Numerical methods for BSDEs and nonlinear PDEs

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FBSDE

Let W_t be a Wiener process, defined on a probability space $(\Omega, \mathcal{F}, P)$, $\mathcal{F}_t$ be its natural filtration. Consider a FBSDE

$$\begin{cases} X_t = x + \int\limits_0^t b(s, X_s, Y_s)ds + \int\limits_0^t \sigma(s, X_s, Y_s)dW_s, \\ Y_t = g(X_T) + \int\limits_t^T f(s, X_s, Y_s, Z_s)ds - \int\limits_t^T Z_s dW_s. \end{cases} \tag{1}$$

$$Y_T = g(X_T) \in \mathbf{L}^2(\mathcal{F}_t).$$

If b, σ, f, g are deterministic and Lipschitz continuous functions, there exist $\mathcal{F}_t$–adapted processes (X_t, Y_t, Z_t) satisfying (1), such that

$$E[\sup_{0 \leqslant t \leqslant T} |\, Y_t|^2 + \int\limits_0^T |\, Z_s|^2 \, ds] < \infty.$$

Nonlinear PDE

If $u(t,x)$ is a classical solution to

$$\begin{cases} u_t + \frac{1}{2}\text{trace}(\sigma\sigma^*(t,x,u)u_{xx}) + \\ \qquad + u_x b(t,x,u) + f(t,x,u,u_x\sigma(t,x,u)) = 0, \\ u(T,x) = g(x). \end{cases} \tag{2}$$

then $Y_t = u(t,X_t), \quad Z_t = v(t,X_t) \triangleq u_x(t,X_t)\sigma(t,X_t,u(t,X_t))$, solve (1). At the other hand one can use (1) to construct less regular solution to (2).

Discretization

A natural time discretization of equation (1) is

$$\begin{cases} X_{i+1}^n \triangleq X_i^n + b(t_i, X_i^n, Y_i^n)h + \sigma(t_i, X_i^n, Y_i^n)\Delta W_{i+1}, \\ Z_i^n \triangleq \dfrac{1}{h} E_{t_i}\{Y_{i+1}^n \Delta W_{i+1}\}, \\ Y_i^n \triangleq E_{t_i}\{Y_{i+1}^n + f(t_i, X_i^n, Y_{i+1}^n, \hat{Z}_i^n)h\}. \end{cases} \quad (3)$$

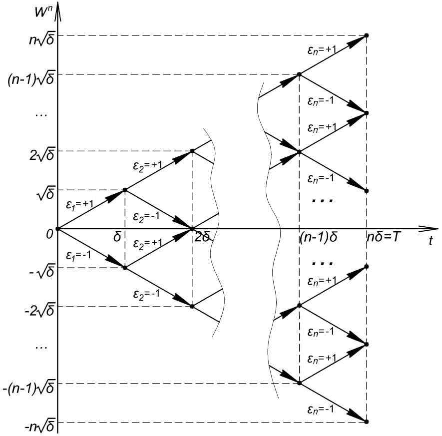
Here $X_0^n \triangleq x$, $Y_n^n \triangleq g(X_n^n)$, $h \triangleq \frac{T}{n}$ and $t_i \triangleq ih, i = 0, 1, ..., n$, and $\Delta W_{i+1} \triangleq W_{t_{i+1}} - W_{t_i}$. E_t denotes the conditional expectation $E\{\cdot|\mathcal{F}_t\}$.

W_t simulation

Set

$$W^n_{t_i} := \sqrt{h}\sum_{j=1}^{i}\varepsilon^n_j,$$

where $\{\varepsilon^n_j\}^n_{j=1}$ are $\{1,-1\}$–valued i.i.d. with $P\{\varepsilon^n_j = 1\} = P\{\varepsilon^n_j = -1\} = 0.5.$



Numerical examples

Consider a simple case

$$dX_t = dW_t,$$

$$dY_t = (Y_t \cdot Z_t)dt + Z_t dW_t, \quad Y_T = \cos(W_T),$$

$y(t) = U(t, x)$ solves the Cauchy problem for the Burgers equation

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = \frac{1}{2}\frac{\partial^2 U}{\partial x^2}, \quad U(T, x) = \cos(x).$$

Explicit solution of Burgers equation

$$U(t,x) = \frac{\int\limits_{-\infty}^{\infty} \frac{(x-\theta)}{\tau} exp(-G(\tau,x,\theta))d\theta}{\int\limits_{-\infty}^{\infty} exp(-G(\tau,x,\theta))d\theta},$$

where

$$G(\tau,x,\theta) = \frac{(x-\theta)^2}{2\tau} + \int\limits_0^\theta \varphi(y)dy, \quad \varphi(y) = U(T,y), \quad \tau = T - t.$$

Explicit solution give us $Y_0 = U(0,0) = 0.5564803023$, and (3) with different value n give

n	100	500	2000	4000
Y_0^n	0.55588	0.55636	0.55645	0.55647

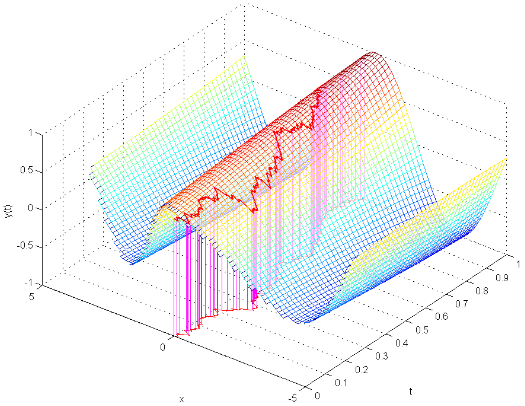


Figure: *The solution surface with one trajectory,* $n = 500$

Reference

Peng, S. and Xu, M. Numerical algorithms for 1-d backward stochastic differential equations: Convergence and simulations. Date posted: 28 Nov 2006; Last revised: 23 Sep 2009. URL: http://arxiv.org/abs/math/0611864

Bender, C. and Zhang, J. (2008). Time discretization and markovian iteration for coupled FBSDEs. *Ann. Appl. Probab.* **18** 143 – 144.