# Stochastic modelling of mortality and financial markets

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# Overview of research

- General objective: pricing and hedging of mortality-linked cash flows
  - Derivatives (e.g. forwards, bonds and swaps) linked to the mortality of a certain population
  - Insurance portfolios, pension fund management

  in incomplete markets
- Stochastic modelling of risk factors
  - Mortality
  - Liabilities
  - Assets
  - Other relevant factors
- Numerical techniques
  - Integration quadratures
  - Numerical optimization

# Mortality-linked cash flows

- Population age structure is shifting towards the old, rendering private insurance companies and governments even more vulnerable to risks of misestimating mortality improvements
- Means of risk management for mortality-linked cash flows is needed
- Mortality-linked securities are financial instruments with cash flows dependent on the mortality of a reference population
- Mortality/longevity markets are developing
- Quantitative methods for risk management of mortality-linked securities are instrumental for the development of such a market

# Mortality-linked cash flows

The value of financial instruments depends essentially on the following subjective factors:

1. **Probability distribution**: description of future development of claims and investment returns, both involving significant uncertainties
   - Dependence between claims and asset returns instrumental in hedging
2. **Risk preferences**: the level of risk at which assets should cover liabilities
3. **Hedging strategy**: investment strategy for the given initial capital
   - Optimal strategy generally cannot be found (cf. Black-Scholes and delta hedging )

# Stochastic modelling of risk factors

- Focus of this talk: modelling the law of a multivariate stochastic process consisting of
  - Mortality
  - Asset returns (interest rates, stock price index)
- Particular emphasis on
  - Connections between mortality and asset returns
  - Long-term development of mortality
- GDP as a link between mortality risk factors and asset returns
- Aro & Pennanen: Stochastic modelling of mortality and its connection with financial markets (manuscript)

# Mortality model

- We employ the general discrete-time stochastic framework for mortality introduced in Aro & Pennanen, *User-friendly approach to stochastic mortality modelling*. European Actuarial Journal, 2011.
- Let $E_{x,t}$ be the size of population aged $[x, x+1)$ (cohort) at the beginning of year $t$
- Denote by $D_{x,t}$ the number of deaths of people aged $[x, x+1)$ during time $[t, t+1)$
- Objective: model the values of $E_{x,t}$ over time $t = 0, 1, 2, \dots$ for a given set $X \subset \mathbb{N}$ of ages
- Assume the conditional distribution of $E_{x+1,t+1} = E_{x,t} - D_{x,t}$ given $E_{x,t}$ is binomial:

$$E_{x+1,t+1} \sim \mathrm{Bin}(E_{x,t}, p_{x,t})$$

where $p_{x,t}$ is the probability that an individual aged $x$ and alive at the beginning of year $t$ is still alive at the end of that year

# Mortality model

- We reduce the dimensionality of $(p_{x,t})_{x \in X}$ by modelling the logistic probabilities by

$$\mathrm{logit}\, p_{x,t} := \ln\left(\frac{p_{x,t}}{1-p_{x,t}}\right) = \sum_{i=1}^{n} v_t^i \phi_i(x),$$

  where $\phi_i(x)$ are user-defined basis functions across cohorts, and $v_t^i$ stochastic risk factors that vary over time
- In other words, $p_{x,t} = p_{v(t)}(x)$, where $v(t) = (v_1(t), \ldots, v_n(t))$, and $p_v : X \to (0,1)$ is the parametric function defined for each $v \in \mathbb{R}^n$ by

$$p_v(x) = \frac{\exp\left(\sum_{i=1}^n v_i \phi_i(x)\right)}{1 + \exp(\sum_{i=1}^n v_i \phi_i(x))}$$

- Modelling the logit transforms instead of $p_{x,t}$ directly guarantees that $p_{x,t} \in (0,1)$.

# Mortality model

- Certain desired properties of $p_{x,t}$, e.g. continuity with respect to $x$, are achieved by corresponding choices of $\phi_i(x)$
  - Incorporation of user preferences and/or population-specific characteristics
- Choice of basis functions assigns interpretations to risk factors
- Concrete interpretations facilitate the modelling of risk factors, which is advantageous the risk management of mortality-linked instruments
- Vector $v_t$ of risk factors is modelled as a stochastic process, based on historical values, expert opinions, or both
- Historical values of $v_t$ are constructed by maximum likelihood estimation, log-likelihood function is strictly concave

# Mortality model

- We consider female mortality dynamics of six large OECD countries: Australia, Canada, France, Japan, UK, US
- Data consists of annual values of cohort sizes $E_{x,t}$ and numbers of deaths $D_{x,t}$ for each country, covering years 1950–2006 (Source: Human mortality database)
- A model with three basis functions and three risk factors is employed

# Mortality model

- The model:

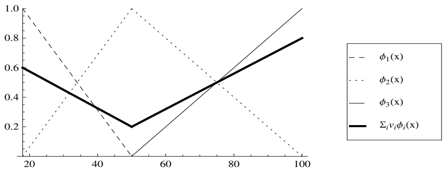
$$\text{logit } p_{x,t} = v_t^1 \phi_1(x) + v_t^2 \phi_2(x) + v_t^3 \phi_3(x),$$

where basis functions are piecewise linear:

$$\phi_1(x) = \begin{cases} 1 - \frac{x-18}{32} & \text{for } x \leq 50 \\ 0 & \text{for } x \geq 50, \end{cases} \quad \phi_2(x) = \begin{cases} \frac{1}{32}(x-18) & \text{for } x \leq 50 \\ 2 - \frac{x}{50} & \text{for } x \geq 50, \end{cases}$$

$$\phi_3(x) = \begin{cases} 0 & \text{for } x \leq 50 \\ \frac{x}{50} - 1 & \text{for } x \geq 50. \end{cases}$$

- The linear combination now also piecewise linear:




- Interpretation: values of $v_t^i$ points on the logit $p_{x,t}$ curve

# Statistical analysis of risk factors: $v_1$

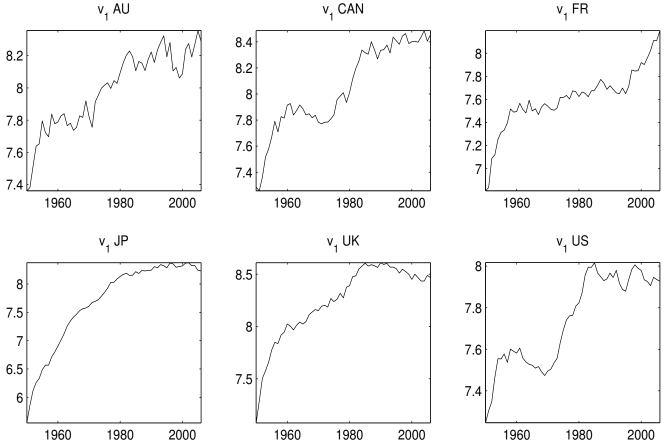



Figure: Historical values for risk factor $v_1$, females. Note the different scales.

# Statistical analysis of risk factors: $v_1$

- Fundamental question: how long can mortality keep improving?
- Mortality risk factors traditionally modelled as random walk with a negative drift, i.e. mortality will decline indefinitely
- Some experts predict that the steady decline in overall mortality during past decades will continue for the foreseeable future, while others suggest that human life expectancy might even decrease.
- In several sample countries, the historical values of $v_t^1$ display a stabilizing tendency
- In order to analyse this phenomenon, we fit the following regression into historical data:

$$\Delta v_t^1 = b + a v_{t-1}^1 + \varepsilon_t = -a\left(-\frac{b}{a} - v_{t-1}^1\right) + \varepsilon_t$$

# Statistical analysis of risk factors: $v_1$

Table: Parameter estimates, t-values and summary statistics for $\Delta v_t^1 = b + av_{t-1}^1$.

| | AU | CAN | F | JP | UK | US |
|---|---|---|---|---|---|---|
| $b$ | 0.778 | 0.446 | 0.569 | 0.614 | 0.902 | 0.437 |
| p-value(t-stat.) | 0.017 | 0.028 | 0.020 | 0.000 | 0.000 | 0.015 |
| $a$ | -0.095 | -0.053 | -0.072 | -0.074 | -0.107 | -0.055 |
| p-value(t-stat.) | 0.020 | 0.036 | 0.026 | 0.000 | 0.000 | 0.018 |
| $R^2$ | 0.097 | 0.079 | 0.089 | 0.627 | 0.461 | 0.100 |
| Adj. $R^2$ | 0.080 | 0.062 | 0.072 | 0.620 | 0.451 | 0.083 |
| p-value(F-Statistic) | 0.020 | 0.036 | 0.026 | 0.000 | 0.000 | 0.018 |

# Statistical analysis of risk factors: $v_1$

Table: Residual test statistics for the regression $\Delta v_t^1 = b + a v_{t-1}^1$.

| | AU | CAN | F | JP | UK | US |
|---|---|---|---|---|---|---|
| Serial correlation (BG) | 0.329 | 0.533 | 0.123 | 0.731 | 0.084 | 0.044 |
| Normality (JB) | 0.450 | 0.364 | 0.170 | 0.484 | 0.692 | 0.481 |
| Heteroskedasticity (BP) | 0.546 | 0.307 | 0.074 | 0.013 | 0.029 | 0.263 |
| ADF t-statistic | -9.191 | -8.374 | -8.105 | -7.11 | -3.722 | -4.474 |
| p-value | $< 0.01$ | $< 0.01$ | $< 0.01$ | $< 0.01$ | $< 0.01$ | $<0.01$ |
| KPSS level | 0.219 | 0.164 | 0.394 | 0.086 | 0.255 | 0.139 |
| p-value | $> 0.1$ | $> 0.1$ | 0.0796 | $> 0.1$ | $> 0.1$ | $>0.1$ |

*Notes.* Serial correlation is tested with the Breusch–Godfrey tests for serial correlation. A small p-value suggests rejecting the zero hypothesis of serial correlation. Normality is tested with the Jarque–Bera test for normality. The Breusch-Pagan tests against heteroskedasticity, with the null hypothesis of homoskedasticity. Stationarity of the residuals is tested with the augmented Dickey-Fuller unit root test (ADF) and Kwiatkowski-Phillips-Schmidt-Shin test (KPSS).

# Statistical analysis of risk factors: $v_2$

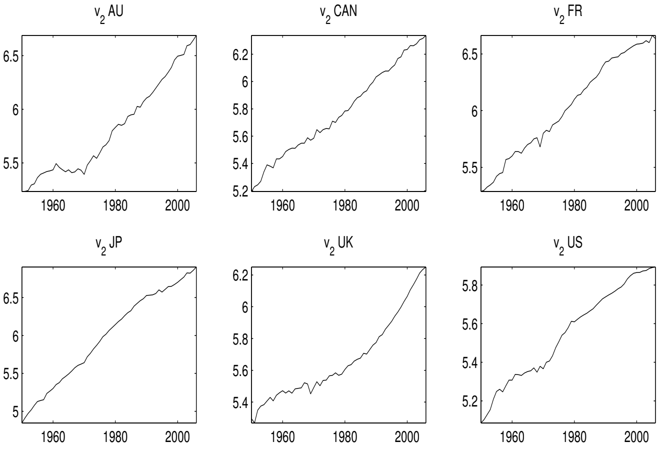



Figure: Historical values for risk factor $v_2$, females. Note the different scales.

# Statistical analysis of risk factors: $v_2$

- Dramatic reduction in coronary disease amongst the middle-aged during the past three decades shows in the rapid growth of risk factor $v_2$ reflecting the survival probability of 50-year-olds
- To which extent do improvements in treatment and possible further reductions in smoking can outweigh the detrimental effects of obesity and other lifestyle-related factors?
- Even thought the historical values (apart from Japan) do not yet show signs of levelling out, eventual stabilizing behaviour similar to that of $v_1$ may also be a future possibility for $v_2$
- To quantify the rate of improvement for $v_2$, we fit the regression

$$\Delta v_t^2 = b + \varepsilon_t,$$

# Statistical analysis of risk factors: $v_3$ and GDP

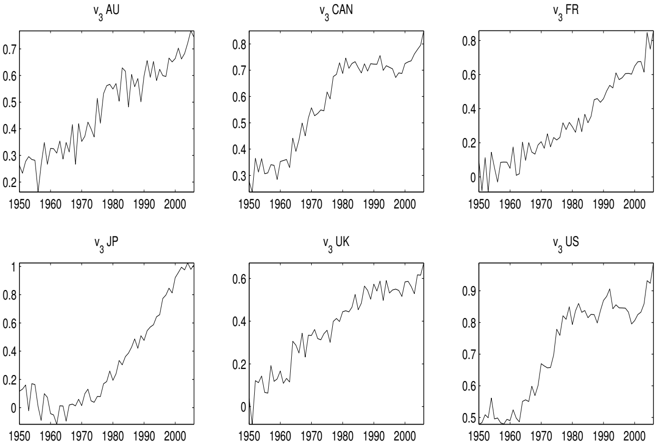



Figure: Historical values for risk factor $v_3$, females. Note the different scales.

# Statistical analysis of risk factors: $v_3$ and GDP

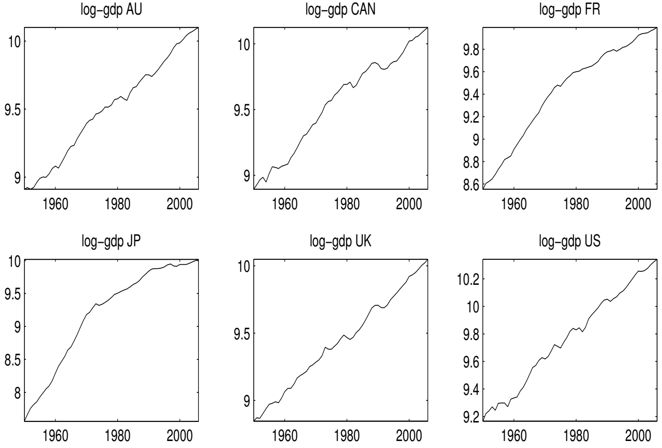



Figure: Historical values for risk factor logarithm of per capita GDP. Note the different scales.

## Statistical analysis of risk factors: $v_3$ and GDP

- Risk factor $v_3$ describes old-age mortality, with which the cash flows of mortality linked instruments are connected
- Long-term dependence between GDP and mortality has been observed (e.g. Preston 1975, Preston 2007).
- Similarities in the general shape of their plots support the observation that $v_3$ and log-per capita GDP may move together in the long run
- We analyse the dependence of $v_3$ on GDP with the regression

$$\Delta v_t^3 = b + a_1 v_{t-1}^3 + a_2 g_{t-1} + \varepsilon_t,$$

where $g_t$ is the log-per capita GDP

# Statistical analysis of risk factors: $v_3$ and GDP

Table: Parameter estimates, t-values and summary statistics for the regression $\Delta v_t^3 = b + a_1 v_{t-1}^3 + a_2 g_{t-1}$

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| | AU | CAN | F | JAP | UK | US |
|---|---|---|---|---|---|---|
| $b$ | -3.062 | -1.686 | -1.826 | -0.332 | -2.894 | -0.664 |
| p-value (t-stat.) | 0.000 | 0.000 | 0.001 | 0.053 | 0.000 | 0.020 |
| $a_1$ | -0.877 | -0.455 | -0.388 | -0.057 | -0.656 | -0.189 |
| p-value (t-stat.) | 0.000 | 0.000 | 0.001 | 0.1941 | 0.000 | 0.015 |
| $a_2$ | 0.367 | 0.205 | 0.208 | 0.040 | 0.334 | 0.083 |
| p-value (t-stat.) | 0.000 | 0.000 | 0.001 | 0.044 | 0.000 | 0.016 |
| | | | | | | |
| $R^2$ | 0.454 | 0.255 | 0.195 | 0.076 | 0.335 | 0.109 |
| Adj. $R^2$ | 0.433 | 0.227 | 0.164 | 0.041 | 0.310 | 0.076 |
| p-value (F-Statistic) | 0.000 | 0.0004 | 0.003 | 0.124 | 0.000 | 0.046 |

# Statistical analysis of risk factors: $v_3$ and GDP

Table: Residual test statistics for $\Delta v_t^3 = b + a_1 v_{t-1}^3 + a_2 g_{t-1}$.

| | 1950–2006 | | | | | |
|---|---|---|---|---|---|---|
| | AU | CAN | F | JAP | UK | US |
| Serial correlation (BG) | 0.000 | 0.002 | 0.000 | 0.000 | 0.090 | 0.26 |
| Normality (JB) | 0.536 | 0.279 | 0.009 | 0.203 | 0.880 | 0.533 |
| Heteroskedasticity (BP) | 0.330 | 0.038 | 0.101 | 0.001 | 0.012 | 0.734 |
| ADF t-statistic | -7.356 | -3.921 | -1.172 | -10.599 | -6.843 | -8.499 |
| p-value | $< 0.01$ | $< 0.01$ | $> 0.1$ | $< 0.01$ | $< 0.01$ | $< 0.01$ |
| KPSS level | 0.120 | 0.311 | 0.463 | 0.084 | 0.263 | 0.150 |
| p-value | $> 0.1$ | $> 0.1$ | 0.050 | $> 0.1$ | $> 0.1$ | $> 0.1$ |

*Notes.* Serial correlation is tested with the Breusch–Godfrey tests for serial correlation. A small p-value suggests rejecting the zero hypothesis of serial correlation. Normality is tested with the Jarque–Bera test for normality. The Breusch-Pagan tests against heteroskedasticity, with the null hypothesis of homoskedasticity. Stationarity of the residuals is tested with the augmented Dickey-Fuller unit root test (ADF) and Kwiatkowski-Phillips-Schmidt-Shin test (KPSS).

# Statistical analysis of risk factors: GDP and financial markets

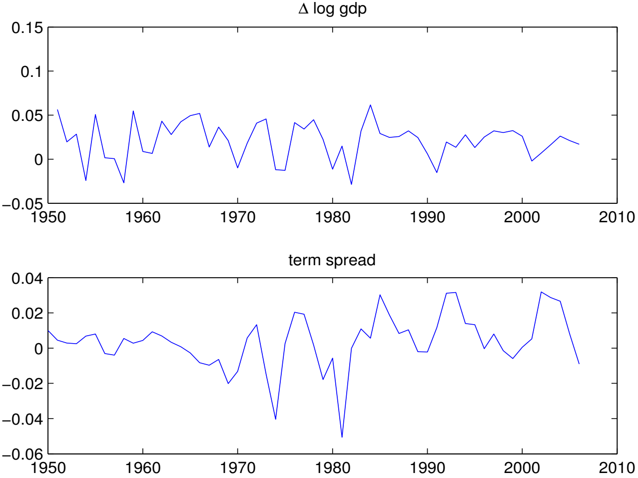


Figure: Differences of US log per capita GDP and term spread

# Statistical analysis of risk factors: GDP and financial markets

- GDP is linked with financial markets (extensive review article: Stock&Watson 2003)
  - Yield curve/term spread
  - Credit spread
  - Stock prices
- As in e.g. Wheelock&Wohar(2009), we analyse the connection between $g_t$ and term spread $s_t$ with the regression

$$\Delta g_t = b + as_{t-1} + \varepsilon_t,$$

where $s_t$ is the difference between the 10-year Treasury rate and the 6-month Certificate of Deposit rate

# Statistical analysis of risk factors: GDP and financial markets

Table: Parameter estimates, t-values and summary statistics for the regression $\Delta g_t = b + as_{t-1}$.

| | | | |
|---|---|---|---|
| $b$ | 0.0185 | Serial Correlation (BG) | 0.4498 |
| p-value(t-statistic) | 0.0000 | Normality (JB) | 0.7417 |
| $a$ | 0.6460 | Heteroskedasticity (BP) | 0.9257 |
| p-value(t-statistic) | 0.0004 | | |
| | | ADF t-statistic | -6.85 |
| $R^2$ | 0.2117 | p-value | $< 0.01$ |
| Adj. $R^2$ | 0.1971 | KPSS level | 0.3374 |
| p-value(F-Statistic) | 0.0004 | p-value | $> 0.1$ |

# Statistical analysis of risk factors: GDP and financial markets

- We apply the established drifting Brownian motion approach, and fit the process

$$\Delta p_t = b + \varepsilon_t$$

into the logarithm of S&P Composite stock price index data $p_t$ for the same observation period

- Residuals of the above equation and of

$$\Delta g_t = b + a s_{t-1} + \varepsilon_t$$

have a correlation coefficient of 0.4843 (significance below 0.001)

# Modelling the risk factors

- Based on our observations, we propose the linear stochastic difference equation model for US:

$$\Delta x = Ax + b + \varepsilon_t$$

for $x = [v_t^1, v_t^2, v_t^3, g_t, s_t, p_t]$ and multivariate Gaussian $\varepsilon_t$,

- The equations are

$$\begin{aligned}
\Delta v_t^1 &= a_{11} v_t^1 + b_1 + \varepsilon_t^1 \\
\Delta v_t^2 &= b_2 + \varepsilon_t^2 \\
\Delta v_t^3 &= a_{33} v_t^3 + a_{34} g_t + b_3 + \varepsilon_t^3 \\
\Delta g_t &= a_{45} s_t + b_4 + \varepsilon_t^4 \\
\Delta s_t &= a_{55} s_t + b_5 + \varepsilon_t^5 \\
\Delta p_t &= b_6 + \varepsilon_t^6
\end{aligned}$$

where $a_{i,j}$ and $b_j$ are components of matrix $A$ and vector $b$

# Simulations

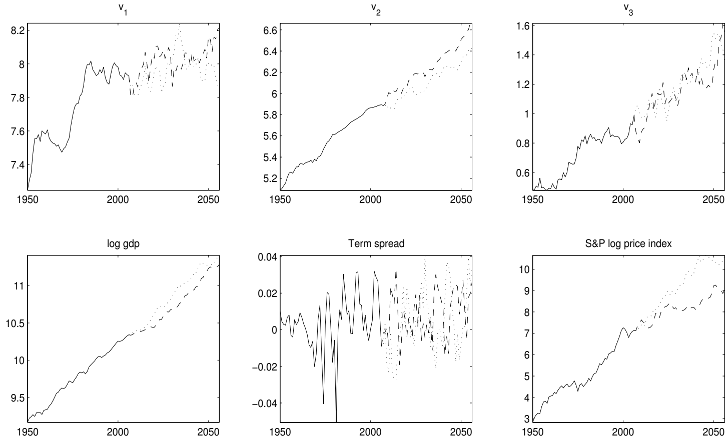


Figure: Simulation sample paths, US females.

# Simulations

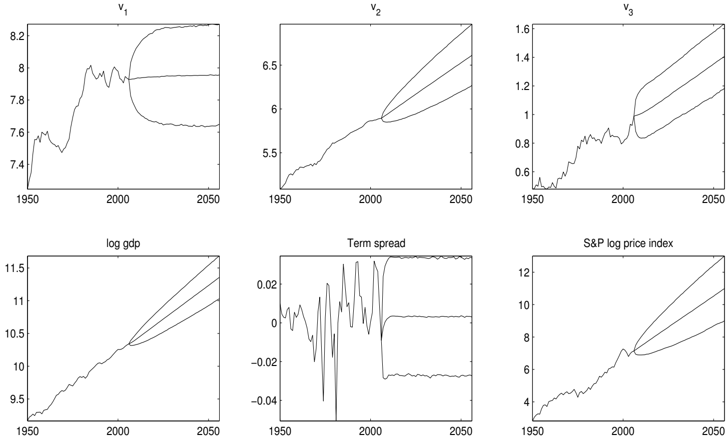


Figure: Medians and 95% confidence intervals for MC simulations (N=10000), US females.

# Conclusions

- Recent development of markets for mortality-linked securities has created a need for quantitative methods for their risk management
- We present a relatively simple model connecting mortality and economic conditions, which provides a solid starting point for solving pricing and hedging problems of mortality-linked cash flows
- Next step: how to choose the investment strategy, given the model and its set of assets?