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On the strong law of large numbers for sequences of dependent random variables

Theorem A. (Kolmogorov) Let $\{X_n\}$ be a sequence of independent identically distributed random variables with finite mathematical expectation EX_1. Then

$$\frac{S_n}{n} \to EX_1 \ \ a.s.$$

where $S_n = \sum_{k=1}^{n} X_k$;

Theorem B. (Kolmogorov) Let $\{X_n\}$ be a sequence of independent random variables with finite variances. If

$$\sum_{n=1}^{\infty} \frac{VarX_n}{n^2} < \infty \tag{1}$$

then

$$\frac{S_n - ES_n}{n} \to 0 \ \ a.s. \tag{2}$$

In the following propositions we consider a sequence of random variables $\{X_n\}$ with finite variances.

Theorem 1 *Let $\{X_n\}$ be a sequence of non-negative random variables. Suppose that*

$$VarS_n \leqslant C \sum_{k=1}^{n} VarX_k \tag{3}$$

for all n where C is a constant,

$$E(S_n - S_m) \leqslant C(n-m) \tag{4}$$

for all sufficiently large $n - m$

and the condition (1) is satisfied. Then

$$\frac{S_n - ES_n}{n} \to 0 \;\; a.s.$$

The following result is a consequence of Theorem 1.

Theorem 2 *Suppose that $X_1, X_2, \ldots$ are pairwise independent random variables (not necessarily non-negative). If*

$$\sum_{k=m+1}^{n} E\left|X_k - EX_k\right| \leqslant C(n-m)$$

for all sufficiently large $n-m$ where C is a constant and the condition (1) is satisfied, then

$$\frac{S_n - ES_n}{n} \to 0 \ \ a.s.$$

Theorem 3 *Let $\{X_n\}$ be a sequence of non-negative random variables. Suppose that*

$$E(S_n - S_m) \leqslant C(n-m) \tag{5}$$

for all sufficiently large $n-m$ where C is a constant,

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} cov(X_i, X_j) \leqslant C \sum_{i,j=1}^{n} |EX_i - EX_j| \,, \tag{6}$$

$$\sum_{j=1}^{\infty} \frac{\sum_{i=1}^{j} |EX_i - EX_j|}{j^2} < \infty \tag{7}$$

and the condition (1) is satisfied, then

$$\frac{S_n - ES_n}{n} \to 0 \;\; a.s.$$

Theorems 1, 2 and 3 are generalizations of some results of N.Etemadi [1]. Conditions of [1] include the uniform boundedness of EX_n.

In the following two propositions the Kolmogorov condition is replaced by more restrictive requirements, but the condition of non-negativity and restrictions related to mathematical expectations are absent.

Theorem 4 *Let* $\{X_n\}$ *be a sequence of random variables. Suppose that*

$$\sum_{n=1}^{\infty} \frac{Var X_n}{n^{2-\varepsilon}} < \infty. \tag{8}$$

for some $\varepsilon > 0$,

$$Var(S_j - S_i) \leqslant C \left(\sum_{k=i+1}^{j} Var X_k \right)^{\gamma} \tag{9}$$

for all i, j *such that* $j > i \geqslant N_0$ *and for some* γ *such that* $1 < \gamma < 3/(3-\varepsilon)$ *where* N_0 *and* C *are constants. Then*

$$\frac{S_n - ES_n}{n} \to 0 \ \ a.s.$$

Theorem 5 *Let $\{X_n\}$ be a sequence of random variables. Suppose that*

$$\sum_{n=1}^{\infty} \frac{VarX_n}{n^2} (\ln n)^2 < \infty, \tag{10}$$

$$Var(S_j - S_i) \leqslant C \sum_{k=i+1}^{j} VarX_k \tag{11}$$

for all i, j such that $j > i \geqslant N_0$ where N_0 and C are constants. Then

$$\frac{S_n - ES_n}{n} \to 0 \ \ a.s.$$

The following result is a generalization of the Kolmogorov theorem on the strong law of large numbers for sequences of independent identically distributed random variables.

Theorem 6 *Let $\{X_n\}$ be a sequence of identically distributed random variables with finite mathematical expectation. Suppose that*

$$Var\left(\sum_{k=1}^{n}(X_k I_{\{0\leqslant X_k\leqslant k\}})\right)\leqslant C\sum_{k=1}^{n}Var(X_k I_{\{0\leqslant X_k\leqslant k\}}), \tag{12}$$

$$Var\left(\sum_{k=1}^{n}(X_k I_{\{-k\leqslant X_k\leqslant 0\}})\right)\leqslant C\sum_{k=1}^{n}Var(X_k I_{\{-k\leqslant X_k\leqslant 0\}}), \tag{13}$$

where C is a constant. Then

$$\frac{S_n}{n}\to EX_1 \ \ a.s.$$

If in Theorem 6 random variables $X_1, X_2, \ldots$ are pairwise independent then conditions (12) and (13) are satisfied.

References

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