Gaussian random matrices and genus expansion

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GUE ensemble

Let X be a random square $n \times n$ Gaussian Hermitian matrix. Entries x_{ij} are random independent (up to transposition) complex-valued variables and $\mathbb{E}x_{ij} = 0$, $\mathbb{E}x_{ij}\overline{x_{ij}} = 1/n$.
Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of X.

Definition

The uniform distribution on $\lambda_1, \lambda_2, \ldots, \lambda_n$ is called empirical spectral distribution of the matrix X.

The main problem

Main Problem

What is the limit (as $n \to \infty$) behavior of the expected spectral distribution μ_n?

Moments of μ_n

$$m_k = \int_{\mathbb{R}} x^k \, d\mu_n(x)$$

$$m_k = \mathbb{E}\frac{\lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k}{n} = \frac{1}{n}\mathbb{E}\,\mathrm{Tr}\,X^k$$

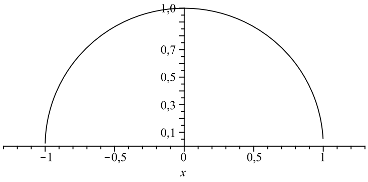
Theorem

Wigner'58

$$m_{2k} = \binom{2k}{k}\frac{1}{k+1} + o(1),$$

and all odd moments are zero.

Semicircle Law Density



Sketch of proof

Trace formula

$$\mathbb{E}\,\mathrm{Tr}\,X^{2k} = \mathbb{E}\sum^{*} x_{i_0 i_1} x_{i_1 i_2} \cdots x_{i_{2k-1} i_0}, \qquad (1)$$

where the sum $\sum^*$ is taken over all possible indices $i_0, i_1, \dots, i_{2k-1}$.

Balanced strings of brackets

The k^{th} Catalan number $\binom{2k}{k}\frac{1}{k+1}$ is the number of balanced strings of brackets with length $2k$.
Each term in the product (1) appears twice. The first appearence of a term corresponds to an opening bracket, and the second corresponds to a closing bracket.

Asymptotic Expansion

Asymptotic Expansion

What is the full asymptotics of $\mathbb{E}\operatorname{Tr} X^{2k}$?

Wick formula

Let $(x_1, x_2, \ldots, x_n)$ be a Gaussian vector and let $f_1, f_2, \ldots, f_{2k}$ be linear functions of $x_1, x_2, \ldots, x_n$. Then

$$\mathbb{E} f_1 f_2 \cdots f_{2k} = \sum \prod \mathbb{E}(f_i f_j),$$

where the sum is taken over all partitions of $\{1, 2, \ldots, 2k\}$ into pairs.

Gluing of a polygon

Polygon

Consider the product $x_{i_0 i_1} x_{i_1 i_2} \cdots x_{i_{2k-1} i_0}$ as a $2k$-gon. Then the partition into pairs corresponds to the pairwise gluing of all edges of this $2k$-gon.

Example

Let $k = 2$. Then $\mathbb{E}\,\mathrm{Tr}\,X^4 = \sum^* \mathbb{E}x_{i_0i_1}x_{i_1i_2}x_{i_2i_3}x_{i_3i_0}$. There are three partitions into pairs, and so there are three gluings. Two of them give a sphere, and one gives a torus.

How many vertices has the glued graph?

After gluing we obtain a graph embedded into some surface. The graph has exactly k edges and only 1 face.
Euler formula

$$V - E + F = 2 - 2g,$$

where g is the genus of the surface.

$$V = k + 1 - 2g$$

Harer-Zagier result

The main result

$$\frac{1}{n}\mathbb{E}\,\mathrm{Tr}\,X^{2k} = \sum_{g=0}^{[k/2]} T(k,g)\frac{1}{n^{2g}},$$

where $T(k,g)$ is the number of ways to glue pairwise all the edges of a $2k$-gon to produce a surface of a given genus g.

Some Generalizations

Problem

Consider a random matrix $W_m = X_1 X_2 \cdots X_m (X_1 X_2 \cdots X_m)^*$, where $X_1, X_2, \ldots, X_m$ are independent Gaussian non-Hermitian matrices. What is the limit behaviour of the spectral distribution of W_m?

The first term

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \operatorname{Tr} W_m^k = FC(m,k) = \frac{1}{mk+1} \binom{mk+k}{k},$$

so-called Fuss-Catalan number.

Some numerical results

The case $m = 2$

$$n^{-1}\mathbb{E}\,\mathrm{Tr}\,W_2 = 1$$

$$n^{-1}\mathbb{E}\,\mathrm{Tr}\,W_2^2 = 3 + n^{-2}$$

$$n^{-1}\mathbb{E}\,\mathrm{Tr}\,W_2^3 = 12 + 21n^{-2} + 3n^{-4}$$

$$n^{-1}\mathbb{E}\,\mathrm{Tr}\,W_2^4 = 55 + 270n^{-2} + 231n^{-4} + 20n^{-6}$$

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Thanks

Thank you for your attention!