Optimal Double-Stopping Problem on Trajectories

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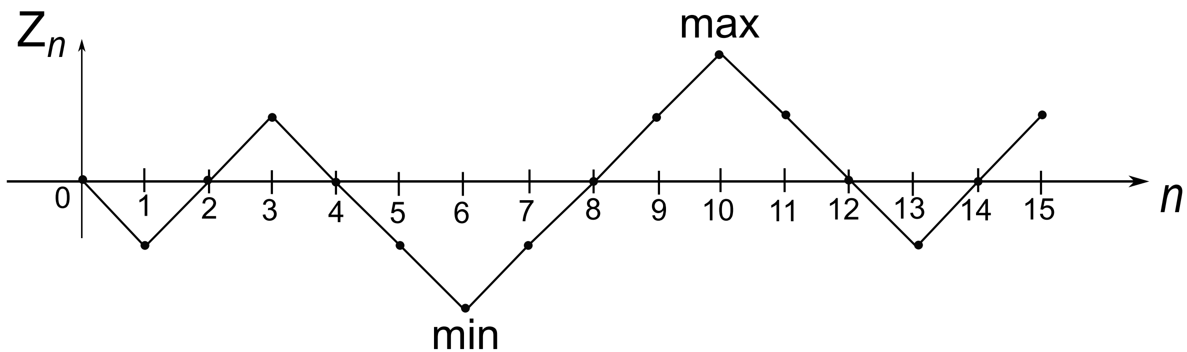
- There are m_0 minus balls and p_0 plus balls in a urn.

- $Z_0 = 0$, $Z_n = \sum_{k=1}^{n} X_k$, $1 \le n \le m_0 + p_0$, where X_k is the value of the ball chosen at the k-th draw, the value -1 is attached to minus ball and value $+1$ to plus ball.

- The objective is to stop with maximum probability at the beginning on the minimum of the trajectory formed by $\{Z_n\}_{n=0}^{m_0+p_0}$ and then on the maximum.



Example of Z_n for $p_0 = 8$, $m_0 = 7$.

The optimal stopping times $(\sigma*, \tau*)$

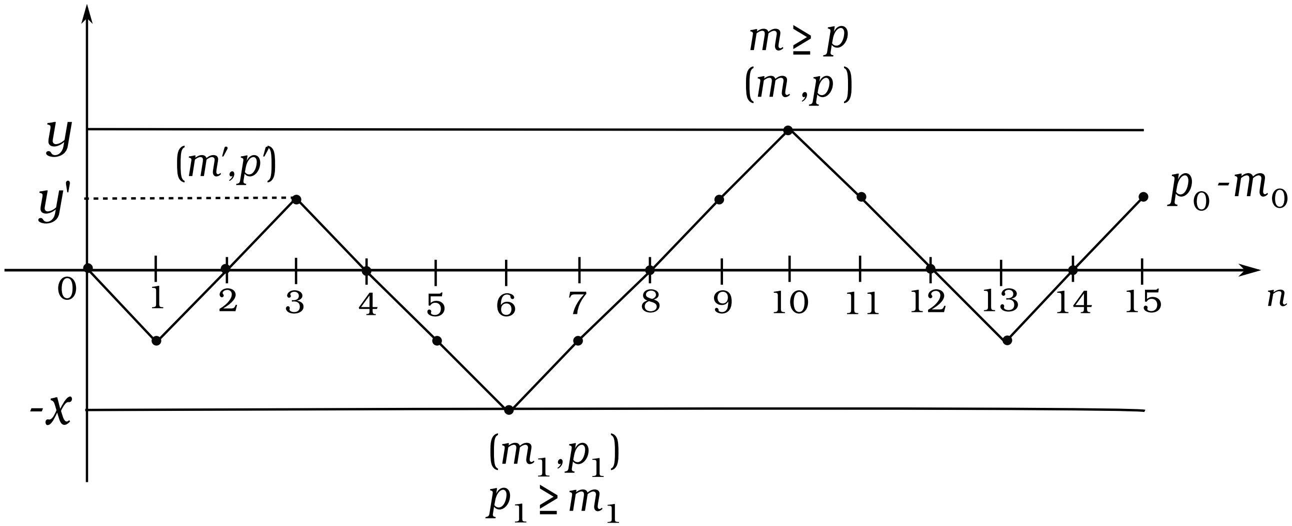
$$P\{Z_{\sigma*} = \min_{0\le n\le m_0+p_0} Z_n,\, Z_{\tau*} = \max_{0\le l\le m_0+p_0} Z_l\}$$
$$= \sup_{(\sigma,\tau)\in C, \sigma<\tau} P\{Z_\sigma = \min_{0\le l\le m_0+p_0} Z_l,\, Z_\tau = \max_{0\le n\le m_0+p_0} Z_n\},$$

where C is the class of all double-stopping times.

Optimal decision rule for the maximum

Let $p_0 > m_0$. Suppose that we have drawn k balls and observed the values of $\{Zn\}_{n=1}^{k}$. We know that there are still remain m minus balls and p plus balls (the urn initially contains $k + m + p = p_0 + m_0$ balls). Let this state be described as (m, p).

Let we stopped on the minimum $-x$ in state (m_1, p_1) on condition that the local maximum $y' = p_0 - m_0 + \max\{m' - p', 0\}$ have been reached in state (m', p').



Z_n for $p_0 = 8$, $m_0 = 7$.

$v(m, p, m_1, p_1)$ is the probability of success in state (m, p)

$s(m, p, m_1, p_1)$ is the probability of success if we stop drawing

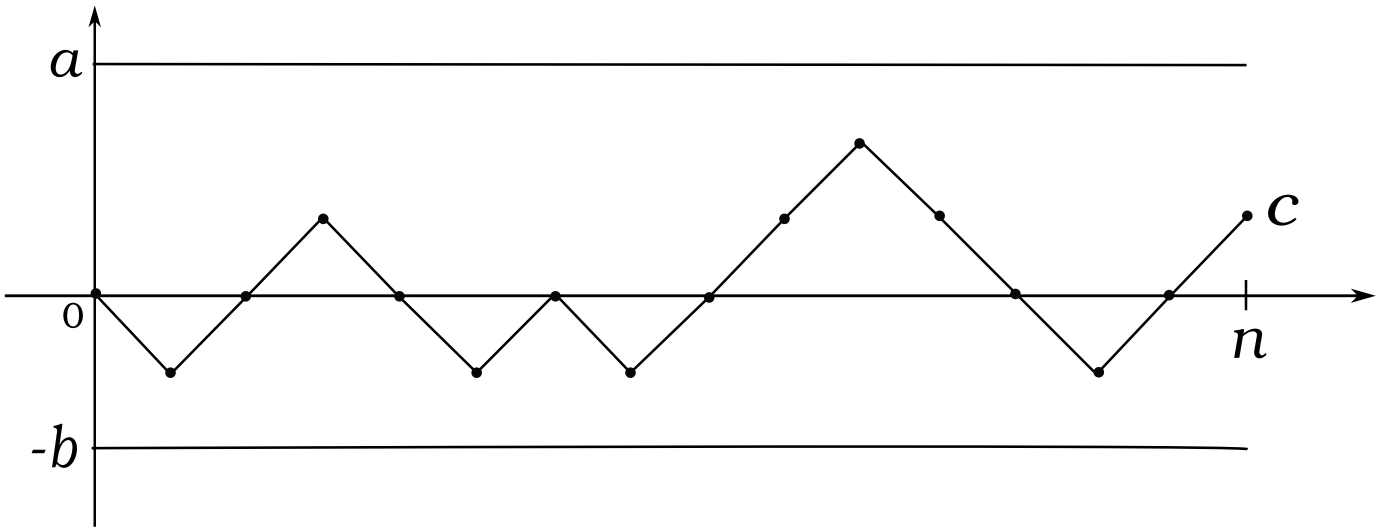
$c(m, p, m_1, p_1)$ is the probability of success if we continue drawing in an optimal way in state (m, p)

$$v(m, p, m_1, p_1) = \max\{s(m, p, m_1, p_1), c(m, p, m_1, p_1)\},$$

where $m - p \geq \max\{m' - p', 0\}$.

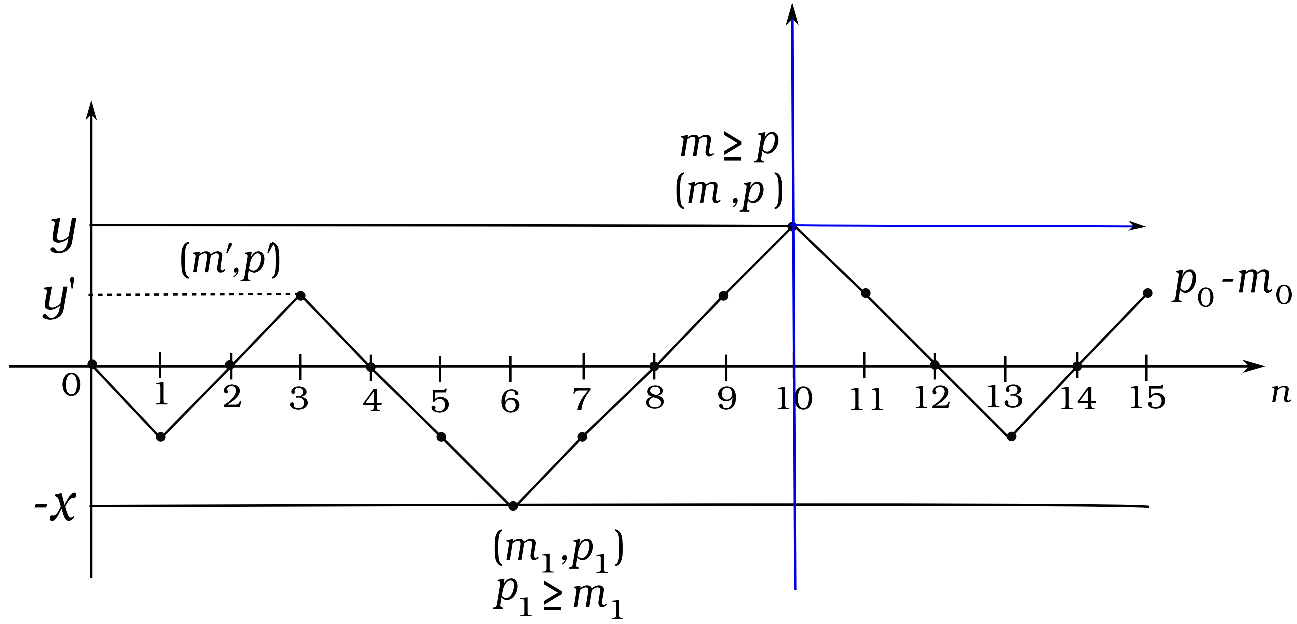
Lemma [Feller, 1984] *Let* $N_{n,x}$ *is the number of trajectories from the origin to the point* (n, x)*. Let* $a > 0, b > 0$ *and* $0 < c < a$*. Then the number of trajectories to the point* (n, c) *which meet neither lines* $x = -b$ *nor* $x = a$ *is equal to*

$$\sum_{k \in Z} N_{n,2k(a+b)+c} - N_{n,2k(a+b)+2a-c}.$$



Notice, if $n = p + m, x = p - m$ then

$$N_{n,x} = \binom{p+m}{p} = \binom{n}{(n+x)/2}.$$



The probability of success if we stop drawing is equal to

$$s(m,p,m_1,p_1) = P(\{-x-1 < Z_i < y+1\}_{k+1}^{p_0+m_0}),\ m-p \geq \max\{m'-p',0\},$$

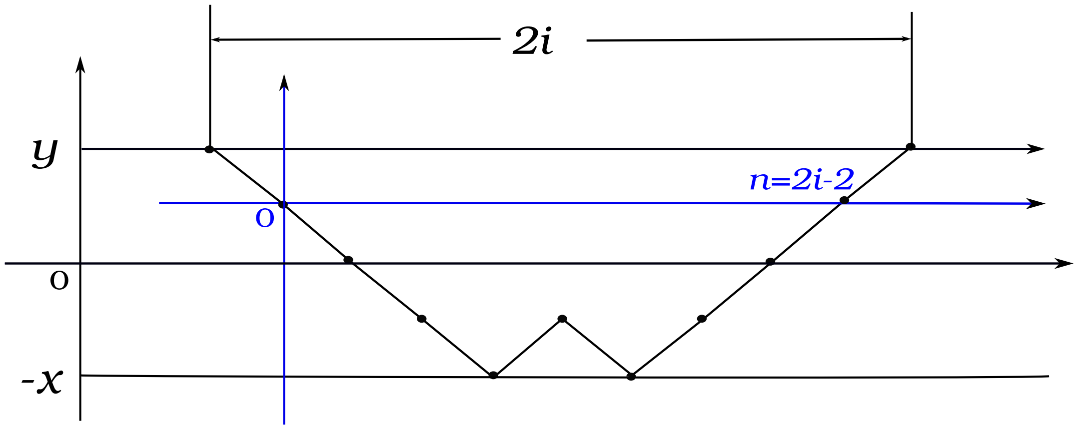
$a = 1,\ b = y+x+1 = m-p+p_1-m_1+1,\ c < 0.$

$$s(m,p,m_1,p_1) = \frac{\sum_{k\in Z}\binom{m+p}{p+k(y+x+2)} - \binom{m+p}{m+k(y+x+2)+1}}{\binom{m+p}{p}}.$$

The probability of success if we continue drawing in an optimal way in state (m,p) is equal to

$$\begin{aligned} c(m,p,m_1,p_1) &= q_1(m,p)s(m,p-1,m_1,p_1) \\ &+ \sum_{i=1}^{p} q_{2i}(m,p,m_1,p_1)s(m-i,p-i,m_1,p_1), \end{aligned}$$

where $q_1(m,p) = \frac{p}{m+p}$,



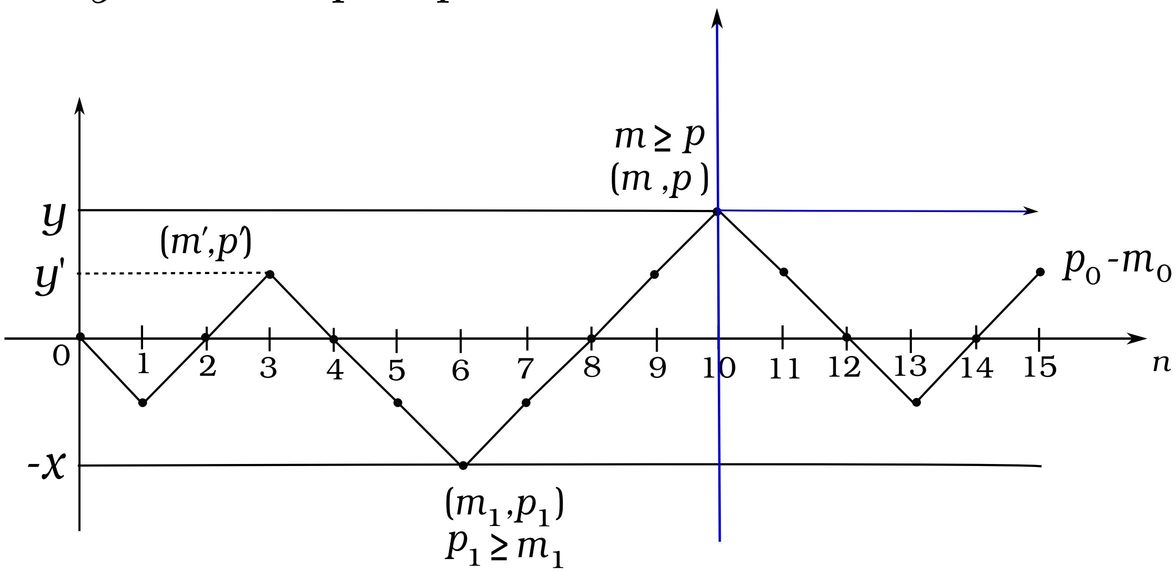
$b = y + x$, $a = 1$, $c = 0$

$$\begin{aligned} q_{2i}(m,p,m_1,p_1) &= P\{\min\{n : Z_n = 0, 0 \le n \le m+p\} = 2i \mid i \text{ minus balls and} \\ & i \text{ plus balls during the first } 2i \text{ drawings}\} \cdot \frac{\binom{m}{i} \cdot \binom{p}{i}}{\binom{m+p}{2i}} \\ &= \frac{\sum_{k \in Z} \binom{2i-2}{i-1+k(y+x+1)} - \binom{2i-2}{i-1+k(y+x+1)+1}}{\binom{2i}{i}} \cdot \frac{\binom{m}{i} \cdot \binom{p}{i}}{\binom{m+p}{2i}}. \end{aligned}$$

We stop if $s(m,p,m_1,p_1) \ge c(m,p,m_1,p_1)$, $m - p \ge \max\{m' - p', 0\}$.

Optimal decision rule for the minimum

The local maximum value is reached in state (m', p'),
$y' = p_0 - m_0 + \max\{m' - p', 0\}$.
Denote $l = x + y' = m - p + p' - m'$.



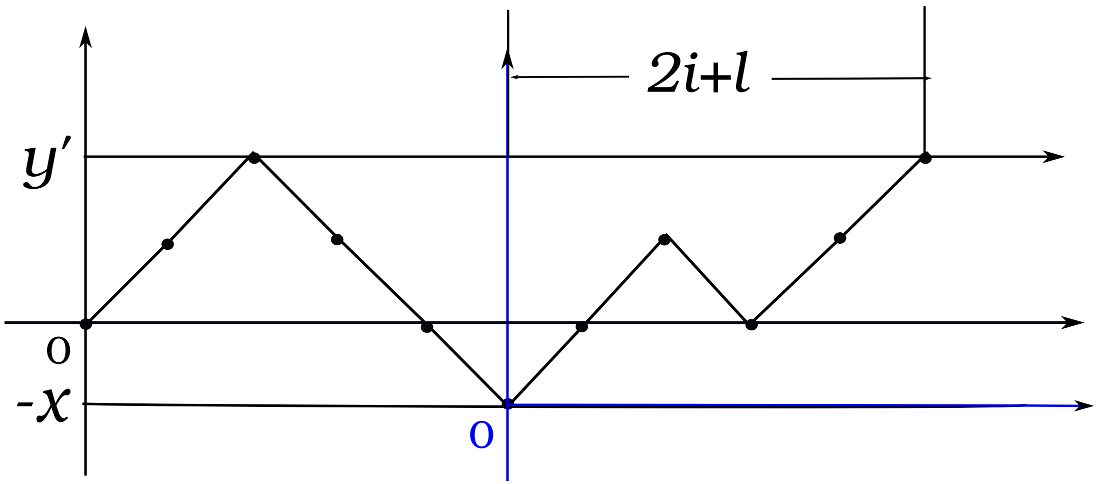
$u(m, p, m', p')$ is the probability of success in state (m, p).
$s_1(m, p, m', p')$ is the probability of success if we stop drawing.
$c_1(m, p, m', p')$ is the probability if we continue drawing in an optimal way.

We obtain that

$$u(m, p, m', p') = \max\{s_1(m, p, m', p'), c_1(m, p, m', p')\},\ p \geq m.$$

The probability of success if we stop is

$$s_1(m,p,m',p') = \sum_{i=0}^{\min\{m,p-l\}} p_{2i+l}(m,p,m',p')v(m-i,p-(i+l),m,p),$$



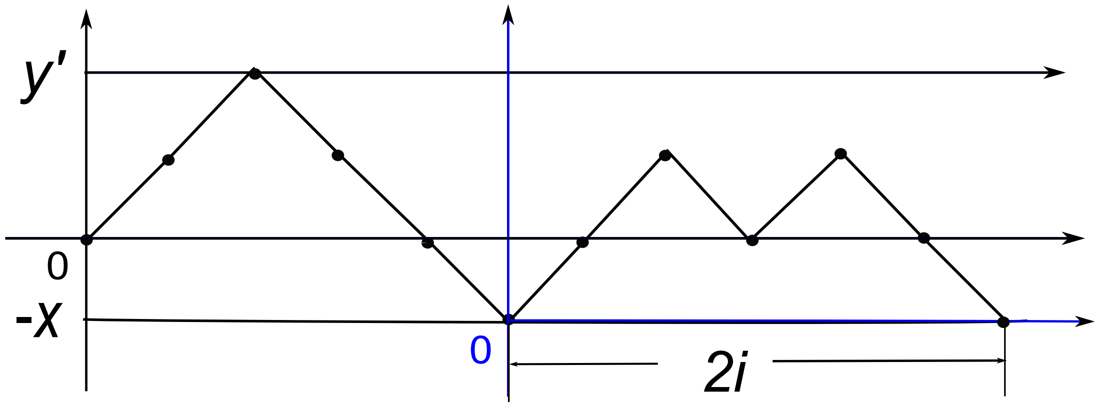
$a=-1$, $b=x+y'$, $c<0$

$$p_{2i+l}(m,p,m',p') = P\{\min\{n: Z_n = l, 0\le n\le m+p\} = 2i+l | i \text{ minus balls and } i+l \text{ plus balls during the first } 2i+l \text{ drawings}\} \cdot \frac{\binom{m}{i}\cdot\binom{p}{i+l}}{\binom{m+p}{2i+l}}$$

$$= \frac{\sum_{k\in Z}\binom{2i+l-1}{i+k(y'+x+1)} - \binom{2i+l-1}{i+l-1+k(y'+x+1)+1}}{\binom{2i+l}{i}} \cdot \frac{\binom{m}{i}\cdot\binom{p}{i+l}}{\binom{m+p}{2i+l}}$$

The probability of success if we continue is

$$c_1(m,p,m',p') = p_1(m,p)s_1(m-1,p,m',p') + \sum_{i=1}^{m} p_{2i}(m,p)s_1(m-i,p-i,m',p'),$$

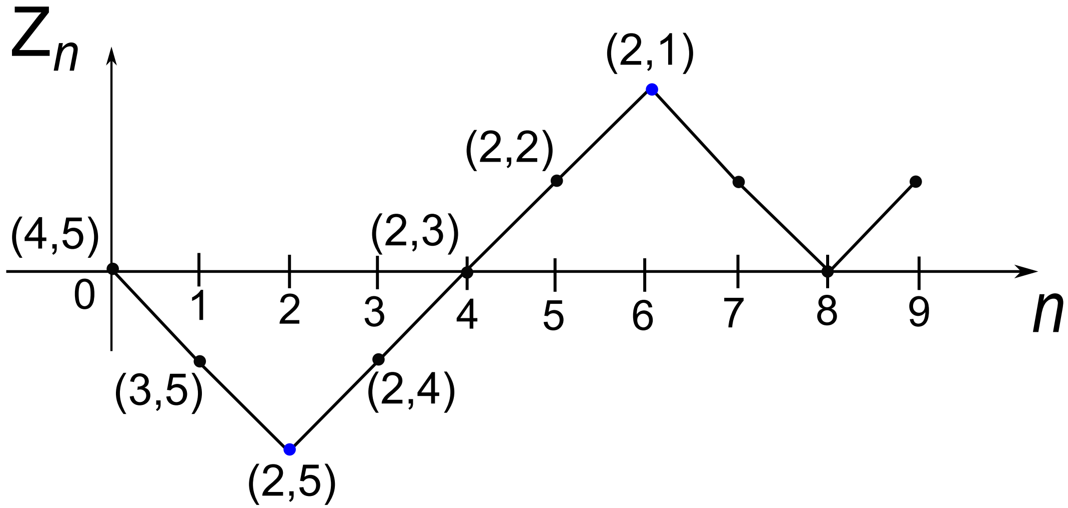


$p_1(m,p) = \frac{m}{m+p}$,

$$p_{2i}(m,p) = \frac{1}{2(2i-1)} \cdot \frac{\binom{m}{i} \cdot \binom{p}{i}}{\binom{m+p}{2i}}.$$

We stop the process if $s_1(m,p,m',p') \geq c_1(m,p,m',p'),\ p \geq m$.

Example. $m_0 = 4,\ p_0 = 5,\ y' = p_0 - m_0 = 1$



1-th stage (4,5)	(3,5) minus ball, $s_1(3,5) = 0.286, c_1(3,5) = 0.429$	continue
2-th stage (3,5)	(2,5) minus ball, $s_1(2,5) = 0.571, c_1(1,5) = 0.476$	**stop**
minimum	$m_{min} = 2, p_{min} = 5, y' = p_0 - m_0 = 1$	
3-th stage (2,5)	(2,4) plus ball	continue
4-th stage (2,4)	(2,3) plus ball, $m < p$	continue
5-th stage (2,3)	(2,2)minus ball, $s(2,2) = 0.333, c(2,2) = 0.667$	continue
6-th stage (2,2)	(2,1) plus ball, $s(2,1) = 0.667, c(2,1) = 0.667$	**stop**
maximim	$m_{max} = 2, p_{max} = 1$	

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