} + \mathbf{s} * \mathbf{u}, \mathbf{u} \in [0,1]^d\}$;

- $X \in \mathcal{CB}_d^\alpha((0,1]^d, c(\cdot), V(\cdot))$ if there exist $0 < \alpha < 2$, and positive continuous functions $c(\mathbf{t}), V(\mathbf{t})$, $\mathbf{t} \in (0,1]^d$ such that $X \in \mathcal{C}_d^\alpha(\mathcal{A}, V(\cdot)) \cap \mathcal{B}_d^\alpha(\mathcal{A}, c(\cdot))$ for any closed $\mathcal{A} \subset (0,1]^d$.

Shifting Condition

We say that a positive function $f(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^d$ satisfies a *shifting condition* if there exist positive constants C and a such that

$$f(\mathbf{s}) \leq C f(\mathbf{v})$$

for all $\mathbf{s}, \mathbf{v}$ such that $\frac{1}{\sqrt{3+d}} \leq \frac{||\,\mathbf{s}\,||}{||\,\mathbf{v}\,||} \leq \sqrt{3+d}, \quad \mathbf{s}, \mathbf{v} \in [0, a]^d \backslash \mathbf{0}_d$.

Let $X \in \mathcal{C}_d^\beta([0,1]^d, M) \cap \mathcal{CB}_d^\alpha((0,1]^d, c(\cdot), V(\cdot))$, $0 < \beta < \alpha < 2$.

For $\beta > \alpha - d$, we prove that under some condition on local Hölder function $V(\cdot)$, the cross regular sequences attain the optimal approximation rate $N^{-(1+\alpha/d)}$.

Observe that $\beta > \alpha - d$ holds for all $\alpha, \beta \in (0,2)$ if $d \geq 2$ and for $d = 1$ if $\beta > \alpha - 1$.

Let $\mathbf{H}(\mathbf{t}) := (H(t_1), \ldots, H(t_d))$,$\mathbf{t} \in [0,1]^d$, and $\mathbf{G}(\mathbf{t}) =: (G(t_1), \ldots, G(t_d))$, $\mathbf{t} \in [0,1]^d$. We formulate the following condition:

(C) Let $V(\mathbf{G}(\cdot))$ be bounded from above by a function $R(\cdot)$ satisfying shifting condition and such that $R(\mathbf{H}(\cdot)) \in L^1[0,b]^d$, for some $b > 0$.

Theorem

Let $X \in \mathcal{C}_d^{\beta}([0,1]^d, M) \cap \mathcal{CB}_d^{\alpha}((0,1]^d, c(\cdot), V(\cdot))$, $\alpha - d < \beta < \alpha$, *be a random field and let* $I(X)$ *be approximated by sMCQ* $I_N(X, T_N)$, *where* T_N *is* $cRS(h, \pi, d)$. *If the local Hölder function* $V(\cdot)$ *satisfies the condition* (C), *then*

$$|| \, I(X) - I_N(X, T_N) \, ||^2 \sim \frac{1}{N^{1+\alpha/d}} \int_{\mathcal{D}} c(\mathbf{t}) b_{\alpha,d}(D(\mathbf{t})) \prod_{m=1}^{d} h(t_m)^{-1} d\mathbf{t}$$

as $N \to \infty$, *where* $D(\mathbf{t}) = (1/h(t_1), \ldots, 1/h(t_d))$.

quasi RS for random processes

Now we consider the case $d = 1$ and $0 < \beta \leq \alpha - 1$, which is not included in the above theorem.

We consider *quasi regular sequences* (qRS) of sampling designs $T_N = T_N(h)$, which are a simple modification of the regular sequences.

We assume that $h(t)$ is continuous for $t \in (0, 1]$, and allow it to be unbounded in $t = 0$. If $h(t)$ is unbounded in $t = 0$, then $h(t) \to +\infty$ as $t \to 0+$. We denote this property of T_N by: T_N is qRS(h). The corresponding quantile density function $g(t)$ is assumed to be continuous for $t \in [0, 1]$ with the convention that $g(0) = 0$ if $h(t) \to +\infty$ as $t \to 0+$.

Let $X \in \mathcal{C}_1^\beta([0,1], M) \cap \mathcal{CB}_1^\alpha((0,1], c(\cdot), V(\cdot))$, $0 < \beta \leq \alpha - 1$. We modify the condition (C) and formulate the following condition for a local Hölder function $V(\cdot)$ and a grid generating density $h(\cdot)$:

(C′) Let $V(G(\cdot))$ and $g(\cdot)$ be bounded from above by functions $R(\cdot)$ and $r(\cdot)$, respectively, where $R(\cdot)$ and $r(\cdot)$ satisfy the shifting condition. Moreover, let $R(H(t))r(H(t))^{-(1+\alpha)} \in L^1[0,b]$, for some $b > 0$, and

$$G(s) = \mathrm{o}\left(s^{(1+\alpha)/(2+\beta)}\right) \text{ as } s \to 0.$$

Theorem

Let $X \in \mathcal{C}_1^{\beta}([0,1], M) \cap \mathcal{CB}_1^{\alpha}((0,1], c(\cdot), V(\cdot))$, $0 < \beta \leq \alpha - 1$, *be a random process and let* $I(X)$ *be approximated by sMCQ* $I_N(X, T_N)$, *where* T_N *is qRS*(h). *Let for the density* $h(\cdot)$ *and local Hölder function* $V(\cdot)$, *the condition* (C′) *hold. Then*

$$\lim_{N\to\infty} N^{1+\alpha} \, || \, I(X) - I_N(X, T_N) \, ||^2 = a_\beta \int_0^1 c(t) h(t)^{-(1+\alpha)} dt.$$

Numerical Experiments

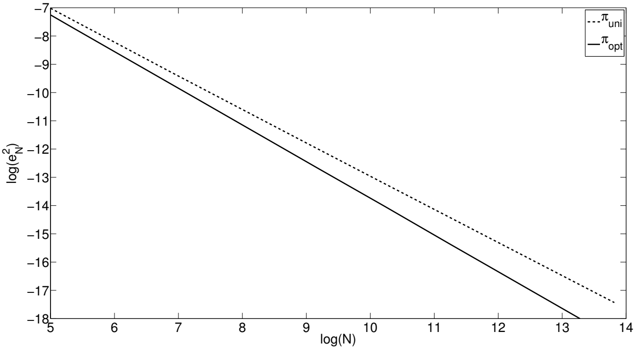
We use h_{uni} to denote a density that results uniform distribution of knots in each direction, i.e., $h_{uni}(t) \equiv 1$, $t \in \mathcal{D}$, and $\pi_{uni}(N)$ to denote the uniform distribution of knots between the components.

Example 1

Let $\mathcal{D} = [0,1]^3$ and $X(\mathbf{t})$ be a fractional Brownian field with covariance function

$$Cov(X(\mathbf{t}), X(\mathbf{s})) = \frac{1}{2}\left(||\,\mathbf{t}\,||_{\boldsymbol{\alpha}} + ||\,\mathbf{s}\,||_{\boldsymbol{\alpha}} - ||\,\mathbf{t}-\mathbf{s}\,||_{\boldsymbol{\alpha}}\right), \qquad \mathbf{s}, \mathbf{t} \in [0,1]^3,$$

where $\boldsymbol{\alpha} = (3/2, 1/2)$ and $\mathbf{l} = (2,1)$.



The plots correspond to the following asymptotic behavior:

$$\begin{aligned} e_N^2(\pi_{uni}) &\sim C_1\, N^{-7/6} + C_2\, N^{-3/2} \sim C_1\, N^{-7/6}, \\ e_N^2(\pi_{opt}) &\sim C_3\, N^{-13/10} \qquad \text{as } N \to \infty, \end{aligned}$$

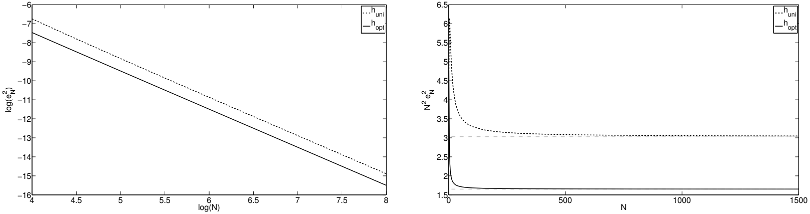
where $C_1 \simeq 0.26$, $C_2 \simeq 0.20$, and $C_3 \simeq 0.48$.

Example 2

Let $Y(t), t \in [0,1]$ be a stochastic process with covariance function $Cov(X(t), X(s)) = \exp(-|s-t|)$ and consider process

$$X(t) = \frac{1}{t+0.1} Y(t), \quad t \in [0,1].$$

Then $X \in \mathcal{B}_1^{\alpha}([0,1], c(\cdot))$ with $\alpha = 1$ and $c(t) = 2/(t+0.1)^2$, $t \in [0,1]$.



The plots correspond to the following asymptotic behavior:

$$\begin{array}{rcl} e_N^2(h_{uni}) & \sim & C_1\,N^{-2}, \\ e_N^2(h_{opt}) & \sim & C_2\,N^{-2} \text{ as } N \to \infty \end{array}$$

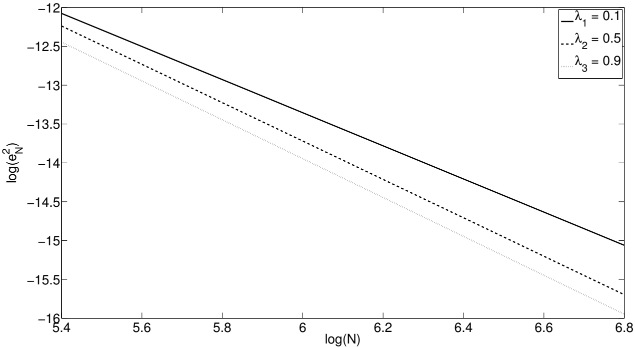
with $C_1 \simeq 3.03$ and $C_2 \simeq 0.86$.

Example 3

Let $X_\lambda(t) = B_{3/2}(t^\lambda)$, $t \in [0,1]$, $0 < \lambda < 1$, where $B_{m,\beta}$, $0 < \beta < 2$, is a fractional Brownian field. Then

$$X_\lambda \in \mathcal{C}_1^{3/2\lambda}([0,1], M) \cap \mathcal{BC}_1^{3/2}((0,1], c(\cdot), V(\cdot))$$

with $M = 1$ and $c(t) = V(t) = \lambda^{3/2} t^{3/2(\lambda-1)}$, $t \in [0,1]$. We consider the behavior of the mean squared errors for $\lambda_1 = 1/10$, $\lambda_2 = 1/2$, and $\lambda_3 = 9/10$.

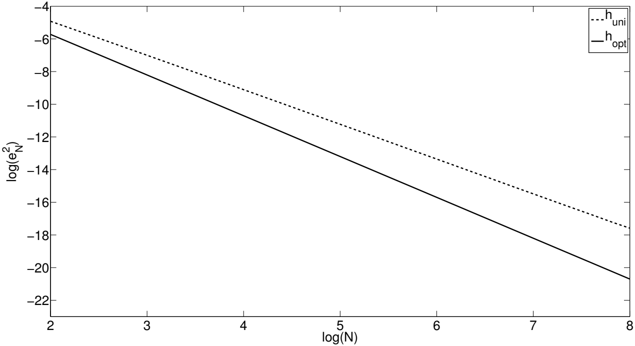


These plots correspond to the following asymptotic behavior:

$$
\begin{array}{rcl}
e_N^2(X_{\lambda_1}, h_{uni}) & \sim & C_1\, N^{-2.15}, \\
e_N^2(X_{\lambda_2}, h_{uni}) & \sim & C_2\, N^{-2.5}, \\
e_N^2(X_{\lambda_3}, h_{uni}) & \sim & C_3\, N^{-2.5} \text{ as } N \to \infty
\end{array}
$$

with $C_1 \simeq 0.64$, $C_2 = 3.69$, and $C_3 \simeq 2.86$.

Consider now the case $\lambda_1 = 1/10$. Using quasi regular sequences of designs we regain the optimal convergence rate.



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Thank you !