Optimal Double-Stopping Problem on Trajectories

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• There are $m_0$ minus balls and $p_0$ plus balls in a urn.

• $Z_0 = 0$, $Z_n = \sum_{k=1}^{n} X_k$, $1 \leq n \leq m_0 + p_0$, where $X_k$ is the value of the ball chosen at the $k$-th draw, the value $-1$ is attached to minus ball and value $+1$ to plus ball.

• The objective is to stop with maximum probability at the beginning on the minimum of the trajectory formed by $\{Z_n\}_{n=0}^{m_0+p_0}$ and then on the maximum.

![Diagram of $Z_n$ trajectory]

Example of $Z_n$ for $p_0 = 8$, $m_0 = 7$.

The optimal stopping times $(\sigma^*, \tau^*)$

$$P\{Z_{\sigma^*} = \min_{0 \leq n \leq m_0+p_0} Z_n, Z_{\tau^*} = \max_{0 \leq l \leq m_0+p_0} Z_l\}$$

$$= \sup_{(\sigma, \tau) \in C, \sigma < \tau} P\{Z_{\sigma} = \min_{0 \leq l \leq m_0+p_0} Z_l, Z_{\tau} = \max_{0 \leq n \leq m_0+p_0} Z_n\},$$

where $C$ is the class of all double-stopping times.
Optimal decision rule for the maximum

Let $p_0 > m_0$. Suppose that we have drawn $k$ balls and observed the values of $\{Z_n\}_{n=1}^k$. We know that there are still remain $m$ minus balls and $p$ plus balls (the urn initially contains $k + m + p = p_0 + m_0$ balls). Let this state be described as $(m, p)$.

Let we stopped on the minimum $-x$ in state $(m_1, p_1)$ on condition that the local maximum $y' = p_0 - m_0 + \max\{m' - p', 0\}$ have been reached in state $(m', p')$.

$Z_n$ for $p_0 = 8, m_0 = 7$. 
$v(m, p, m_1, p_1)$ is the probability of success in state $(m, p)$

$s(m, p, m_1, p_1)$ is the probability of success if we stop drawing

$c(m, p, m_1, p_1)$ is the probability of success if we continue drawing in an optimal way in state $(m, p)$

\[
v(m, p, m_1, p_1) = \max\{s(m, p, m_1, p_1), c(m, p, m_1, p_1)\},
\]

where $m - p \geq \max\{m' - p', 0\}$. 
Lemma [Feller, 1984] Let $N_{n,x}$ is the number of trajectories from the origin to the point $(n, x)$. Let $a > 0, b > 0$ and $0 < c < a$. Then the number of trajectories to the point $(n, c)$ which meet neither lines $x = -b$ nor $x = a$ is equal to

$$
\sum_{k \in \mathbb{Z}} N_{n, 2k(a+b)+c} - N_{n, 2k(a+b)+2a-c}.
$$

Notice, if $n = p + m, x = p - m$ then

$$
N_{n,x} = \binom{p+m}{p} = \binom{n}{(n+x)/2}.
$$
The probability of success if we stop drawing is equal to

\[ s(m, p, m_1, p_1) = P\left(\{-x - 1 < Z_i < y + 1\}_{k+1}^{p_0+m_0}\right), \quad m - p \geq \max\{m' - p', 0\}, \]

\[ a = 1, \quad b = y + x + 1 = m - p + p_1 - m_1 + 1, \quad c < 0. \]

\[ s(m, p, m_1, p_1) = \sum_{k \in \mathbb{Z}} \frac{m+p}{(p+k(y+x+2))} - \frac{m+p}{(m+k(y+x+2)+1)}. \]
The probability of success if we continue drawing in an optimal way in state \((m, p)\) is equal to
\[
c(m, p, m_1, p_1) = q_1(m, p) s(m, p - 1, m_1, p_1) \\
+ \sum_{i=1}^{p} q_2 i(m, p, m_1, p_1) s(m - i, p - i, m_1, p_1),
\]
where \(q_1(m, p) = \frac{p}{m + p},\)

\[
b = y + x, \ a = 1, \ c = 0 \]
\[
q_{2i}(m, p, m_1, p_1) = P\{\min\{n : Z_n = 0, 0 \leq n \leq m + p\} = 2i| i \text{ minus balls and } i \text{ plus balls during the first } 2i \text{ drawings}\} \cdot \frac{\binom{m}{i} \cdot \binom{p}{i}}{\binom{m+p}{2i}} \\
= \sum_{k \in \mathbb{Z}} \frac{\binom{2i - 2}{i - 1 + k(y + x + 1)} - \binom{2i - 2}{i - 1 + k(y + x + 1) + 1}}{\binom{2i}{i}} \cdot \frac{\binom{m}{i} \cdot \binom{p}{i}}{\binom{m+p}{2i}}.
\]

We stop if \(s(m, p, m_1, p_1) \geq c(m, p, m_1, p_1), \ m - p \geq \max\{m' - p', 0\}.\)
The local maximum value is reached in state \((m', p')\),
\[ y' = p_0 - m_0 + \max\{m' - p', 0\}. \]
Denote \(l = x + y' = m - p + p' - m'\).

We obtain that
\[ u(m, p, m', p') = \max\{s_1(m, p, m', p'), c_1(m, p, m', p')\}, \quad p \geq m. \]
The probability of success if we stop is

\[ s_1(m, p, m', p') = \min\{m, p-l\} \sum_{i=0}^{m'} p_{2i+l}(m, p, m', p') v(m-i, p-(i+l), m, p), \]

\[ a = -1, \quad b = x + y', \quad c < 0 \]

\[ p_{2i+l}(m, p, m', p') = P\{\min\{n: Z_n = l, 0 \leq n \leq m + p\} = 2i + l | i \text{ minus balls and } i + l \text{ plus balls during the first } 2i + l \text{ drawings}\} \cdot \frac{\binom{m}{i} \cdot \binom{p}{i+l}}{\binom{m+p}{2i+l}} \]

\[ = \sum_{k \in Z} \binom{2i+l-1}{i+k(y'+x+1)} - \binom{2i+l-1}{i+l-1+k(y'+x+1)+1} \cdot \binom{m}{i} \cdot \binom{p}{i+l} \cdot \frac{\binom{m+p}{2i+l}}{\binom{m+p+l}{2i+l}} \]
The probability of success if we continue is
\[ c_1(m, p, m', p') = p_1(m, p)s_1(m-1, p, m', p') + \sum_{i=1}^{m} p_{2i}(m, p)s_1(m - i, p - i, m', p'), \]
\[ p_1(m, p) = \frac{m}{m+p}, \]
\[ p_{2i}(m, p) = \frac{1}{2(2i-1)} \cdot \frac{(m)_i \cdot (p)_i}{(m+p)_{2i}}. \]

We stop the process if \( s_1(m, p, m', p') \geq c_1(m, p, m', p'), \ p \geq m. \)
Example. $m_0 = 4$, $p_0 = 5$, $y' = p_0 - m_0 = 1$

<table>
<thead>
<tr>
<th>Stage</th>
<th>Description</th>
<th>$s_1$</th>
<th>$c_1$</th>
<th>Decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-th stage (4,5)</td>
<td>(3,5) minus ball</td>
<td>$s_1(3,5) = 0.286$, $c_1(3,5) = 0.429$</td>
<td>continue</td>
<td></td>
</tr>
<tr>
<td>2-th stage (3,5)</td>
<td>(2,5) minus ball</td>
<td>$s_1(2,5) = 0.571$, $c_1(1,5) = 0.476$</td>
<td>stop</td>
<td></td>
</tr>
<tr>
<td>minimum</td>
<td>$m_{min} = 2$, $p_{min} = 5$, $y' = p_0 - m_0 = 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3-th stage (2,5)</td>
<td>(2,4) plus ball</td>
<td></td>
<td>continue</td>
<td></td>
</tr>
<tr>
<td>4-th stage (2,4)</td>
<td>(2,3) plus ball, $m &lt; p$</td>
<td></td>
<td>continue</td>
<td></td>
</tr>
<tr>
<td>5-th stage (2,3)</td>
<td>(2,2) minus ball</td>
<td>$s(2,2) = 0.333$, $c(2,2) = 0.667$</td>
<td>continue</td>
<td></td>
</tr>
<tr>
<td>6-th stage (2,2)</td>
<td>(2,1) plus ball</td>
<td>$s(2,1) = 0.667$, $c(2,1) = 0.667$</td>
<td>stop</td>
<td></td>
</tr>
<tr>
<td>maximum</td>
<td>$m_{max} = 2$, $p_{max} = 1$</td>
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<td></td>
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</tr>
</tbody>
</table>
REFERENCES


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